A NONSTANDARD PROOF OF THE JORDAN CURVE THEOREM

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In this paper a proof of the Jordan curve theorem will be presented. Some familiarity with the basic notions of nonstandard analysis is assumed. The rest of the paper is self-contained except for some standard theorems about polygons.

The theorem will be proved in what ought to be a natural way: by approximation by polygons. This method is not usually found in the standard proofs since the approximating sequence of polygons is often unwieldy. But by using nonstandard analysis, one can approximate a Jordan curve by a single polygon that is infinitesimally close to the curve. This allows types of reasoning which are extremely difficult and unnatural on sequences of polygons.

Preliminaries. The basic concepts of nonstandard analysis and some acquaintance with polygons are assumed. Some basic definitions and theorems of point set topology are also assumed.

Throughout this paper the following notations and conventions will be used:

1. All discussion, unless otherwise stated, is assumed to be about a nonstandard model of the Euclidean plane. “Otherwise stated” will often mean that the notion or concept will be prefaced by the word “standard”.

2. A standard concept and its extension will be denoted by the same symbol. If it is necessary to distinguish between them, reference to the model in which they are to be interpreted will be made.

3. If $A$ and $B$ are sets of points and $x$ is a point, then $|x, A|$ denotes the distance from $x$ to $A$ and $|B, A| = \inf_{x \in B} |x, A|$. (Thus if $A \cap B \neq \emptyset$ then $|A, B| = 0$.) $|x, y|$ will denote the distance from the point $x$ to the point $y$.

4. $f$ will denote a fixed continuous function on $[0,1]$ into the Euclidean plane with the property that $x < y$ and $f(x) = f(y)$ if and only if $x = 0$ and $y = 1$. $C$ will denote the range of $f$.

5. $x \approx y$ will mean that the distance from $x$ to $y$ is infinitesimal. If $x$ is near-standard then $\delta(x)$ will denote the standard $y$ such that $x \leq y$.

6. If $x$ and $y$ are points then $xy$ will denote the ordered, closed line segment that begins at $x$ and ends at $y$.

7. If $x$ and $y$ are points then $\text{intv}(x, y)$ is the set of all points $z$ of $xy$ such that $z \neq x$ and $z \neq y$.
2. The approximation of $C$. In this section the Jordan curve $C$ will be approximated by a polygon $P$. Not all approximations of $C$ by polygons are suitable for our purposes. A particular sort of approximation, called a *good* approximation, will be constructed. A good approximation not only approximates the point set $C$, but also has associated with it a function $h: P \to [0,1]$, that approximates $f^{-1}$ in the following sense: the point set $C$ has the property that if $x, y \in C$ then $x \equiv y$ if and only if $f^{-1}(x) \equiv f^{-1}(y)$ or $|f^{-1}(x) - f^{-1}(y)| \equiv 1$.

DEFINITION 2.1. A *good approximation* for $C$ is a simple closed polygon $P$ and a function $h: P \to [0,1]$ which assumes only (nonstandardly) finitely many values such that,

1. if $x \in C$ then there is a $y \in P$ such that $x \equiv y$,
2. if $x \in P$ then there is a $y \in C$ such that $x \equiv y$,
3. if $x, y \in P$ and $h(x) \equiv h(y)$ then $x \equiv y$,
4. if $x, y \in P$ and $x \equiv y$ then $h(x) \equiv h(y)$ or $|h(x) - h(y)| \equiv 1$,
5. there are points $K = \{k_0, \ldots, k_s\}$ (where $s$ is an infinite natural number) such that $P = (\bigcup_{i < s} k_ik_{i+1}) \cup k_0k_0$ and such that:

(a) if $x = k_0$ then $h(x) = 0$,
(b) if for $i < s$, $x \in k_ik_{i+1}$ and $x \neq k_i$ then $h(k_i) < h(x) = h(k_{i+1})$,
(c) if $x \neq k_0$ and $x \in k_ik_0$ then $h(x) = 1$.

