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This paper is a study of reflexivity of tensor products of Banach spaces and the related topic of reflexivity of the space $\mathscr{L}(X, Y)$ (the space of bounded linear operators from X to Y with operator norm). If X and Y are Banach spaces with Schauder bases, then necessary and sufficient conditions for $X \otimes_\pi Y$, $X \otimes_\varepsilon Y$, and $\mathscr{L}(X, Y)$ to be reflexive are given, and examples of infinite dimensional spaces X and Y for which $X \otimes_\pi Y$, $X \otimes_\varepsilon Y$, and $\mathscr{L}(X, Y)$ are reflexive are constructed.

In this paper we study the reflexivity of tensor products of Banach spaces and the related topic of reflexivity of the space $\mathscr{L}(X, Y)$ (the space of bounded linear operators from X to Y with operator norm). In particular we study the tensor products of the classical l^p and $L^p[0, 1]$ spaces.

If α is a crossnorm on the tensor product [9, p. 9] and $X \otimes_\alpha Y$ denotes the completion of the algebraic tensor product $X \otimes Y$ in the α-norm, then each of X and Y is isometrically embedded in $X \otimes_\alpha Y$. Consequently, if $X \otimes_\alpha Y$ is reflexive then each of X and Y must also be reflexive. In general the converse is not true, as Schatten [9, p. 138] and Grothendieck [4, p. 49] have shown. We will study this problem when $\alpha = \pi$ and $\alpha = \varepsilon$ (i.e., for α the greatest and least crossnorm), obtaining a complete characterization of reflexive spaces $X \otimes_\pi Y$ and $X \otimes_\varepsilon Y$ in the case where X and Y have Schauder bases.

In § 3 we derive a necessary and sufficient condition for $X \otimes_\pi Y$ to be reflexive in the case where X and Y have bases. As corollaries to this result we obtain a necessary and sufficient condition for $X \otimes_\varepsilon Y$ to be reflexive if X and Y have bases and we construct examples of reflexive spaces $X \otimes_\varepsilon Y$ and $X \otimes_\pi Y$ in which both X and Y are infinite dimensional.

In § 4 we give a necessary and sufficient condition for the space $\mathscr{L}(X, Y)$ to be reflexive when X and Y have bases. Using results of Lindenstrauss and Pelczynski on Hilbertian operators [7] we are able to obtain another sufficient condition for the reflexivity of $\mathscr{L}(X, Y)$ when X and Y have bases and to exhibit infinite dimensional spaces X and Y for which $\mathscr{L}(X, Y)$ is reflexive.

Section 5 is concerned with the reflexivity of certain special tensor products and with some unusual examples. A remark on bases for tensor product spaces is also made.

Section 6 contains several unsolved problems related to the subject matter of the previous sections and remarks concerning their solution.

2. Notation and preliminary results. The only spaces considered in this paper will be Banach spaces. If X is a given space we will denote its dual space by X^*. The closed linear span of a sequence (x_i) in X is denoted by $[x_i]$.

A sequence (x_i) in X is called a basis for X (basic sequence in X) if for each x in X (for each x in $[x_i]$) there exists a unique sequence of scalars (a_i) such that $x = \sum_{i=1}^{\infty} a_i x_i$, convergence in the norm topology of X. Two bases (x_i) and (y_i) for X and Y are called *similar* if $\sum_i a_i x_i$ converges if and only if $\sum_i a_i y_i$ converges. Equivalently, there exists an isomorphism $T: X \to Y$ such that $T(x_i) = y_i$.

The sequence of linear functionals (f_i) in X^* defined by

$$f_n(x) = f_n\Big(\sum_i a_i x_i\Big) = a_n$$

is called the *associated sequence of coefficient functionals* and we denote the basis by (x_i, f_i). One may show that (f_i) is a basic sequence in X^* whose coefficient functionals in $[f_i]^*$ are similar to (x_i) in X. Hence we write (f_i, x_i) is a basic sequence in X^*.

