

Pacific Journal of Mathematics

**THE ABEL SUMMABILITY OF CONJUGATE MULTIPLE
FOURIER-STIELTJES INTEGRALS**

S. P. PHILIPP, VICTOR LENARD SHAPIRO AND WILLIAM HALL SILLS

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Let $K(x) = \Omega(x/|x|)|x|^{-k}$ where $\Omega(\xi), |\xi| = 1$, is a real valued function which is in $\text{Lip } \alpha, 0 < \alpha < 1$, on the unit $(k-1)$ -sphere S in k -dimensional Euclidean space, $E_k, k \geq 2$ with the additional property that $\int_S \Omega(\xi) d\sigma(\xi) = 0$ where σ is the natural surface measure for S . ($K(x)$ is usually called a Calderón-Zygmund kernel in $\text{Lip } \alpha$.) Let μ be a Borel measure of finite total variation on E_k and set $\hat{\mu}(y) = (2\pi)^{-k} \int_{E_k} e^{-i(y,w)} d\mu(w)$. Also designate the principal-valued Fourier transform of K by $\hat{K}(y)$ and the principal-valued convolution of K with μ by $\tilde{\mu}(x)$. Define $I_R(x) = (2\pi)^k \int_{E_k} e^{-i(y, x/R)} \hat{K}(y) \hat{\mu}(y) e^{i(y,x)} dy$. Then if k is an even integer or if $k = 3$, the following result is established: $\lim_{R \rightarrow \infty} I_R(x) = \tilde{\mu}(x)$ almost everywhere.

In [5] V. L. Shapiro proved that the conjugate Fourier-Stieltjes integral of a finite Borel measure μ in the plane E_2 , taken with respect to a Calderón-Zygmund kernel $K(x)$ in $\text{Lip } \alpha, 1/2 < \alpha < 1$, is almost everywhere Abel summable to the principal-valued convolution $K * \mu$. The purpose of this paper is to extend this result to E_3 and to even-dimensional E_k for $K(x)$ in $\text{Lip } \alpha, 0 < \alpha < 1$. The first author will obtain the corresponding result for the odd-dimensional cases $k = 2s + 1, s \geq 2$, in a paper to appear, by the use of special functions. Also, the results of the present paper should be compared with Theorem 2 of [6, p. 44].

2. Definitions and notation. For $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$ put $(x, y) = x_1 y_1 + \dots + x_k y_k, |x| = (x, x)^{1/2}$ and $B(x, t) = \{y: |x - y| < t\}$. We will work with a fixed Calderón-Zygmund kernel $K(x) = \Omega(x/|x|)|x|^{-k}$ where $\Omega(\xi), |\xi| = 1$, is a real-valued function defined on the unit $(k-1)$ -dimensional sphere S in Euclidean space $E_k, k \geq 2$, and $\int_S \Omega(\xi) d\sigma(\xi) = 0$, where σ is the natural surface measure for S [2, Chapter 11]. We define $K(x)$ to be in $\text{Lip } \alpha$ if $|\Omega(\xi) - \Omega(\eta)| = O(|\xi - \eta|^\alpha)$ for some $\alpha, 0 < \alpha < 1$. The Fourier transform of a Borel measure μ in E_k of finite total variation is denoted as usual by

$$(1) \quad \hat{\mu}(y) = (2\pi)^{-k} \int_{E_k} e^{-i(y,w)} d\mu(w)$$

and by the principal-valued convolution $\tilde{\mu}(x)$ we mean

$$(2) \quad \lim_{t \rightarrow 0} \int_{E_k - B(x,t)} K(x - y) d\mu(y)$$

which is known to exist and be finite almost everywhere [1, p. 118]. The formal conjugate Fourier-Stieltjes integral of μ is given by

$$(3) \quad (2\pi)^k \int_{E_k} e^{i(x,y)} \hat{\mu}(y) \hat{K}(y) dy$$

where

$$\hat{K}(y) = (2\pi)^{-k} \lim_{t \rightarrow 0; T \rightarrow \infty} \int_{B(0,T) - B(0,t)} e^{-i(y,x)} K(x) dx$$

is the principal-valued Fourier transform. We will denote the Abel means of (3) by

$$(4) \quad I_R(x) = (2\pi)^k \int_{E_k} e^{-|y|/R} e^{i(x,y)} \hat{\mu}(y) \hat{K}(y) dy, \quad R > 1.$$

