LACUNARY SERIES AND PROBABILITY

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In this note we continue some investigations connecting a lacunary series $A$ of real numbers

$$A: 1 \leq \lambda_1 < \cdots < \lambda_k < \cdots, q \lambda_k \leq \lambda_{k+1} \quad (1 < q)$$

and a probability measure $\mu$ on $(-\infty, \infty)$ satisfying

$$(1) \quad \mu([a, a + h]) \ll h^\beta$$

for all intervals $[a, a + h]$ of length $h < 1$, and a fixed exponent $0 < \beta < 1$. (The notation $X \ll Y$ is a substitute for $X = O(Y)$.) Measures $\mu$ occur in the theory of sets of fractional Hausdorff dimension.

In the following statements $S$ is a subset of $(-\infty, \infty)$ of Lebesgue measure 0, depending only on $\mu$ and $A$.

**Theorem 1.** For $r = 2, 4, 6, \cdots$ and $t \in S$, there is a constant $B_r(t)$ so that

$$\int_{-\infty}^{\infty} | \sum a_k \cos (\lambda_k t x + b_k) | r \mu(dx) \leq B_r(t) (\sum |a_k|^2)^{r/2}.$$  

Here $B_r(t)$ is independent of the sequences $(a_j)$ and $(b_k)$.

**Theorem 2.** For $t \in S$ the normalized sums

$$\langle \sum \cos (\lambda_k t x + b_k) \rangle^{1/2}$$

tend in law (with respect to the probability $\mu$) to the normal law. Here the convergence is uniform for all sequences $(b_k)$.

Theorem 1 is a random form of a fact apparently known from the advent of the study of lacunary series; Theorem 2 bears the same relation to the work of Salem and Zygmund [4]. Probability enters critically in the theorems because $\beta < 1$: for any increasing sequence $A$ there is a measure $\mu$ fulfilling (1) for every $\beta < 1$ and such that the $t$-set defined in Theorem 1 is of first category.

1. In this section and later we use the notations

$$e(y) = e^{iy}, \quad \mu(y) = \int_{-\infty}^{\infty} e(yx) \mu(dx),$$

$-\infty < y < \infty$. In the following estimation $|y| > 1$.

$$I = \int_{1}^{2} |\hat{\mu}(ty)|^2 dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e(tyx_1 - tyx_2) dt \cdot \mu(dx_1) \mu(dx_2)$$

$$\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \inf (1, 2|yx_1 - yx_2|^{-1}) \mu(dx_1) \mu(dx_2).$$
Let \( r > 0 \) be the integer defined by \( 2^{-r} < |y|^{-1} \leq 2^{1-r} \); we sum the integrand over the sets
\[
(|x_1 - x_2| > 1), (1 > |x_1 - x_2| \geq \frac{1}{2}), \ldots, (2^{1-r} > |x_1 - x_2| > 2^{-r})
\]
and finally over the set \((2^{-r} > |x_1 - x_2|)\). In each case the product measure can be estimated by (1) and Fubini's Theorem; summing up we obtain \( I \ll |y|^{-\beta} \). A more convenient form is valid for all real \( y \):
\[
(2) \quad \int_1^2 |\hat{\mu}(ty)| dt \ll (1 + |y|)^{-1/2\beta}.
\]

2. To prove Theorem 1 we require an elementary lemma.

**Lemma.** Let \((v_k)_{k=1}^\infty\) be a sequence of real numbers and \( r \) a positive integer. Let \( T \) be the sum of the moduli of all Fourier-Stieltjes coefficients
\[
\hat{\mu}(d_1 v_{k_1} + d_2 v_{k_2} + \cdots)
\]
where \( 1 \leq k_1 < k_2 < \cdots, d_1, d_2, \cdots \) are integers \( \neq 0 \), and
\[
|d_1| + |d_2| + \cdots \leq 2r;
\]
the number of integers \( d_1, d_2, \cdots \) varies between 1 and 2\( r \). Then
\[
\int |\sum a_k e(v_k x)|^r \mu(dx) \leq (1 + T)(r!)(\sum |a_k|^2)^r.
\]