DEFINITION 2.2. Part (5) of Definition 2.1 gives a natural method of ordering $P$ in terms of $\{k_0, \ldots, k_s\}$. For $x, y \in P$ and $x \neq y$ define $x < y$ if and only if (1) $x = k_0$, or (2) $x \in k_ik_{i+1}$ and $y \in k_jk_{j+1}$ and $i < j$, or (3) $x \in K_iK_{i+1}$ and $y \in K_iK_0$ and $y \neq K_0$, or (4) $x$ and $y$ belong to the same ordered segment $(k_ik_{i+1}$ or $k_0k_0$) and $x$ comes before $y$ in the ordering of that segment. (Also note that if $x, y \in P$ and $h(x) < h(y)$ then $x < y$.)

THEOREM 2.1. A good approximation for $C$ exists.

Proof. Let $M$ be an infinite natural number and

$$\beta = \max \{\beta_1, \beta_2\}$$

and $\beta = \max \{|a_i|, |a_{i+1}|\}$. Since $f$ is standard continuous, $\beta$ is infinitesimal.

Divide $[0,1]$ into $M$ equal intervals $[t_i, t_{i+1}]$ with $t_0 = 0$ and $t_{M} = 1$. For $i < M$ let $a_i = f(t_i)$. Then $|a_i, a_{i+1}| \leq \beta$. The points $k_i$ will be defined inductively. $k_0 = a_0$, $k_{i+1} = a_p$ if and only if $|a_p, k_i| \leq \beta$ and for each $j > p$ either $|a_j, k_i| > \beta$ or $|f^{-1}(a_j) - f^{-1}(k_i)| \leq \frac{1}{M}$, where
$f^{-1}(x)$ is the least $t$ such that $f(t) = x$. Since there are only "finitely" many $a_i$, there will be a last $k_i$ defined. This last element is denoted by $k_i$. The points $k_i$ will have the following properties:

(A1) $f^{-1}(k_i) \ni 1$,
(A2) if $a \in C$ then there is an $i \leq l$ such that $k_i \ni a$,
(A3) for each $i \leq l$ there is an $a \in C$ such that $k_i \ni a$,
(A4) if $f^{-1}(k_i)$ is not in the monad of 1 and $j > i + 1$ then $|k_i, k_j| > \beta$,
(A5) if $i + 1 < k$ then $k_i k_{i+1} \cap k_{i+1} k_{i+2} = \{k_{i+1}\}$,
(A6) if $0 \leq p < i - 2$ and $f^{-1}(k_p)$ is not in the monad of 1 then $k_p k_{p+1} \cap k_{i} k_{i+1} = \emptyset$,
(A7) if $f^{-1}(k_p)$ is not in the monad of 0, $f^{-1}(k_q) \ni 1$, and $p \leq q - 2$, then $k_p k_{p+1} \cap k_{q-i} k_q = \emptyset$.

Proof of (A1). Let $k_i = a_p$. $|a_p, a_{\omega}|$ must be infinitesimal or otherwise $k_{i+1}$ would exist. This means that $f^{-1}(a_p) \ni 1$ or $f^{-1}(a_p) \ni 0$. If $f^{-1}(a_p) \ni 0$ then $k_{i+1}$ would exist. Therefore $f^{-1}(a_p) \ni 1$.

Proof of (A2). By the method in which the $a_i$ were defined, there is an index $p$ such that $a_p \ni a$. Let $j$ be the largest "natural number" $\leq l$ such that $f^{-1}(k_j) \ni f^{-1}(a_p)$. If $f^{-1}(a_p) - f^{-1}(k_j)$ were not infinitesimal then $f^{-1}(k_{j+1}) < f^{-1}(a_p)$. Therefore $f^{-1}(a_p) - f^{-1}(k_j)$ is infinitesimal and thus $k_j \ni a_p \ni a$.

Proof of (A3). Let $a = k_i$.

Proof of (A4). Immediately follows from the definition of $k_{i+1}$.

Proof of (A5). Suppose not. Let $b \in k_i k_{i+1} \cap k_{i+1} k_{i+2}$ and $b \neq k_{i+1}$. Then $b, k_i, k_{i+1}, k_{i+2}$ are collinear. Since $|k_i, k_{i+1}| > \beta$, $k_{i+1} \in k_i k_{i+1}$ and $k_i \in k_{i+1} k_{i+2}$. But this can only happen to collinear points $k_i, k_{i+1}, k_{i+2}$ if and only if $k_i k_{i+1} \cap k_{i+1} k_{i+2} = \{k_{i+1}\}$.