With $\lambda = (k - 2)/2$, P_n^λ will designate the Gegenbauer polynomials defined by the equation

$$(5) \quad (1 - 2\rho \cos \theta + \rho^2)^{-\lambda} = \sum_{n=0}^{\infty} \rho^n P_n^\lambda(\cos \theta), \quad 0 \leq \rho < 1.$$

These functions allow us to form the Laplace series $\sum_{n=1}^{\infty} Y_n(\xi)$ of surface harmonics attached to $\Omega(\xi)$ on the unit sphere S in E_k by means of the equation

$$(6) \quad Y_n(\xi) = \frac{\Gamma(\lambda)(n + \lambda)}{2\pi^{\lambda+1}} \int_S P_n^\lambda[(\xi, \eta)] \Omega(\eta) d\sigma(\eta)$$

(see [2, Chapter 11]). Formulas (5) and (6) give the Poisson integral representation

$$(7) \quad \sum_{n=1}^{\infty} \rho^n Y_n(\xi) = \frac{\Gamma(\lambda + 1)}{2\pi^{\lambda+1}} \int_S \frac{(1 - \rho^2) \Omega(\eta) d\sigma(\eta)}{(1 - 2\rho(\xi, \eta) + \rho^2)^{\lambda+1}}$$

which is valid for $0 \leq \rho < 1$. The assumptions on $\Omega(\xi)$ imply that $Y_0(\xi) = 0$.

3. The main theorem. Our principal theorem is

THEOREM 1. *Let $K(x) = \Omega(x/|x|)/|x|^k$ be a Calderón-Zygmund kernel in $\text{Lip } \alpha$, $0 < \alpha < 1$. Let μ be a Borel measure in E_k of finite total variation. Let $k = 3$ or $k = 2s$ where s is a positive integer. Then $\lim_{R \rightarrow \infty} I_R(x) = \tilde{\mu}(x)$ almost everywhere.*

Our proof will closely follow the original proof in [5]. We shall use, in addition, generalizations of certain statements in [5] obtained by V. L. Shapiro in [6]. Before outlining the proof, we will need some lemmas.

4. **Basic lemmas.** Throughout the balance of this paper, $\sum_{n=1}^{\infty} Y_n(\xi)$ will designate the Laplace series for $\Omega(\xi)$ on the unit sphere S in E_k . We will denote $\sup \{Y_n(\xi): |\xi| = 1\}$ by $\|Y_n\|_{\infty}$. The proof of the following lemma is given in [6, p. 69].

LEMMA 1. (i) For each $\gamma, 0 < \gamma < \alpha, \sum_{n=1}^{\infty} \|Y_n\|_{\infty} n^{\gamma} / n^{(k-1)/2} < \infty$.
 (ii) $\hat{K}(y)$ exists everywhere and if

$$y \neq 0, \hat{K}(y) = \sum_{n=1}^{\infty} (-i)^n Y_n(y/|y|) \Gamma(n/2) / 2^k \pi^{k/2} \Gamma((n+k)/2).$$

Also, $\hat{K}(0) = 0$ and the series converges absolutely and uniformly.

Next we set

$$H_n^k(R) = \{ \Gamma(n/2) / 2^{k/2} \Gamma((n+k)/2) \} \int_0^{\infty} e^{-t/R} t^{k/2} J_{n+(k/2)-1}(t) dt,$$

$R > 1; n = 1, 2, \dots; k = 2, 3, \dots$, where $J_{n+\lambda}(t), \lambda = (k-2)/2$, is a Bessel function of the first kind of order $n + \lambda$. The $H_n^k(R)$ arise naturally in the computation of $I_R(x)$.

LEMMA 2. (i) $0 \leq H_n^k(R) \leq 1, \lim_{R \rightarrow \infty} H_n^k(R) = 1$,
 (ii) $0 \leq H_n^k(R) \leq \text{Const. } R^k n^{-k/2}$,
 (iii) $\sum_{n=1}^{\infty} \|Y_n\|_{\infty} H_n^k(R) = O(R^k)$ as $R \rightarrow \infty$.