A basis (x_i, f_i) is called *semi-normalized* if

$$0 < \inf_i \|x_i\| \leqq \sup_i \|x_i\| < +\infty .$$

Throughout this paper *all bases will be assumed to be semi-normalized.*

The notation $X = Y$ will mean X is linearly homeomorphic (isomorphic) to Y, and $X \subset Y$ will mean X is isomorphic to a closed subspace of Y. The symbol $\mathscr{L}(X, Y)$ will denote the space of bounded linear operators from X to Y with operator norm.

We denote by $X \otimes_\varepsilon Y$ the completion of the algebraic tensor product $X \otimes Y$ in the norm

$$\Big\|\sum_{i=1}^{n} x_i \otimes y_i\Big\| = \sup_{\substack{\|f\| \leqq 1, f \in X^* \\ \|g\| \leqq 1, g \in Y^*}} \Big|\sum_{i=1}^{n} f(x_i) g(y_i)\Big| ,$$

and by $X \otimes_\pi Y$ the completion in the norm

$$\Big\|\sum_{i=1}^{n} x_i \otimes y_i\Big\| = \inf\Big\{\sum_{j=1}^{k} \|x'_j\|\|y'_j\| : \sum_{j=1}^{k} x'_j \otimes y'_j = \sum_{i=1}^{n} x_i \otimes y_i\Big\} .$$

The following results in the theory of tensor product spaces are crucial to our work and will be used repeatedly throughout this paper, often without specific reference.

(A) If M is a closed subspace of X and N is a closed subspace of Y, then $M \otimes_\varepsilon N \subset X \otimes_\varepsilon Y$ [9, p. 35].

(B) $X^* \otimes_\varepsilon Y^* \subset (X \otimes_\pi Y)^*$ [9, p. 43].

(C) The space $(X \otimes_\pi Y)^*$ is isometrically isomorphic to the space of all continuous linear mappings from X to Y^* [9, p. 45].

(D) The space $X^* \otimes_\varepsilon Y^*$ is isometrically isomorphic to the space of all continuous linear mappings from X to Y^* which can be approximated in operator norm by finite dimensional maps [9, p. 50].

Hence if Y^* has a basis, then $X^* \otimes_\varepsilon Y^*$ is the space of all compact maps from X to Y^* [9, p. 51].

(E) If X^* has a basis, then $(X \otimes_\varepsilon Y)^* = X^* \otimes_\pi Y^*$ [5].

(F) If (x_i, f_i) is a basis for X and (y_i, g_i) is a basis for Y, then the sequence of tensors $(x_i \otimes y_j)$ ordered in the following way

$$\begin{array}{ccc|c}
\multicolumn{1}{c|}{x_1 \otimes y_1} & \multicolumn{1}{c|}{x_1 \otimes y_2} & x_1 \otimes y_3 & \cdots \\
\cline{1-1}
x_2 \otimes y_1 & \multicolumn{1}{c|}{x_2 \otimes y_2} & x_2 \otimes y_3 & \cdots \\
\cline{1-2}
x_3 \otimes y_1 & x_3 \otimes y_2 & x_3 \otimes y_3 & \cdots \\
\cline{1-3}
\vdots & \vdots & \vdots &
\end{array}$$

is a basis for $X \otimes_\varepsilon Y$ and $X \otimes_\pi Y$ and the associated sequence of coefficient functionals is the sequence $(f_i \otimes g_j)$ [3].

The basis $(x_i \otimes y_j, f_i \otimes g_j)$ described in (F) is called the tensor product of the bases (x_i) and (y_i), or the *tensor product basis*.

3. Reflexive tensor products.

In this section we characterize those spaces $X \otimes_\pi Y$ and $X \otimes_\varepsilon Y$ which are reflexive (for the case where X and Y both have bases). We will need the following lemma and its corollary.

LEMMA 3.1. *Let (x_i, f_i) and (y_i, g_i) be bases for Banach spaces X and Y respectively. Then the basis $(x_i \otimes y_j, f_i \otimes g_j)$ for $X \otimes_\varepsilon Y$ is shrinking if and only if each of (x_i, f_i) and (y_i, g_i) is shrinking.*

Proof. It follows trivially from the fact that every subsequence of a shrinking basis is shrinking that if $(x_i \otimes y_j)$ is shrinking then each of (x_i) and (y_i) is also shrinking.