Proof of (A6). Assume that $k_p k_{p+1} \cap k_i k_{i+1} \neq \emptyset$. Then $k_p k_i, k_{i+1} k_{i+2}, k_{p+1} k_{i+1}, k_{i+1} k_p$ form the sides of a quadrilateral. Without loss of generality assume that the angle at $k_p \geq \pi/2$ radians. Then in the triangle with vertices at $k_i, k_p$, and $k_{i+1}$, the angle at $k_p \geq \pi/2$ radians. This makes $k_i k_{i+1}$ the longest side. Therefore $|k_p, k_i| \leq |k_i, k_{i+1}| \leq \beta$. This contradicts (A4).

Proof of (A7). Similar to (A6).

If $i < k$, $x \in k_i k_{i+1}$, and $x \neq k_i$ define $h(x) = f^{-1}(k_{i+1})$.

$P_i = \bigcup_{i < j} k_i k_{i+1}$ and $h$ almost form a good approximation for $C$. Unfortunately $P_i$ is not a closed polygon and may intersect itself in the monad of $k_i$. To form a simple closed polygon from $P_i$ we let $k_j$ be such that $h(k_j)$ is not in the monad of 0 or 1. Let

$P_z = \bigcup_{i \geq j+1} k_i k_{i+1}$ and $P_z = \bigcup_{j \leq i < l} k_i k_{i+1}$.

By (A5), (A6), and (A7) $P_z$ and $P_z$ are simple polygonal paths. $P_z$ and $P_z$ can be ordered in such a way that $P_z$ starts at $k_0$ and ends
at \( k_j \) and \( P_3 \) starts at \( k_i \) and ends at \( k_1 \). Let \( E = k_i k_0 \cap P_3 \). Since \( k_i k_0 \) and \( P_3 \) are closed sets and \( k_i \in P_1 \cap k_j k_0 \), let \( e \) be the smallest member of \( E \) in the ordering induced by \( P_3 \). Then

\[
P_4 = \{ x \in P_3 | k_j \leq x \leq e \} \cup ek_0
\]

is also a simple polygonal path. For \( x \in ek_0 \) define \( h(x) = 1 \). \( P_4 \) can be ordered in such a way that \( P_4 \) starts at \( k_j \) and ends at \( k_0 \). Since \( k_i \in P_3 \cap P_4 \), let \( e_i \) be the greatest member in the ordering of \( P_i \) of \( P_3 \cap P_4 \). Then \( P = \{ x \in P_3 | e_i \leq x \leq k_j \} \cup \{ x \in P_4 | k_j \leq x \leq e_i \} \) is a simple closed polygon. Let \( h_i(e_i) = 0 \) and for \( x \in P - \{ e_i \} \) let \( h_i(x) = h(x) \). Then \( P \) and \( h_i \) will form a good approximation for \( C \) by (A2) and (A3) and this construction.

**Notation.** Throughout the rest of this paper, we let the polygon \( P \) and the function \( h \) be a fixed good approximation for \( C \).

3. The inside and outside of \( C \). The following theorem and other easily established facts about polygons will be used without proof in this paper. A proof of the following theorem can be found in [1].

**Theorem 3.1.** *(The Jordan curve theorem for polygons.)* A simple closed polygon \( Q \) divides the Euclidean plane into three non-empty disjoint sets, the polygon itself and two open components. One of the components is bounded and is called the inside of \( Q \), and the other component is unbounded and called the outside of \( Q \).

**Definition 3.1.** The inside of \( C \) is the set of all standard points \( x \) such that \( x \) is inside \( P \) and \( |P, x| \) is not infinitesimal.

**Definition 3.2.** The outside of \( C \) is the set of all standard points \( x \) such that \( x \) is outside \( P \) and \( |P, x| \) is not infinitesimal.