The first statement of (i) is proved in [6, Lemma 24, p. 64]. Also, as in formula (25) of [6, p. 56], we may express $H_n^k(R)$ by use of Euler's integral representation for hypergeometric functions as follows:

$$(8) \quad H_n^k(R) = (B(1/2, (n-1)/2))^{-1} \int_0^1 t^{-1/2} (1-t)^{(n-3)/2} (1+1/tR^2)^{-(n+k)/2} dt,$$

where $B(p, q)$ is the usual Beta function. From this follows the second statement of (i). Part (ii) is a consequence of the inequalities $|J_{n+\lambda}(t)| \leq t^{\lambda}$ [8, p. 60, Ex. 5] and $\Gamma(n/2) / 2^{k/2} \Gamma((n+k)/2) \leq \text{Const. } n^{-k/2}$ [8, p. 58]. Part (iii) is a consequence of (ii) and Lemma 1, (i).

In what follows we will set $\rho = \sqrt{1 + 1/R^2} - 1/R, R > 1$. We note that $0 < \rho < 1$ and $\rho \rightarrow 1$ as $R \rightarrow \infty$. Our proof of the main theorem is based upon showing that $\sum_{n=1}^{\infty} H_n^k(R) Y_n(\xi)$ behaves somewhat like $\sum_{n=1}^{\infty} \rho^n Y_n(\xi)$. Next we state some lemmas which relate ρ^n to the $H_n^k(R)$. In the case that k is a positive even integer, $H_n^k(R)$ can be computed in closed form. Consider, for example, the formula

$\int_0^\infty e^{-at} J_\nu(t) dt = ((1 + a^2)^{1/2} - a)^\nu / (1 + a^2)^{1/2}$, $a > 0, \nu > -1$ [7, p. 202]. By differentiating the integral and replacing a by R^{-1} and ν by the appropriate integer, one shows that

$$(9) \quad H_n^2(R) = n^{-1} \int_0^\infty e^{-t/R} t J_n(t) dt = \rho^n (1 + 1/R^2)^{-1} \{1 + n^{-1} (R\sqrt{1 + 1/R^2})^{-1}\}$$

and that

$$(10) \quad \begin{aligned} H_n^4(R) &= \frac{1}{n(n+2)} \int_0^\infty e^{-t/R} t^2 J_{n+1}(t) dt \\ &= \rho^{n+1} (\sqrt{1 + 1/R^2})^{-3} \left\{ 1 + \frac{3(n+1)}{n(n+2)} (R\sqrt{1 + 1/R^2})^{-1} \right. \\ &\quad \left. + \frac{3}{n(n+2)} (R\sqrt{1 + 1/R^2})^{-2} \right\}, \end{aligned}$$

and so on. The general formula for $H_n^{2s}(R) = (n(n+2) \cdots (n+2s-2))^{-1} \int_0^\infty e^{-t/R} t^s J_{n+s-1}(t) dt$, $s \geq 1$, is obtained by induction. We formalize this in the next lemma, whose proof we leave to the reader.

LEMMA 3. For $s = 1$ put $C_0^s(n) = 1, C_1^s(n) = 1/n$. For $s \geq 2$ let the coefficients $C_j^s(n), n \geq 1, 1 \leq j \leq s$ be determined by

$$(11) \quad \begin{aligned} C_j^s(n) &= b(n, s-1) \{ (j+s-1) C_{j-1}^{s-1}(n+1) \\ &\quad + (n+s-1) C_j^{s-1}(n+1) - (j+1) C_{j+1}^{s-1}(n+1) \} \end{aligned}$$

where $b(n, s) = (n+1)(n+3) \cdots (n+2s-1) / (n(n+2) \cdots (n+2s))$ and where we agree to set $C_0^{s-1}(n+1) = 1$ and $C_{s-1}^{s-1}(n+1) = C_{s+1}^{s-1}(n+1) = 0$. Then

$$(12) \quad H_n^{2s}(R) = \rho^{n+s-1} (\sqrt{1 + 1/R^2})^{-(s+1)} \left\{ 1 + \sum_{j=1}^s C_j^s(n) (R\sqrt{1 + 1/R^2})^{-j} \right\}.$$