Conversely, if (x_i, f_i) and (y_i, g_i) are shrinking, then by definition (f_i) is a basis for X^* and (g_i) is a basis for Y^*. It follows that $(f_i \otimes g_j)$ is a basis for $X^* \otimes_\pi Y^* = (X \otimes_\varepsilon Y)^*$ (by (E) in § 2), and by definition $(x_i \otimes y_i, f_i \otimes g_j)$ is shrinking.

COROLLARY 3.2. *Let (x_i, f_i) and (y_i, g_i) be bases for X and Y respectively. Then the basis $(x_i \otimes y_j, f_i \otimes g_j)$ for $X \otimes_\pi Y$ is boundedly complete if and only if each of (x_i) and (y_i) is boundedly complete.*

Proof. As in 3.1, if $(x_i \otimes y_j)$ is boundedly complete it follows trivially that each of (x_i) and (y_i) is boundedly complete.

Conversely, if (x_i) and (y_i) are boundedly complete then the associated sequences of coefficient functionals (f_i) and (g_i) are shrinking. Hence by Lemma 3.1 $(f_i \otimes g_j)$ is a shrinking basis for $[f_i] \otimes_\varepsilon [g_i] \subset (X \otimes_\pi Y)^*$. But then $(x_i \otimes y_j, f_i \otimes g_j)$ is boundedly complete.

We can now state and prove the main result of this section.

THEOREM 3.3. *Let X and Y be reflexive Banach spaces with bases. Then $X \otimes_\pi Y$ is reflexive if and only if every continuous linear map $T: X \to Y^*$ is compact.*

Proof. Let (x_i, f_i) and (y_i, g_i) be bases for X and Y respectively. Then by the result of James [6] (f_i) is a basis for X^* and (g_i) is a basis for Y^*.

Suppose $X \otimes_\pi Y$ is reflexive. Then the basis $(x_i \otimes y_j, f_i \otimes g_j)$ is shrinking, implying that $(f_i \otimes g_j)$ is a basis for $(X \otimes_\pi Y)^*$. Now since $(f_i \otimes g_j)$ is also a basis for $[f_i] \otimes_\varepsilon [g_i] = X^* \otimes_\varepsilon Y^*$, a closed subspace of $(X \otimes_\pi Y)^*$, it must be that $X^* \otimes_\varepsilon Y^*$ and $(X \otimes_\pi Y)^*$ coincide. It then follows from the results of Schatten ((C) and (D) of § 2) that every $T: X \to Y^*$ is compact.

On the other hand, if every $T: X \to Y^*$ is compact then reversing the above argument we have that $X^* \otimes_\varepsilon Y^*$ and $(X \otimes_\pi Y)^*$ coincide so $(f_i \otimes g_j)$, being a basis for $X^* \otimes_\varepsilon Y^*$, is a basis for $(X \otimes_\pi Y)^*$. That is, the basis $(x_i \otimes y_j)$ for $X \otimes_\pi Y$ is shrinking. By Corollary 3.2 this basis is also boundedly complete, and it follows from the theorem of James [6] that $X \otimes_\pi Y$ is reflexive.

In order to use Theorem 3.3 to obtain a characterization of reflexive spaces $X \otimes_\varepsilon Y$ we will need the following simple lemma.

LEMMA 3.4. *Let X and Y be reflexive Banach spaces with bases. Then $X \otimes_\varepsilon Y$ is reflexive if and only if $X^* \otimes_\pi Y^*$ is reflexive.*

Proof. By (E) of § 2, $(X \otimes_\varepsilon Y)^* = X^* \otimes_\pi Y^*$. The lemma is now immediate since a Banach space is reflexive if and only if its dual is reflexive.

COROLLARY 3.5. *Let X and Y be reflexive Banach spaces with bases. Then $X \otimes_\varepsilon Y$ is reflexive if and only if every continuous linear map $T: X^* \to Y$ is compact.*

Proof. By Lemma 3.4, $X \otimes_\varepsilon Y$ is reflexive if and only if $X^* \otimes_\pi Y^*$ is reflexive. But since X^* and Y^* each have bases, it follows from Theorem 3.3 that $X^* \otimes_\pi Y^*$ is reflexive if and only if every

$T: X^* \to (Y^*)^* = Y$ is compact.