**Theorem 3.2.** Each standard point is either in the inside of \( C \), in the outside of \( C \), or is on \( C \).

**Proof.** Suppose that \( x \) is a standard point and \( x \) is not in the inside of \( C \) and \( x \) is not in the outside of \( C \). Then either \( x \) is inside \( P \) and \( |P, x| \leq 0 \), or \( x \) is outside \( P \) and \( |P, x| \leq 0 \), or \( x \) is on \( P \). In any case, \( |P, x| \leq 0 \). Since \( P \) is a good approximation for \( C \), \( |C, x| \leq 0 \). Thus there is a point \( y \) on \( C \) such that \( x \approx y \). Let \( t = f^{-1}(y) \). Let \( \theta_t \) be the standard real number in \([0, 1]\) that is nearstandard to \( t \). Since \( f \) is a standard continuous function, \( f(\theta_t) \) is standard, and
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$f(\omega t)$ is on $C$, and $x \equiv f(\omega t)$. Since there is only one standard point in each monad, $x = f(\omega t)$. Thus $x \in C$.

**THEOREM 3.3.** The inside of $C$ is bounded in the standard topology and the outside of $C$ is unbounded in the standard topology.

**Proof.** It is true in the standard model that there is a real number $r$ such that for some fixed point $x_0$, $|x_0, f(t)| < r$ for all $t \in [0, 1]$. Therefore in the nonstandard model, $|x_0, C| < r$. Since $P$ is a good approximation for $C$, $|x_0, P| < r + 1$. This implies that the inside of $C$ is bounded. The outside of $C$ is unbounded since there are standard points outside of $P$ that are greater than any given standard distance.

**THEOREM 3.4.** The inside of $C$ and the outside of $C$ are open.

**Proof.** Let $a$ be a point in the inside of $C$. Let $|a, C| = r$. Then $\omega r > 0$. Let $E = \{x | x$ is standard and $|a, x| < \omega r\}$. Then $E$ is a standard open set containing $a$, and $E$ is contained in the inside of $C$. Similarly for the outside of $C$.

**DEFINITION 3.3.** If $t, t' \in [0, 1]$ let

$$D(t, t') = \min \{|t', t|, |1 + t', t|, |1 + t, t'|\}.$$  

Note that if $a, b \in P$ then $a \approx b$ if and only if $D(h(a), h(b)) \equiv 0$.

**LEMMA 3.1.** Let $a, b \in P$ and $a < b$, $P_1 = \{x | a \leq x \leq b\}$, $P_2 = P - P_1$. Assume that $D(h(a), h(b))$ is not infinitesimal, $x \in P_1$, $y \in P_2$, and $x \equiv y$. Then $x \equiv a$ or $x \equiv b$.

**Proof.** Suppose not. Since $x \in P_1$ and $a \leq x \leq b$, $h(a) \leq h(x) \leq h(b)$. By hypothesis, $x$ is not infinitesimally close to $a$ and $x$ is not infinitesimally close to $b$. Therefore $h(x)$ is not infinitesimally close to $h(a)$ or $h(b)$. Since $y \in P_2$, $h(y) \leq h(a)$ or $h(y) \geq h(b)$.

**Case 1.** $h(y) \leq h(a)$. Since $x \equiv y$, either $h(x) - h(y) \equiv 0$ or $h(x) - h(y) \equiv 1$. But $h(x) - h(y)$ is not infinitesimal since $h(x) - h(y) \geq h(x) - h(a)$ which is not infinitesimal. $h(x) - h(y)$ is not in the monad of 1. For if it were, $h(x) \equiv 1$ and since $h(b) > h(x)$, $h(b) \equiv 1$. But then $h(x) \equiv h(b)$. A contradiction.

**Case 2.** $h(y) \geq h(b)$. Similar to Case 1.

**THEOREM 3.5.** The inside of $C$ is nonempty.
Proof. Let $L$ be the set of line segments that have endpoints on $P$ and are contained in $(\text{inside } P) \cup P$. Since the function $h$ takes on only "finitely" many values, there is a fixed segment, $ab$, in $L$ such that $D(h(a), h(b))$ is a maximum.