Next let $S(\xi, 1 - \rho) = \{\eta: |\eta| = 1, (\xi, \eta) > \cos(1 - \rho)\}$, $|\xi| = 1, 0 < 1 - \rho < 1$, denote the spherical cap centered at ξ of curvilinear radius $1 - \rho$. Fix the North pole of S at ξ and write $\sum_{n=1}^\infty \rho^n Y_n(\xi) - \Omega(\xi)$ in the Poisson integral form

$$\frac{\Gamma(\lambda + 1)}{2\pi^{\lambda+1}} \int_S \frac{(1 - \rho^2)(\Omega(\eta) - \Omega(\xi)) d\sigma(\eta)}{(1 - 2\rho(\xi, \eta) + \rho^2)^{\lambda+1}},$$

$\lambda = (k - 2)/2$. Using the standard argument [10, p. 90 and Th. 3.15] we split the integral over the sets $S(\xi, 1 - \rho), S - S(\xi, 1 - \rho)$ and use the inequality $(1 - \rho^2)(1 - 2\rho(\xi, \eta) + \rho^2)^{-(\lambda+1)} \leq \text{Const.} (1 - \rho) \times (1 - (\xi, \eta))^{-(\lambda+1)}, 1/2 \leq \rho < 1$, in the second integral to obtain, for $\Omega(\xi)$ in Lip α ,

$$(13) \quad \left| \sum_{n=1}^{\infty} \rho^n Y_n(\xi) - \Omega(\xi) \right| = 0(1 - \rho)^\alpha$$

uniformly in ξ as $\rho \rightarrow 1$.

LEMMA 4. *Let $C_j^s(n)$, $1 \leq j \leq s$; $n \geq 1$ be as in (11). Let $0 \leq \rho < 1$. Then $|\sum_{n=1}^{\infty} \rho^n Y_n(\xi) C_j^s(n)| = 0(1)$ uniformly in ρ, ξ .*

To establish the lemma we note that the recursion formula (11) implies that the coefficients $C_j^s(n)$ are ratios of polynomials in n with integer coefficients and that the denominators are products of unrepeated factors of the form $n + p$, p a nonnegative integer. Also, because $b(n, s) = 0(n^{-1})$ and $C_1^1(n) = 1/n$, an obvious induction argument shows that $C_j^s(n) = 0(n^{-j})$ as $n \rightarrow \infty$. It follows that each $C_j^s(n)$ can be written as a finite sum of the form $\sum A_p^q/(n + p)^q$, the A_p^q being independent of n . Hence, in order to establish the lemma it is enough to prove that for q a positive integer $\sum_{n=1}^{\infty} \rho^n Y_n(\xi)/(n + p)^q$ is uniformly bounded in ρ, ξ . This follows at once from induction, integration, Lemma 1, and the fact that by Lemma 3, $\rho^{p-1} \sum_{n=1}^{\infty} \rho^n Y_n(\xi)$ is uniformly bounded for $1/2 < \rho < 1$ and ξ in S .

LEMMA 5. *Let $K(x) = \Omega(x/|x|)/|x|^k$ be a Calderón-Zygmund kernel in $\text{Lip } \alpha$, $0 < \alpha < 1$ on the unit sphere S in E_k . Let $\xi = x/|x|$ and suppose $k = 2s$ where s is a positive integer. Then*

$$\left| \sum_{n=1}^{\infty} H_n^{2s}(R) Y_n(\xi) - \Omega(\xi) \right| = 0(R^{-\alpha})$$

uniformly in ξ as $R \rightarrow \infty$.

To establish the lemma, let $0 \leq \rho < 1$ and put

$$I_1 = \left| \sum_{n=1}^{\infty} (H_n^{2s}(R) - \rho^{n+s-1}(\sqrt{1 + 1/R^2})^{-(s+1)}) Y_n(\xi) \right|,$$

$$I_2 = \left| \sum_{n=1}^{\infty} (\rho^{n+s-1}(\sqrt{1 + 1/R^2})^{-(s+1)} - \rho^n) Y_n(\xi) \right|,$$