Using Theorem 3.3 and Corollary 3.5 it is now easy to give examples of reflexive tensor products in the ε and π topologies. Of course, if either X or Y is finite dimensional and the other has a basis, then both $X \otimes_\pi Y$ and $X \otimes_\varepsilon Y$ are reflexive. Less trivially, we have the following two propositions.

PROPOSITION 3.6. *Let $1 < p, r < +\infty$. Then $l^p \otimes_\pi l^r$ is reflexive if and only if $p > r/r - 1$.*

Proof. By Theorem 3.3 $l^p \otimes_\pi l^r$ is reflexive if and only if every continuous linear mapping $T: l^p \to l^{r/r-1}$ is compact. However this last is true if and only if $p > r/r - 1$ by a theorem by Pitt [8] (see also [1]).

PROPOSITION 3.7. *Let $1 < p, r < +\infty$. Then $l^p \otimes_\varepsilon l^r$ is reflexive if and only if $p/p - 1 > r$.*

4. Hilbertian operators and reflexive spaces $\mathscr{L}(X, Y)$. Our first result in this section characterizes those spaces X and Y with bases for which $\mathscr{L}(X, Y)$ is reflexive.

THEOREM 4.1. *Let X and Y be reflexive Banach spaces with bases. Then $\mathscr{L}(X, Y)$ is reflexive if and only if every continuous linear map $T: X \to Y$ is compact.*

Proof. As Schatten has shown ((C) of § 2),

$$(X \otimes_\pi Y^*)^* = \mathscr{L}(X, Y^{**}) = \mathscr{L}(X, Y) .$$

But $(X \otimes_\pi Y^*)^*$ is reflexive if and only if $X \otimes_\pi Y^*$ is reflexive. Therefore it follows from Theorem 3.3 that $\mathscr{L}(X, Y)$ is reflexive if and only if every continuous linear map $T: X \to Y^{**} = Y$ is compact.

An immediate consequence of Theorem 4.1 is

COROLLARY 4.2. *Let $1 < p, r < +\infty$. Then $\mathscr{L}(l^p, l^r)$ is reflexive if and only if $p > r$.*

Proof. By Pitt's theorem [8], every $T: l^p \to l^r$ is compact if and only if $p > r$. Apply Theorem 4.1.

Recall that a mapping $T: X \to Y$ is called Hilbertian [7] if there exists a Hilbert space H and mappings $T_2 X \to H$, $T_2: H \to Y$ for which $T = T_2 T_1$.

THEOREM 4.3. *Let X and Y be reflexive Banach spaces with* *bases such that every continuous linear map $T: X \to Y$ is Hilbertian and at least one of the following holds*:

(i) *every $S: X \to l^2$ is compact.*

(ii) *every $S: l^2 \to X$ is compact.*

Then $\mathscr{L}(X, Y)$ is reflexive.

Proof. Let $T \in \mathscr{L}(X, Y)$. Then since T is Hilbertian, $T = T_2 T_1$ (where T_2 and T_1 are as above). If (i) holds, then since the range of T_1 is contained in a separable subspace of a Hilbert space H we must have T_1, and therefore T, is compact. By Theorem 4.1 $\mathscr{L}(X, Y)$ is reflexive.

A similar argument establishes the result if (ii) holds.

An immediate consequence of Theorem 4.3 and Pitt's theorem is

COROLLARY 4.4. *Let $2 < p < +\infty$ and let X be a reflexive Banach space with a basis such that every $T: l^p \to X$ is Hilbertian. Then $\mathscr{L}(l^p, X)$ is reflexive.*

Similarly we have

COROLLARY 4.5. *Let $1 < p < 2$ and let X be a reflexive Banach space with a basis such that every $T: X \to l^p$ is Hilbertian. Then $\mathscr{L}(X, l^p)$ is reflexive.*

Lindenstrauss and Pelczynski have proven the following deep result [7]:

If X is an $\mathscr{L}^p$-space [7] *with $2 \leqq p \leqq +\infty$ and Y is an $\mathscr{L}^r$-space with $1 \leqq r \leqq 2$, then every $T: X \to Y$ is Hilbertian.*

Since every $L^p(\mu)$ space is an $\mathscr{L}^p$-space [7], we obtain the following corollaries of 4.4 and 4.5.

