

# Pacific Journal of Mathematics

**ON A PARTITION PROBLEM OF H. L. ALDER**

GEORGE E. ANDREWS

## ON A PARTITION PROBLEM OF H. L. ALDER

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**We study  $\Delta_d(n) = q_d(n) - Q_d(n)$ , where  $q_d(n)$  is the number of partitions of  $n$  into parts differing by at least  $d$ , and  $Q_d(n)$  is the number of partitions of  $n$  into parts congruent to 1 or  $d + 2 \pmod{d + 3}$ . We prove that  $\Delta_d(n) \rightarrow +\infty$  with  $n$  for  $d \geq 4$ , and that  $\Delta_d(n) \geq 0$  for all  $n$  if  $d = 2^s - 1$ ,  $s \geq 4$ .**

In 1956, H. L. Alder proposed the following problem [1].

“Let  $q_d(n)$  = the number of partitions of  $n$  into parts differing by at least  $d$ ; let  $Q_d(n)$  = the number of partitions of  $n$  into parts congruent to 1 or  $d + 2 \pmod{d + 3}$ ; let  $\Delta_d(n) = q_d(n) - Q_d(n)$ . It is known that  $\Delta_1(n) = 0$  for all positive  $n$  (Euler’s identity),  $\Delta_2(n) = 0$  for all positive  $n$  (one of the Rogers-Ramanujan identities),  $\Delta_3(n) \geq 0$  for all positive  $n$  (from Schur’s theorem which states  $\Delta_3(n)$  = the number of those partitions of  $n$  into parts differing by at least 3 which contain at least one pair of consecutive multiples of 3). (a) Is  $\Delta_d(n) \geq 0$  for all positive  $d$  and  $n$ ? (b) If (a) is true, can  $\Delta_d(n)$  be characterized as the number of a certain type of restricted partitions of  $n$  as is the case for  $d = 3$ ?”

This problem was again mentioned in [2; p. 743] as still being open. A recent general result on partitions with difference conditions [3] allows us to give some partial answers to Alder’s problem.

First we derive a partition theorem which is somewhat analogous to the type of result asked for by Alder.

**THEOREM 1.** *Let  $\nu$  be the largest integer such that  $2^{\nu+1} - 1 \leq d$ . Let  $\mathcal{L}_d(n)$  denote the number of partitions of  $n$  into distinct parts  $\equiv 1, 2, 4, \dots$ , or  $2^\nu \pmod{d}$ . Then*

$$q_d(n) \geq \mathcal{L}_d(n).$$

We may utilize some asymptotic formulae of Meinardus [4], [5] to prove

**THEOREM 2.** *For any  $d \geq 4$ ,  $\lim_{n \rightarrow \infty} \Delta_d(n) = +\infty$*

Finally, Theorem 1 may be utilized to prove a result which settles Alder’s problem in an infinite number of cases

**THEOREM 4.** *If  $d = 2^s - 1$  and  $s = 1, 2$ , or  $\geq 4$ , then  $\Delta_d(n) \geq 0$  for all  $n$ .*

The proof of Theorem 4 relies on the following result which is of independent interest.

**THEOREM 3.** *Let  $S = \{a_i\}_{i=1}^{\infty}$  and  $T = \{b_i\}_{i=1}^{\infty}$  be two strictly increasing sequences of positive integers such that  $b_1 = 1$  and  $a_i \geq b_i$  for all  $i$ . Let  $\rho(S; n)$  (resp.  $\rho(T; n)$ ) denote the number of partitions of  $n$  into parts taken from  $S$  (resp.  $T$ ). Then*

$$\rho(T; n) \geq \rho(S; n)$$

for all  $n$ .

**2. Proof of Theorem 1.** In Theorem 1 of [3] set  $N = d$ ,  $a(1) = 1$ ,  $a(2) = 2, \dots, a(\nu + 1) = 2^\nu$ . Thus in the notation of [3],  $D(A_N; n)$  becomes  $\mathcal{L}_d(n)$ . Now  $D(A_N; n) = E(A'_N; n)$  where the latter partition function is the number of partitions of  $n$ :

$$n = b_1 + b_2 + \dots + b_s,$$

$$b_i \equiv 1, 2, 3, 4, \dots, 2^{\nu+1} - 1 \pmod{d}$$

with

$$b_i - b_{i+1} \geq dw(\beta_d(b_{i+1})) + v(\beta_d(b_{i+1})) - \beta_d(b_{i+1}).$$

Here  $\beta_d(m)$  is the least positive residue of  $m \pmod{d}$ ,  $w(m)$  is the number of powers of 2 in the binary representation of  $m$  and  $v(m)$  is the least power of 2 in the binary representation of  $m$ . Consequently if  $b_{i+1} \equiv 2^j \pmod{d}$ ,  $0 \leq j \leq \nu$ ,

$$dw(\beta_d(b_{i+1})) + v(\beta_d(b_{i+1})) - \beta_d(b_{i+1}) = d \cdot 1 + 2^j - 2^j = d.$$

If  $b_{i+1} \not\equiv 2^j \pmod{d}$ ,  $0 \leq j \leq \nu$ , then

$$\begin{aligned} dw(\beta_d(b_{i+1})) + v(\beta_d(b_{i+1})) - \beta_d(b_{i+1}) \\ \geq 2 \cdot d + 1 - (2^{\nu+1} - 1) \geq 2d + 1 - d = d + 1. \end{aligned}$$

Thus the difference condition is always  $b_i - b_{i+1} \geq d$  or stronger. Therefore  $E(A'_N; n) \leq q_d(n)$  and Theorem 1 follows.