$$I_3 = \left| \sum_{n=1}^{\infty} \rho^n Y_n(\xi) - \Omega(\xi) \right|.$$

Recall that $\rho = \sqrt{1 + 1/R^2} - 1/R$. It is easy to see that $0(R^{-\alpha})$ as $R \rightarrow \infty$ is equivalent to $0((1 - \rho)^\alpha)$ as $\rho \rightarrow 1$. Thus, $I_3 = 0(R^{-\alpha})$ follows from (13). The same bound for I_2 follows from $|\rho^{s-1}(\sqrt{1 + 1/R^2})^{-(s+1)} - 1| = 0(R^{-1})$ and (13). By formula (12) of Lemma 3 and by Lemma 4, I_1 is dominated by a finite sum of terms of the form

$$\text{Const. } (R\sqrt{1 + 1/R^2})^{-j} \left| \sum_{n=1}^{\infty} \rho^n Y_n(\xi) C_j^s(n) \right|, 1 \leq j \leq s,$$

all of which are $0(R^{-1})$.

Lemma 5 is needed to prove the main theorem in the even-dimensional cases. For the case E_3 we shall have need of

LEMMA 6. *Let $K(x) = \Omega(x/|x|)/|x|^3$ be a Calderón-Zygmund kernel in $\text{Lip } \alpha, 0 < \alpha < 1$, in E_3 . Let $\xi = x/|x|$ and $0 < \gamma < \alpha$, then $|\sum_{n=1}^\infty H_n^3(R) Y_n(\xi) - \Omega(\xi)| = O(R^{-\gamma})$ uniformly in ξ as $R \rightarrow \infty$.*

To prove the lemma we put $A_0 = 0$ and $A_n = \sum_{k=1}^n \|Y_k\|_\infty$. We sum $(1 - \rho) \sum_{n=1}^N \|Y_n\|_\infty \rho^n$ by parts to obtain $(1 - \rho)^2 \sum_{n=1}^{N-1} A_n \rho^n + (1 - \rho) \rho^N A_N$. By Lemma 1, (i) $\sum_{n=1}^\infty \|Y_n\|_\infty n^\gamma/n = C < \infty$. Since $A_N \leq \sum_{n=1}^N \|Y_n\|_\infty (N/n)^{1-\gamma} \leq N^{1-\gamma} C$, we have

$$(1 - \rho) \sum_{n=1}^\infty \rho^n \|Y_n\|_\infty \leq (1 - \rho)^2 C \sum_{n=1}^\infty n^{1-\gamma} \rho^n \leq (1 - \rho)^2 \text{Const.} (1 - \rho)^{-2+\gamma},$$

where we have used the inequality $\sum_{n=1}^\infty n^\beta \rho^n \leq \text{Const.} 1/(1 - \rho)^{1+\beta}$, $\beta > 0, 0 \leq \rho < 1$. Next we observe from (8) that $H_n^k(R)$ is decreasing as a function of k , in particular, $H_n^4(R) \leq H_n^3(R) \leq H_n^2(R)$. By (10) and (9) we have $\rho^{n+1}/(1 + 1/R^2)^{3/2} \leq H_n^3(R) \leq \rho^{n-1}/(1 + 1/R^2)^{3/2}$. It follows that

$$\begin{aligned} |H_n^3(R) - \rho^n| &\leq |\rho^{n+1}/(1 + 1/R^2)^{3/2} - \rho^n| + |\rho^{n-1}/(1 + 1/R^2)^{3/2} - \rho^n| \\ &\leq \rho^n \text{Const.} R^{-1} \leq \text{Const.} (1 - \rho) \rho^n. \end{aligned}$$

Therefore,

$$\begin{aligned} \left| \sum_{n=1}^\infty H_n^3(R) Y_n(\xi) - \Omega(\xi) \right| &\leq \left| \sum_{n=1}^\infty (H_n^3(R) - \rho^n) Y_n(\xi) \right| + O(R^{-\alpha}) \\ &\leq \text{Const.} (1 - \rho) \sum_{n=1}^\infty \|Y_n\|_\infty \rho^n + O(R^{-\alpha}) \\ &= O(R^{-\gamma}) + O(R^{-\alpha}) = O(R^{-\gamma}). \end{aligned}$$