Case 1. $h(b) - h(a)$ is not infinitesimal. Without loss of generality, assume that $a < b$. Let $c$ be the midpoint of $ab$, $P_1 = \{x | a \leq x \leq b\}$, and $P_2 = (P - P_1) \cup \{a, b\}$. Since $c$ is not infinitesimally close to $a$ or $b$, it follows from Lemma 3.1 that $c$ is not infinitesimally close to both $P_1$ and $P_2$. Without loss of generality, suppose that $c$ is not infinitesimally close to $P_1$. Let $r = |c, P_1|$ and $A = ab \cap P_1$. Since $A$ is a closed set, let $a_i$ and $b_i$ be those members of $A$ such that in the ordering of $ab$, for each $x, y \in A$ if $x < c$ and $y > c$ then $x \leq a_i < c < b_i \leq y$. (See Figure 3.1.) Then $a_i b_i \cup \{x \in P_1 | a_i \leq x \leq b_i\}$ can easily be shown to be a simple closed polygon, $Q$, with the property that the inside of $Q$ is inside $P$. Further, $P_2 \cap \text{inside } Q = \emptyset$. If $d$ is inside $Q$ and is the endpoint of the segment of length $r/3$, drawn perpendicular through $c$ to the side $a_ib_i$ of $Q$, then it follows
that $d$ is inside $P$ and that $|d, P| \geq r/3$. Since $r/3$ is not infinitesimal, $d$ is inside $C$.

**Case 2.** $D(h(a), h(b))$ is infinitesimal. Without loss of generality, assume that $a < b$ and $h(a)$ is not in the monad of $0$. Let

$$B = \{x \in P | ax \in L \text{ and } h(x) = h(b)\}.$$ 

In the ordering "<" of $P$ let $c = \sup B$. By continuity, $ac \in L$. Since $c \geq b$, $h(c) \geq h(b)$. Since $h(a)$ is not in the monad of 0, and $D(h(a), h(c))$ is infinitesimal, and $h(c) \geq h(b)$, and $h(b) - h(a)$ is a maximum it follows that $h(c) = h(b)$. If $e \in P$ and $e > c$ then $ae \in L$. For if $e \in P$ and $e > c$ and $ae \in L$, then either $h(e) = h(c)$, contradicting the definition of $c$, or $h(e) > h(c)$, contradicting the maximality of $D(h(a), h(b))$ together with $h(a)$ not being in the monad of 0. Similarly a point $d \in P$ can be found such that for all $e < d$, $ea \not\in L$ and $da \in L$. (See Figure 3.2.) Let $Q_1 = \{x \in P | x \leq d \text{ or } x \geq c\}$. Let $Q = da \cup ae \cup Q_1$. Then $Q$ is a simple closed polygon with inside $Q \subset$ inside $P$. Since $a$ is a vertex of $Q$, it follows that a segment $aq$ can be drawn such that intv $(a, q)$ is inside $Q$ and $q$ is on $Q$. It then follows that $q \in Q$. Thus $aq \in L$. But by the method in which $c$ and $d$ were chosen, this is impossible.

4. Connectivity of the inside and outside of $C$.

**DEFINITION 4.1.** If $A \subset B$ and $B$ is a standard compact set, let $^\circ A = \{^\circ x | x \in A\}$.

**LEMMA 4.1.** If $A$ is a nonempty connected set and $A \subset B$ where $B$ is a standard compact set, then $^\circ A$ is connected in the standard model.

**Proof.** Suppose $^\circ A$ is not connected in the standard model. Then in the standard model, there are open sets $W$ and $V$ such that $W \cap ^\circ A \neq \emptyset$, $V \cap ^\circ A \neq \emptyset$, $W \cap V = \emptyset$, and $^\circ A \subset W \cup V$. But $x \in W$ in the standard model implies that the monad of $x$ is contained in $W$ in the nonstandard model. Therefore in the nonstandard model, $W$ and $V$ are open sets, $W \cap A \neq \emptyset$, $V \cap A \neq \emptyset$, $W \cap V = \emptyset$, and $A \subset W \cup V$. Hence $A$ is not connected. But this contradicts the hypothesis that $A$ is connected. Therefore $^\circ A$ is connected in the standard model.