COROLLARY 4.6. *Let $2 < p < +\infty$ and $1 < r \leqq 2$. Then*

$$\mathscr{L}(l^p, L^r[0, 1])$$

is reflexive.

COROLLARY 4.7. *Let $1 < p < 2$ and $2 \leqq r < +\infty$. Then*

$$\mathscr{L}(L^r[0, 1], l^p)$$

is reflexive.

As a consequence of the above corollaries and Theorems 4.1, 3.3, and 3.5, we have

COROLLARY 4.8. *Let* $2 < p < +\infty$ *and* $1 < r \leqq 2$. *Then*

$$l^{p/p-1} \otimes_\varepsilon L^r[0, 1]$$

and $l^p \otimes_\pi L^{r/r-1}$ $[0, 1]$ *are reflexive.*

COROLLARY 4.9. *Let* $1 < p < 2$ *and* $2 \leqq r < +\infty$. *Then*

$$L^{r/r-1}[0, 1] \otimes_\varepsilon l^p$$

and $L^r[0, 1] \otimes_\pi l^{p/p-1}$ *are reflexive.*

5. Some special tensor products. The purpose of this section is to show that certain types of tensor product spaces can never be reflexive. We also make some comments on tensor product bases.

Our first results are a consequence of the theorems of § 3.

PROPOSITION 5.1. *Let* X *be a reflexive infinite dimensional Banach space with a basis. Then neither* $X \otimes_\pi X^*$ *nor* $X \otimes_\varepsilon X^*$ *is reflexive.*

Proof. Neither the evaluation map $J: X \rightarrow X^{**}$ nor the identity map $I: X^* \rightarrow X^*$ is compact. Hence by Theorem 3.3 and Corollary 3.5 neither $X \otimes_\pi X^*$ nor $X \otimes_\varepsilon X^*$ is reflexive.

Somewhat surprisingly, the tensor product of Hilbert spaces is not reflexive.

PROPOSITION 5.2. *Let* H_1 and H_2 *be infinite dimensional Hilbert spaces. Then neither* $H_1 \otimes_\varepsilon H_2$ *nor* $H_1 \otimes_\pi H_2$ *is reflexive.*

Proof. Clearly $l^2 \subset H_1$ and $l^2 \subset H_2$. Therefore $l^2 \otimes_\varepsilon l^2 \subset H_1 \otimes_\varepsilon H_2$. But by Corollary 3.5, $l^2 \otimes_\varepsilon l^2$ is not reflexive and it follows that $H_1 \otimes_\varepsilon H_2$ cannot be reflexive.

Now $H_1 \otimes_\varepsilon H_2 = H_1^* \otimes_\varepsilon H_2^*$, and $H_1^* \otimes_\varepsilon H_2^* \subset (H_1 \otimes_\pi H_2)^*$. It follows that $(H_1 \otimes_\pi H_2)^*$, and hence $H_1 \otimes_\pi H_2$, is not reflexive.

(Proposition 5.2 was proved by Grothendieck in a different manner [4, p. 49]).

In light of 3.6, 3.7, 4.8, and 4.9 the following proposition is interesting.

PROPOSITION 5.3. *Let* $1 < r < +\infty$ *and* $1 < s < +\infty$. *Then* $L^r[0, 1] \otimes_\varepsilon L^s[0, 1]$ *and* $L^r[0, 1] \otimes_\pi L^s[0, 1]$ *are not reflexive.*

Proof. It is well known that $l^2 \subset L^p[0,1]$ for all $1 < p < +\infty$ [7]. Hence $l^2 \otimes_\varepsilon l^2 \subset L^r[0,1] \otimes_\varepsilon L^s[0,1]$ for all $1 < r < +\infty, 1 < s < +\infty$ and by Proposition 5.2 $L^r[0,1] \otimes_\varepsilon L^s[0,1]$ is not reflexive.