5. Proof of the main theorem. Let $(D_{\text{Sym}}\mu)(x)$ denote the symmetric derivative of μ [4, p. 175, Ex. 1]. Let $|E|$ denote the Lebesgue measure of E . If the total variations of the measures $\mu(E) - (D_{\text{Sym}}\mu)(x)|E|$ are denoted by $\int_E |d\mu(y) - (D_{\text{Sym}}\mu)(x)dy|$ then it follows as in the proof of Lebesgue's Theorem [4, Th. 8.8] that

$$(14) \quad \lim_{t \rightarrow 0} |B(x, t)|^{-1} \int_{B(x, t)} |d\mu(y) - (D_{\text{Sym}}\mu)(x)dy| = 0$$

almost everywhere. Thus, in order to prove Theorem 1, it is sufficient to prove that at each point x for which (14) holds,

$$(15) \quad \lim_{R \rightarrow \infty} \left\{ I_R(x) - \int_{E_{k-B(x, 1/R)}} K(x - y) d\mu(y) \right\} = 0.$$

With no loss in generality we will assume that $x = 0$. Set $x = 0$ in (4) and interchange the order of integration using (1). Next introduce spherical coordinates $r^{k-1}drd\sigma(\xi') = dy$ where $\xi' = y/|y|$ and $r = |y|$ and use Lemma 1, (ii) to obtain

$$I_R(0) = \int_{E_k} d\mu(w) \int_0^\infty r^{k-1} e^{-r/R} dr \sum_{n=1}^\infty (-i)^n \Gamma(n/2) / 2^k \pi^{k/2} \Gamma((n+k)/2) \cdot \int_S e^{-ir|w|(\xi', \eta)} Y_n(\xi') d\sigma(\xi'),$$

where $\eta = w/|w|$. By [9, p. 368 (2)] (with $\nu = \lambda = (k-2)/2$) the integral over S is $(2\pi)^{2+1} (-i)^n J_{n+\lambda}(r|w|)(r|w|)^{-\lambda} Y_n(\eta)$. Next, interchange summation and the integral in r . Letting Δ_R denote the term in brackets in (15) we obtain

$$\Delta_R = \int_{E_k} \sum_{n=1}^\infty H_n^k(R|w|) Y_n(\xi) |w|^{-k} d\mu(w) - \int_{E_k - B(0, 1/R)} \Omega(\xi) |w|^{-k} d\mu(w)$$

where $\xi = -w/|w|$. Next we write $\Delta_R = J_1 + J_2 + J_3$ where

$$J_1 = \int_{B(0, 1/R)} \sum_{n=1}^\infty Y_n(\xi) H_n^k(R|w|) |w|^{-k} d\mu(w),$$

$$J_2 = \int_{E_k - B(0, T)} \left[\sum_{n=1}^\infty Y_n(\xi) H_n^k(R|w|) - \Omega(\xi) \right] |w|^{-k} d\mu(w), \text{ and}$$

$$J_3 = \int_{B(0, T) - B(0, 1/R)} \left[\sum_{n=1}^\infty Y_n(\xi) H_n^k(R|w|) - \Omega(\xi) \right] |w|^{-k} d\mu(w),$$

$\xi = -w/|w|$. If $d\mu(w)$ is replaced by dw in J_1 or J_3 the resulting integral is zero. This follows from the uniform convergence of the series and $\int_S \Omega(\xi) d\sigma(\xi) = 0$. By Lemma 2, (ii),

$$|J_1| \leq \text{Const. } |B(0, 1/R)|^{-1} \int_{B(0, 1/R)} |d\mu(w) - (D_{\text{Sym}} \mu)(0) dw| = o(1)$$

as $R \rightarrow \infty$. In the case $k = 2s$, Lemma 5 gives $|J_2| \leq \text{Const. } T^{-k}$ where T can be taken arbitrarily large. For J_3 we again use Lemma 5 to obtain

$$|J_3| \leq \text{Const. } R^{-\alpha} \int_{B(0, T) - B(0, 1/R)} |w|^{-(k+\alpha)} |d\mu(w) - (D_{\text{Sym}} \mu)(0) dw|.$$

The proof of the fact that for fixed T , $J_3 = o(1)$ as $R \rightarrow \infty$ is similar to that given in [5, p. 14]. In the case $k = 3$, we replace α in the above integrals by γ , where γ is chosen so that $0 < \gamma < \alpha$, and use Lemma 6.

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Received April 16, 1969. This research was sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Grant Nos. AF-AFOSR 694-66 and 69-1689.

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Pacific Journal of Mathematics

Vol. 36, No. 1

November, 1971

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