**Proof.** We first expand \((\sum a_k e(v_k x))^r\) by the multinomial formula, obtaining a sum of terms
\[
r!(e_1 e_2 \cdots e_r)^{1-r} a_{k_1}^r \cdots a_{k_r}^r e(e_1 v_{k_1} x + \cdots + e_r v_{k_r} x).
\]
Of course \( 1 \leq k_1 < \cdots < k_r \), and the \( r \)-tuple \((e_1, \cdots, e_r)\) is variable, subject to the equality \( e_1 + \cdots + e_r = r \). Next to this expansion we place that of the conjugate, using exponents \( f_1, \cdots, f_r \). Multiplying these expansions and integrating with respect to \( \mu \), we collect the integrals in two steps.

First we consider terms in the product in which \((e_1, \cdots, e_r) = (f_1, \cdots, f_r)\). Making a term-by-term comparison with \((\sum |a_k|^2)^r\), we find a sum \( \leq r!(\sum |a_k|^2)^r\).

For the remaining terms we note the factor \( \hat{\mu}(e_i v_{k_i} - f_i v'_{k_i} + \cdots) \) attached to the number \(|a_{k_1}^{e_1+f_1} \cdots|\), and note that the former number is counted in \( T \). Thus the sum here is \( \leq (r!)^2 \max |a_k|^{2r} \), and the proof is complete.
To prove Theorem 1 it will be enough to give a proof for sequences $A$ with a gap $q \geq 2r$, for in any case $A$ is a union of $1 + \lceil \log q / \log 2r \rceil$ sequences with gaps of this size. According to the lemma, it is sufficient to show that for almost all $t$, the sum $T$ is finite, where $T$ is calculated for the sequence $v_k = t\lambda_k$. Thus $T$ is a sum of numbers

$$|\hat{\mu}(td_1\lambda_{k_1} + \cdots + td_s\lambda_{k_s})|,$$

where $d_i \neq 0$, $\cdots$, $d_s \neq 0$, $|d_1| + \cdots + |d_s| \leq 2r$. Because $q \geq r$ and $|d_1| + \cdots + |d_{s-1}| \leq 2r - 1$,

$$|d_1\lambda_{k_1} + \cdots + d_s\lambda_{k_s}| \geq \frac{1}{r}\lambda_{k_s},$$

whence

$$\int_1^2 |\hat{\mu}(td_1\lambda_{k_1} + \cdots + td_s\lambda_{k_s})| dt \ll k_{k_s}^{-1/2}. $$

But the number of forms $d_1\lambda_{k_1} + \cdots + d_s\lambda_{k_s}$ having a certain $k = k_s$ is $\ll k^r$. Thus $\int_1^2 T dt < \infty$ because $\sum_{k=1}^{\infty} k^r \lambda_k^{-1/2j} < \infty$. This proves Theorem 1 for the interval $1 < t < 2$ and the same argument is plainly valid for $(-\infty, \infty)$.

3. In the proof of Theorem 2 it is again necessary to estimate sums like $T$, but it is no longer possible to make such sums converge. Instead, we must estimate their rate of increase.

**Lemma.** Let $d_1 \neq 0$, $\cdots$, $d_s \neq 0$ be integers and

$$p = |d_1| + \cdots + |d_s|.$$  

The number of $s$-tuples $1 \leq k_1 < \cdots < k_s \leq N$ for which

(3)  

$$|d_1\lambda_{k_1} + \cdots + d_s\lambda_{k_s} - \lambda| \leq 2^j \quad (j = 1, 2, 3, \cdots)$$

is bounded as follows for all real $\lambda$ and $N \geq 1$:

(a)  

$$\leq B(p, q)j^p \quad \text{if } p = 1 \text{ or } p = 2.$$ 

(b)  

$$\leq B(p, q)j^pN^{1/2(p-1)} \quad \text{if } p > 2.$$ 

**Proof.** The argument for $s = 1$ is very simple and is contained implicitly in that now given for $s = 2$, $p \geq 2$. Here we distinguish two cases, according as $|d_1\lambda_{k_1}| \leq q^{-1}|d_2\lambda_{k_2}|$, or not. In the first case we can write

$$d_1\lambda_{k_1} + d_2\lambda_{k_2} = (1 + \theta)d_2\lambda_{k_2}, \quad |\theta| \leq q^{-1} < 1.$$ 

Let $k < k^*$ be two values of $k_2$ occurring in this case. Then

$$|\lambda_k(1 + \theta) - \lambda_{k^*}(1 + \theta^*)| \leq 2^{j+1}$$
\[ \lambda_{k^*} \leq (\lambda_k + 2^{j+1})(1 - q^{-1})^{-2}. \]

From this it follows that \( k^* - k \ll j \), so that \( k_2 \) is restricted to \( \ll j \) values. Once \( k_2 \) is chosen, \( k_1 \) is similarly confined, and so the first case distinguished before gives a contribution \( \ll j^2 \). Moreover this case always obtains when \( |d_1| \leq |d_2| \), and in particular when \( s = 2, p = 2 \); thus (a) is proved. Again, if \( |d_1\lambda_{k_1}| > q^{-1}d_2\lambda_{k_2} \) then

\[ k_1 < k_2 \leq k_1 + \log |d_1|/\log q \]

and \((k_1, k_2)\) is restricted to \( \ll N \) values. Because \( p > 2 \), this is consistent with (b).

When \( s \geq 3 \) we choose an integer \( A = A_{q,s} \) so that \( 2A^{-p}p \leq 1 \) and first estimate the number of solutions of (3) wherein \( k_{s-1} + A < k_s \). Then

\[ d_1\lambda_{k_1} + \cdots + d_s\lambda_{k_s} = (1 + \theta)d_s\lambda_{k_s}, \quad |\theta| \leq \frac{1}{2}. \]

We find as above that \( k_s \) can assume \( \ll j \) different values, and once \( k_s \) is fixed we find by induction (on \( p \) or on \( s \)) that the remaining choices are \( \ll j^{p-1}N^{1/2(p-2)} \) in number. Finally, if \( k_{s-1} < k_s \leq k_{s-1} + A \), then \((k_1, k_s)\) has at most \( AN \) values, and for each one of these the number of choices is \( \ll j^{p-2}N^{1/2(p-3)} \). This proves the lemma.

Much more precise estimates are given by Erdős and Gál, but these don’t seem to be applicable \([1]\).

4. In the proof of Theorem 2 we use the multinomial expansion of \((\sum_{k \leq N} \cos (t\lambda_k x + b_k))^r \) into a finite combination of sums (with coefficients to be considered later)

\[ \sum_{1 \leq k_1 < \cdots < k_s \leq N} \cos^{e_1}(t\lambda_{k_1} x + b_{k_1}) \cdots \cos^{e_s}(t\lambda_{k_s} x + b_{k_s}). \]

Here \( e_1 \geq 1, \cdots, e_s \geq 1, \) and \( e_1 + \cdots + e_s = r \). This sum is \( \leq N^s \) in modulus, and so it can be neglected if \( s < \frac{1}{2}r \). When \( r \) is even, say \( r = 2v \), there occurs a dominant contribution determined by the choice \( s = v, e = \cdots = e_v = 2 \). This requires closer argument and we exclude it for the moment; in every \( s \)-tuple \((e_1, \cdots, e_s)\) remaining at least one component must be odd.