**LEMMA 4.2.** Let $e, g$ be points on $P$ such that intv $(e, g)$ is inside $P$. Let $a, b$ be points on $eg$ such that $|a, P|, |b, P|$, and $|a, b|$ are not infinitesimal. Then there is a connected set, $A$, such that $A$ is inside $P$, $a, b \in A$, and $|A, P|$ is not infinitesimal.
Proof. Assume the hypotheses. Without loss of generality assume that \( e \) and \( g \) are on the \( Y \)-axis. Let \( w \) be an arbitrary point of \( ab \). Let \( c \) and \( d \) be points of \( P \) such that \( cd \) is parallel to the \( X \)-axis, \( w \in cd \), and \( \text{intv}(c, d) \) is inside \( P \). It will be shown that \( (c, d) \) is not infinitesimal. There are three cases to consider: (i) \( w \equiv a \), (ii) \( w \equiv b \), (iii) \( w \) is not infinitesimally closed to either \( a \) or \( b \). Cases (i) and (ii) immediately follow from the assumption that \( |a, P| \) and \( |b, P| \) are not infinitesimal. (iii) will be shown by contradiction. Assume that \( c \equiv d \). Without loss of generality we may assume that \( c \) is in the left half-plane and \( d \) is in the right half-plane, and that in the ordering of \( P \), \( c < d \). For convenience we assume that \( h(d) \) is not in the monad of 1. Let \( w = (0, \tau, \omega) \) be an arbitrary point of \( abc \) and \( d = (d, \tau, \omega) \) be points on \( P \) such that \( \text{intv}(c, \omega) \) is inside \( P \). Let \( u(w) = (1/2(d_i - c_i), \omega_i) \). Then, by construction, \( u(w) \) is inside \( P \) and \( |u(w), P| \) is not infinitesimal. Let

\[
\begin{align*}
\text{min}(|w, a|, |w, b|) & \leq |w, \tau, \omega| \leq |d, \tau, \omega|,
\end{align*}
\]

it follows that \( |d, \tau, \omega| \) is not infinitesimal. Hence \( h(d) - h(\tau, \omega) \) is not infinitesimal—a contradiction.

One again, let \( w = (0, \tau, \omega) \) be an arbitrary point of \( abc \) and \( d = (d, \tau, \omega) \) be points on \( P \) such that \( cd \) is parallel to the \( X \)-axis and \( \text{intv}(c, d) \) is inside \( P \). Let \( u(w) = (1/2(d_i - c_i), \omega_i) \). Then, by construction, \( u(w) \) is inside \( P \) and \( |u(w), P| \) is not infinitesimal. Let

\[
A = au(a) \cup \{u(w) \mid w \in ab\} \cup bu(b).
\]

Then \( A \) is a connected set that is inside \( P \) and is such that \( a \in A \), \( b \in A \), and \( |A, P| \) is not infinitesimal.

**Lemma 4.3.** Let \( a_1, a_2 \) be distinct points inside \( C \). Then there is a connected set \( A \) such that \( a_1 \in A \), \( a_2 \in A \), \( A \) is inside \( P \), and \( |A, P| \) is not infinitesimal.

**Proof.** Let \( a_1, a_2 \) be distinct points inside \( C \). Let \( c, d \) be a segment through \( a_i \) such that \( |a_i, c, d_i| \) is not infinitesimal, \( c_i \) and \( d_i \), are on \( P \), and \( \text{intv}(c_i, d_i) \) is inside \( P \). Similarly, let \( c, d \) be a segment through \( a_2 \) that is parallel to \( c, d \), and such that \( c \) and \( d \) are on \( P \) and \( \text{intv}(c, d) \) is inside \( P \). Without loss of generality, it may be assumed that \( c_i < c_2 < d_2 < d_i \).

