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A space X is of countable type if for every compact $C \subset X$, there exists a compact set K having a countable basis with $C \subset K$. X is of point-countable type if there exists a covering of compact subsets of X, each having a countable basis. It is shown that in a Hausdorff space of countable type, a compact set has a countable basis if and only if it is a G_δ-set. Similarly, for Hausdorff spaces of point-countable type, a point has a countable basis if and only if it is a G_δ-set.

1. Terminology. Notation and terminology will follow that of Dugundji [2]. By a neighborhood of a set A, we will mean an open set containing A.

If X is a space and $A \subset X$, a collection $\mathscr{D}$ of neighborhoods of A is called a *basis at* A if and only if for every neighborhood 0 of A, there exists $D \in \mathscr{D}$ with $A \subset D \subset 0$.

If X is a space and $A \subset X$, then A is said to be of *countable character* if and only if there exists a countable basis at A.

A space X is said to be of *countable type* if for every compact $C \subset X$, there exists a compact set K of countable character with $C \subset K$.

A space X is said to be of *point-countable type* if there exists a covering of compact subsets of X, each having countable character.

2. Discussion and theorems. Every first countable space, as well as every locally compact Hausdorff space, is of point-countable type, while spaces of point-countable type are, in turn, k-spaces. Compact spaces are trivially of countable type, but these two concepts are fairly far removed from each other since a metric space is of countable type.

The following lemmas will be needed. Lemma 2, which was first noted by Arhangel'skiĭ [1], can be verified by a slight modification of Wicke's proof of Lemma 1. The author is indebted to Howard Cook for some valuable suggestions.

LEMMA 1. (*Wicke*). *In a Hausdorff space X, the following properties are equivalent*:

(i) *X is of point-countable type.*

(ii) *If 0 is an open set in X and $x \in 0$, there exists a compact set B of countable character such that $x \in B$ and $B \subset 0$.*

LEMMA 2. (*Arhangel'skii*). *Suppose X is a Hausdorff space of* *countable type, U is an arbitrary compact subset, and 0 is any of its neighborhoods. Then there exists a compact set C of countable character such that $U \subset C \subset 0$.*

LEMMA 3. *Let X be a Hausdorff space and let U and V be compact subsets of countable character. Then $U \cap V$ is also a compact set of countable character.*

Proof. That $U \cap V$ is compact is obvious. Denote the members of the countable bases at U and V by U_n and V_n, respectively, and assume that the collections $\{U_n\}$ and $\{V_n\}$ are descending. It will be shown that the collection $\{U_n \cap V_n\}$ forms a local basis at $U \cap V$. Thus, let 0 be any neighborhood of $U \cap V$. Then $U - 0$ and $V - 0$ are disjoint compact sets, and hence there exist disjoint open sets U^* and V^* with $U - 0 \subset U^*$ and $V - 0 \subset V^*$. Since $U^* \cup 0$ is a neighborhood of U, there exists an integer m with $U \subset U_m \subset U^* \cup 0$. Similarly, there exists an integer n with $V \subset V_n \subset V^* \cup 0$. Letting $k = \max\{m, n\}$, it follows that $U \cap V \subset U_k \cap V_k \subset 0$; for if this is not true then there must exist a point $p \in U_k \cap V_k - 0$ which implies that $p \in U^* \cap V^*$, contradicting the disjointness of U^* and V^*.

For $n \geqq 1$, it follows from Lemma 2 that there exists a compact set C'_n of countable character such that $U \subset C'_n \subset G_n$. Let $C_n = \bigcap_{i=1}^{n} C'_i$. By Lemma 3, each C_n is also a compact set of countable character.

THEOREM 1. *Let X be a Hausdorff space of countable type and let U be any compact subset which is also a G_δ-set. Then U has a countable basis.*

Proof. By hypothesis, there exist neighborhoods G_n of U such that $U = \bigcap_{n=1}^{\infty} G_n$. Construct a sequence $\{C_n\}$ of compact sets in the following manner:

By Lemma 2, there exists a compact set C_1 of countable character such that $U \subset C_1 \subset G_1$. For $n > 1$, it also follows from Lemma 2 that there exists a compact set C'_n of countable character such that $U \subset C'_n \subset G_n$. Let $C_n = [\bigcap_{i=1}^{n-1} C_i] \cap C'_n$. From a previous remark, each C_n is also a compact set of countable character.

Let $\{U_{m,n}\}$ be a countable basis at C_m. Clearly, $U \subset U_{m,n}$ for every pair (m, n), and furthermore, $U \subset \bigcap_{m,n} U_{m,n} \subset \bigcap_{n=1}^{\infty} G_n = U$. Hence, $\bigcap_{m,n} U_{m,n} = U$. It will now be shown that the collection $\{U_{m,n}\}$ is a basis at U. Indeed, if it is not, then there exists a neighborhood K of U such that $U_{m,n} - K \neq \phi$ for every pair (m, n). This forces $C_m - K \neq \phi$ for each integer m because, if not, then $C_m \subset K$ for some m, and hence there exists an integer n with $C_m \subset U_{m,n} \subset K$ which is contrary to our assumption. Since $C_m - K$ is a decreasing sequence of nonempty compact sets, $\bigcap_{m=1}^{\infty}[C_m - K] \neq \phi$. But if $p \in \bigcap_{m=1}^{\infty}[C_m - K]$, then $p \in \bigcap_{m,n} U_{m,n}$ which implies that $p \in U$. This is impossible since $p \in X - K$ and $U \subset K$. Thus, $\{U_{m,n}\}$ is a basis at U, and the theorem is proved.

COROLLARY. *In a Hausdorff space of countable type, a compact set has a countable basis if and only if it is a G_δ-set.*

THEOREM 2. *Let X be a Hausdorff space of point-countable type, and let $p \in X$ be any point which is a G_δ-set. Then p has a countable basis.*

Proof. In the proof of Theorem 1, use Lemma 1 instead of Lemma 2 and substitute "point p" in place of U.

COROLLARY. *A Hausdorff space is first countable if and only if it is of point countable type and each point is a G_δ-set.*

COROLLARY. *A locally compact Hausdorff space is first countable if and only if each point is a G_δ-set.*

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