To exploit the last remark we expand

\[ \cos^{e_1}(t\lambda_{k_1} x + b_{k_1}) \cdots \cos^{e_s}(t\lambda_{k_s} x + b_{k_s}) \]

into a linear combination of exponentials \( e((tx)(d_1\lambda_{k_1} + \cdots + d_r\lambda_{k_r})) \), wherein \( 1 \leq |d_1| + \cdots + |d_s| \leq r \).

We can handle the dominant term in almost the same way, using the identity \( 2\cos^2u = 1 + \cos 2u \). In the multinomial formula there
occurs the factor \( r! 2^{-v}(v = \frac{1}{2}r) \). Hence the dominant term contains
the constant 1 with a coefficient
\[
2^{-v} \cdot r! 2^{-v} \cdot \binom{v}{r} = 2^{-v} r! (v!)^{-1} N^v + O(N^{v-1})
\]
Now the \( r \)th moment
\[
m_r = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^r e^{-u^2/2} du = 2^{-v} r! (v!)^{-1}
\]
Thus the constant term is \( 2^{-v} N^v m_r + O(N^{v-1}) \), and this is correct because the ‘norming’ constant is \( (\frac{1}{2}N)^{-1/2} \).

In the dominant term there occur other exponentials, but each of them is of the type considered above. It remains now to be proved that the random error, say \( R_N \), encountered in the moment of
\[
\sum_{k \leq N} \cos(t \lambda_k x + b_k)
\]
is almost surely \( o(N^v) \) as \( N \to +\infty \). But in fact these errors are Fourier-Stieltjes coefficients
\[
|\hat{\mu}(td_1 \lambda_{k_1} + \cdots + td_s \lambda_{k_s})|
\]
where \( 1 \leq k_1 < \cdots < k_s \leq N \) and \( 1 \leq |d_1| + \cdots + |d_s| \leq r \). From the previous lemma and from the estimation (2), we find that
\[
\int_1^2 R_N dt \ll N^{v-1/2}
\]
and therefore, by Chebyshev’s inequality, \( R_{N^2} = o(N^{3v}) \) almost surely. Because \( (N + 1)^3 = N^3 + o(N^3) \) this completes the proof.

It is not difficult to formulate and prove a similar theorem for the union of sequences \( tA \cup sA \), where \((t, s)\) is a point in the plane. When \( \mu \) is absolutely continuous, however, we can suppress one of the variables and obtain a central-limit theorem for sums
\[
\sum_{k \leq N} \cos(\lambda_k x + b_k) + \sum_{k \leq N} \cos(\lambda_k tx + b'_k)
\]
The central-limit phenomenon here is false for certain sequences \( A \) and certain values of \( t \): \( \lambda_k = 2^k \) and \( t = 2 \). The existence of even one \( t > 1 \) rendering the central-limit theorem false is presumably a strong restriction on a lacunary sequence.

5. We conclude by stating a theorem and a conjecture related to it. As before \( S \) is a set of measure 0 in \(( -\infty, \infty)\) depending only on \( A \) and \( \mu \).

**Theorem 3.** For each \( t \in S \), each closed set \( E \), and each \( s > 0 \),
there is an integer \( N = N(t, \varepsilon, E) \) such that

\[
\left| \sum_{k \in N} a_k e(\lambda_k tx) \right|^2 \mu(dx) - \mu(E) \sum_{k \in N} |a_k|^2 \leq \varepsilon \sum_{k \in N} |a_k|^2.
\]

The proof is very similar to that of Theorem 1, and to some extent depends upon Theorem 1; however, it is necessary here to use the estimate (a) of the lemma in § 3.

**Corollary.** If \( \sum |a_k|^2 = +\infty \), then \( \sum_{1}^{\infty} a_k e(\lambda_k tx) \) diverges almost everywhere with respect to \( \mu \).

It is natural to conjecture that \( \sum_{1}^{\infty} a_k e(\lambda_k tx) \) converges almost everywhere, provided \( \sum |a_k|^2 < \infty \).

*Added in proof.* This follows from theorems on orthogonal series.

**References**


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