Let \( P_1 = \{x \in P \mid c_i \leq x \leq c_2 \} \) and \( P_2 = \{x \in P \mid d_2 \leq x \leq d_i \} \). (See Figure 4.1.) \( |P_1, P_2| \) is not infinitesimal. For if \( x_1 \) and \( x_2 \) are arbitrary members of \( P_1 \) and \( P_2 \) respectively, then \( c_i \leq x_1 \leq c_2 \leq d_2 \leq x_2 \leq d_i \), and hence \( h(c_i) \leq h(x_1) \leq h(c_2) \leq h(d_2) \leq h(x_2) \leq h(d_i) \). Since \( c, d \) passes through \( a_i \) and \( |a_i, P| \) is not infinitesimal, \( h(d_i) - h(c_i) \) is not in the
monad of 0 or 1. Similarly \( h(d_2) - h(c_2) \) is not in the monad of 0 or 1. But, \( h(d_i) - h(c_i) \geq h(x_i) - h(x_i) \geq h(d_2) - h(c_2) \). Therefore \( h(x_0) - h(x_0) \) is not in the monad of 0 or 1. Thus \( |x_1, x_2| \) is not infinitesimal. Therefore \( |P_1, P_2| \) is not infinitesimal.

Let \( r = \min \{|P_1, P_2|, |c_1d_1, c_2d_2|, |a_1, P|, |a_2, P|\} \). Without loss of generality, it may be assumed that \( c_1d_1 \) is on the Y-axis. Since each line segment \( l \) that is parallel to \( c_1d_1 \) has the same X-coordinate for each of its points, we shall say that the coordinate of \( l \) is the X-coordinate of its points. Let \( v \) be the X-coordinate of \( c_2d_2 \). Without loss of generality we may assume that \( v \) is positive. Let

\[
L = \{ cd | c \in P_1, d \in P_2, \ \text{intv}(c, d) \ \text{is inside} \ P, \ \text{and} \ cd \ \text{is parallel to} \ c_1d_1 \}.
\]

Let \( B = \{ x | \text{there is an} \ l \in L \text{ such that} \ x \in l \text{ and} \ |x, P| > r/4 \} \). Let \( E \) be the largest connected subset of \( B \) that contains \( a_i \). Let

\[
T = \{ t | \text{there is a} \ z \in E \text{ such that} \ t \text{ is an X-coordinate of} \ z \}.
\]

Let \( s \) be the supremum of \( T \). \( s = v \). For if \( s \) were less than \( v \), let \( cd \in L \) and such that \( s = X\)-coordinate of \( cd \). Let \( p \) be the midpoint of \( cd \). If \( |p, P| \geq r/4 \) then the disk \( D \) about \( p \) of radius \( r/4 \) intersects \( E \). Then \( (E \cup D) \cap B \) would be a connected subset of \( B \) containing \( a_i \) that is larger than \( E \). This is impossible since \( E \) is the largest connected subset of \( B \) containing \( a_i \). Thus \( |p, P| < r/4 \). Since \( |p, P_1| \geq r/2 \) and \( |p, P_2| \geq r/2 \), there is a point \( q \in P - (P_1 \cup P_2) \) such that \( |p, q| \leq r/4 \). Since \( Q = c_1d_1 \cup P_1 \cup c_2d_2 \cup P_2 \) is a simple closed polygon, and \( p \) is inside \( Q \) and \( q \) is outside \( Q \), \( pq \cap Q \neq \emptyset \). This can only happen if \( pq \cap c_1d_1 \neq \emptyset \) or \( pq \cap c_2d_2 \neq \emptyset \). Assume that \( pq \cap c_1d_1 \neq \emptyset \). (The case of \( pq \cap c_2d_2 \neq \emptyset \) follows similarly.) Then \( |p, c_1d_1| < r/4 \). Since \( c \in P_1 \) and \( d \in P_2 \) and \( cd \) is parallel to \( c_1d_1 \), there is a point \( w \in cd \) such that \( |w, a_1| < r/4 \). Since \( |a_1, P| \geq r \), the disk \( D_1 \) of radius \( r/4 \) about the point \( w \) has the property that \( |D_1, P| > r/4 \). Also, \( D_1 \cap E \neq \emptyset \) and \( (B \cap D_1) - E \neq \emptyset \). Therefore \( (B \cap D_1) \cup E \) is a larger, connected subset of \( B \) than \( E \). A contradiction.

\[\text{FIGURE 4.1.}\]
Since \( s = v, E \cap c_d^2 \neq \emptyset \). Let \( e \in E \cap c_d^2 \). Since \( |E, P| \geq r/4 \), \( e \) is not infinitesimally close to \( P \). By Lemma 4.2, there is a connected set \( F \) such that \( e \in F, a_2 \in F, |F, P| \) is not infinitesimal, and \( F \) is inside \( P \). Let \( A = E \cup F \). Then \( a_1, a_2 \in A, A \) is connected, \( |A, P| \) is not infinitesimal, and \( A \) is inside \( P \).