Now since $L^r[0,1]$ and $L^s[0,1]$ each has a basis,

$$L^r[0,1] \otimes_\pi L^s[0,1] = [L^{r/r-1}[0,1] \otimes_\varepsilon L^{s/s-1}[0,1])^* .$$

Therefore it follows from the above consideration that

$$L^r[0,1] \otimes_\pi L^s[0,1]$$

is not reflexive.

PROPOSITION 5.4. *Let X be a reflexive space and X_1 a subspace of X having a basis. Then $X_1 \otimes_\pi X^*$ is not reflexive.*

Proof. Suppose $X_1 \otimes_\pi X^*$ is reflexive. Then $(X_1 \otimes_\pi X^*)^*$ is reflexive, and since $X_1^* \otimes_\epsilon X_1 \subset X_1^* \otimes_\varepsilon X^{**} \subset (X_1 \otimes_\pi X^*)^*$ this implies that $X_1 \otimes_\varepsilon X_1^*$ is reflexive, a contradiction to Proposition 5.1.

COROLLARY 5.5. *Let X be a reflexive infinite dimensional Banach space. Then there exists an infinite dimensional reflexive space Y such that $Y \otimes_\varepsilon X$ is not reflexive.*

Proof. By a result of Gelbaum [2] the space X contains a basic sequence (x_i). Let $Y = [x_i]^*$. Then if $Y \otimes_\epsilon X$ were reflexive, $[x_i] \otimes_\pi X^* = [x_i]^{**} \otimes_\pi X = ([x_i]^* \otimes_\varepsilon X)^* = (Y \otimes_\varepsilon X)^*$ would be reflexive, a contradiction to Proposition 5.4.

The usual bases studied in tensor product spaces $X \otimes_\alpha Y$ are the so called "tensor product bases" of Gelbaum and de Lemadrid [3], i.e. bases of the form $(x_i \otimes y_j)$ where (x_i) is a basis for X, (y_i) is a basis for Y, and the sequence $(x_i \otimes y_j)$ is ordered according to (F) in § 2. For a large class of spaces we can prove the existence of bases which are not of this form.

PROPOSITION 5.6. *Let X and Y be reflexive spaces with bases* such that *$X \otimes_\varepsilon Y$ is not reflexive. Then $X \otimes_\varepsilon Y$ has a basis which is not a tensor product $(x_i \otimes y_j)$ of bases (x_i) in X and (y_i) in Y.*

Proof. Since X and Y are reflexive, every basis in X and every basis in Y is shrinking. By Lemma 3.1 every tensor product basis is shrinking in $X \otimes_\varepsilon Y$. Therefore if every basis for $X \otimes_\varepsilon Y$ was a tensor product basis, every basis would be shrinking and $X \otimes_\epsilon Y$ would be reflexive [10], a contradiction.

In particular, according to Proposition 5.3, the spaces $L^r[0.1] \otimes_\varepsilon L^s[0, 1]$ have such bases.

We remark that *Proposition* 5.6 *remains true if* ε *is replaced by* π since if every basis for a space with a basis is boundedly complete, then the space is reflexive [10]. Apply Corollary 3.2.

In particular, the spaces $L^r[0, 1] \otimes_\pi L^s[0, 1]$ have bases which are not tensor product bases.

6. Unsolved problems.

I. Do there exist infinite dimensional reflexive spaces X and Y such that both $X \otimes_\varepsilon Y$ and $X \otimes_\pi Y$ are reflexive?

II. Do there exist infinite dimensional reflexive spaces X and Y such that $\mathscr{L}(X, Y)$ and $\mathscr{L}(Y, X)$ are reflexive?

According to Theorem 4.1, if each of X and Y has a basis this problem is equivalent to the problem:

II'. If every $T: X \to Y$ and every $S: Y \to X$ is compact (S and T bounded linear operators), must one of X or Y be finite dimensional? This question, for arbitrary Banach spaces X and Y, has been asked by Pelczynski at the Sopot Conference.

III. Does there exist an infinite dimensional reflexive space X such that $X \otimes_\varepsilon X^*$ or $X \otimes_\pi X^*$ is reflexive?

(We have shown in Proposition 5.1 that the answer is "no" if X has a basis. Therefore a positive answer (which is unlikely) for separable X would settle the "basis problem").

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