**Theorem 4.1.** The inside of \( C \) is connected (in the standard model).

*Proof.* Suppose not. Let \( X = \text{inside } C \). Then in the standard model, there are open sets \( V \) and \( W \) such that \( V \cap X \neq \emptyset, W \cap X \neq \emptyset, W \cap V = \emptyset, \) and \( X \subset W \cup V \). Let \( a \in V \cap X \) and \( b \in W \cap X \). Then in the nonstandard model, by Lemma 4.3 there is a connected set \( A \) such that \( a, b \in A, A \) is inside \( P, \) and \( |A, P| \) is not infinitesimal. Note that since \( |A, P| \) is not infinitesimal, \( ^oA \) is inside \( C \). That is, \( ^oA \subseteq X \). Also, by Lemma 4.1, \( ^oA \) is connected in the standard model. Therefore in the standard model,

\[
V \cap ^oA \neq \emptyset, \ W \cap ^oA \neq \emptyset, \ V \cap W = \emptyset,
\]

and \( ^oA \subset V \cup W \). Hence \( ^oA \) is not connected. A contradiction.

**Lemma 4.4.** Let \( a, b \) be points in the outside of \( C \). Then there is a line \( l \) that intersects \( P \) and such that

1. \( |a, l| \) and \( |b, l| \) are not infinitesimal, and
2. \( a \) and \( b \) are in the same half-plane determined by \( l \).

*Proof.* Let \( a, b \) be points in the outside of \( P \) and \( c \) a point in the inside of \( C \). Then \( a, b, \) and \( c \) are standard and \( |a, P|, |b, P|, \) and \( |c, P| \) are not infinitesimal. Let \( k \) be the line through \( a \) and \( c \). Let \( H_1 \) and \( H_2 \) be the closed half-planes determined by \( k \). Without loss of generality, suppose that \( b \in H_1 \). Since \( |c, P| \) is not infinitesimal, there is a point \( d \) that is inside \( P \) and such that \( |d, H_1| \) is not infinitesimal. Let \( l \) be the line through \( d \) that is parallel to \( k \). Then \( |a, l| \) and \( |b, l| \) are not infinitesimal and \( a \) and \( b \) are in the same half-plane determined by \( l \).

**Theorem 4.2.** The outside of \( C \) is connected (in the standard model).

*Proof.* Let \( a \) and \( b \) be points in the outside of \( C \). It only needs to be shown that there is a connected set in the outside of \( C \) that contains \( a \) and \( b \). By Lemma 4.4, let \( l \) be a line through \( P \) such that \( a \) and \( b \) are in the same closed half-plane, \( H \), that is determined
by $l$. Then a square $S$ can be found with the following four properties:

1. one side of $S$ is on $l$,  
2. $l \cap P$ is contained on a side of $S$,  
3. $a$ and $b$ are inside $S$ and $|a, S|$ and $|b, S|$ are not infinitesimal,  
4. $S \subset H$.

Let $T = (\text{inside of } S) \cap (\text{outside of } P)$. Then it is easy to show that $T$ is an open set that has a simple closed polygon, $Q$, as a boundary. In other words, inside of $Q = T$, $a$ and $b$ are in the inside of $Q$, $|a, Q|$ and $|b, Q|$ are not infinitesimal. Now, $^\circ Q = \{^\circ x \mid x \in Q\}$ can easily be seen to be a Jordan curve in the standard model (composed partly of the square $^\circ S = \{^\circ x \mid x \in S\}$ and partly of $C$) with $a$ and $b$ in the inside of $^\circ Q$. Thus by Theorem 4.1, in the standard model there is a polygonal path, $A$, which is inside $^\circ Q$ and such that $a, b \in A$. But since the inside of $^\circ Q$ is the outside of $C$, it follows that $a$ and $b$ belong to a connected set that is in the outside of $C$.

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