**3. Proof of Theorem 2.** Meinardus has proved a general theorem on asymptotic formulae for partitions with repetitions [4]. Following the notation of Meinardus [4; pp. 388-389], we see that to treat  $Q_d(n)$ , we must have his

$$a_n = \begin{cases} 1 & \text{if } n \equiv 1, d + 2 \pmod{d + 3} \\ 0 & \text{otherwise.} \end{cases}$$

Under these circumstances, Meinardus's  $D(s)$  satisfies

$$D(s) = (d + 3)^{-s} \left( \zeta \left( s, \frac{1}{d + 3} \right) + \zeta \left( s, \frac{d + 2}{d + 3} \right) \right),$$

where  $\zeta(s, a) = \sum_{n=1}^{\infty} (n + a)^{-s}$ , the Hurwitz zeta function [6; Ch. XIII],  $\alpha$ , the abscissa of convergence for  $D(s)$  is 1, and  $A$ , the residue at  $s = 1$  is  $2/d + 3$ .

$$g(\tau) = \frac{e^{-\tau} + e^{-(d+2)\tau}}{1 - e^{-(d+3)\tau}}.$$

One may now easily verify that Meinardus's analytic conditions on  $D(s)$  and  $g(\tau)$  are fulfilled, thus

$$(3.1) \quad \log Q_d(n) \sim 2\pi\sqrt{\frac{n}{3d + 9}}.$$

In [5], Meinardus has derived the asymptotic formula

$$(3.2) \quad \log q_d(n) \sim 2\sqrt{A_d n},$$

where

$$A_d = \frac{d}{2} \log^2 \alpha_d + \sum_{r=1}^{\infty} \frac{(\alpha_d)^{rd}}{r^2},$$

and  $\alpha_d$  is real  $> 0$ ,  $\alpha_d^d + \alpha_d - 1 = 0$ .

If we put  $\alpha_d = e^{-\lambda_d}$ , so that  $e^{-d\lambda_d} + e^{-\lambda_d} = 1$ , then

$$\begin{aligned} A_d &= \frac{d}{2} \lambda_d^2 + \sum_{r=1}^{\infty} \frac{\alpha_d^{d \cdot r}}{r^2} > \frac{d}{2} \lambda_d^2 + \alpha_d^d \\ &= \frac{d}{2} \lambda_d^2 + 1 - e^{-\lambda_d} > \frac{d}{2} \lambda_d^2 + \lambda_d - \frac{1}{2} \lambda_d^2 \\ &= \frac{d-1}{2} \lambda_d^2 + \lambda_d. \end{aligned}$$

Now the following table shows that

$$A_d > \pi^2/(3d + 9) \text{ for } 4 \leq d \leq 14$$

TABLE 1.

$d$	$\lambda_d >$	$\frac{d-1}{2} \lambda_d^2 >$	$A_d >$	$\frac{\pi^2}{3d+9} <$
4	0.32	0.153	0.473	0.471
5	0.28	0.15	0.43	0.42
6	0.25	0.15	0.40	0.37
7	0.22	0.14	0.36	0.33
8	0.20	0.14	0.34	0.30
9	0.19	0.14	0.33	0.28
10	0.18	0.14	0.32	0.26
11	0.16	0.12	0.28	0.24
12	0.15	0.12	0.27	0.22
13	0.15	0.13	0.28	0.21
14	0.14	0.12	0.26	0.20

For  $d \geq 15$ , we have

$$e^{-d(2/d)} + e^{-2/d} > e^{-2} + 1 - 2/d > 1,$$

Hence,  $\lambda_d > 2/d$  and

$$A_d > \frac{d-1}{2} \left(\frac{2}{d}\right)^2 + 2/d = \frac{1}{d}(4 - 2/d) > \frac{10}{3d} > \frac{\pi^2}{3d+9}.$$

Thus for all  $d \geq 4$ ,

$$A_d > \frac{\pi^2}{3d+9}.$$

Hence comparing (3.1) with (3.2) we find

$$\lim_{n \rightarrow \infty} (\log q_d(n) - \log Q_d(n)) = +\infty.$$

Thus  $\lim_{n \rightarrow \infty} \Delta_d(n) = \lim_{n \rightarrow \infty} q_d(n)(1 - Q_d(n)/q_d(n)) = +\infty$ .

and we have Theorem 2.

I would like to thank the referee for aid in simplifying and extending Theorem 2.

**4. Proof of Theorem 3.** Let us define  $S_i = \{a_1, a_2, \dots, a_i\}$  and  $T_i = \{b_1, b_2, \dots, b_i\}$ . We shall proceed to prove by induction on  $i$  that  $\rho(T_i; n) \geq \rho(S_i; n)$ ; this will establish Theorem 3 for if we choose  $I$  such that  $a_i > n, b_i > n$ , then  $\rho(T; n) = \rho(T_i; n) \geq \rho(S_i; n) = \rho(S; n)$ .

First we remark that  $\rho(T_i; n)$  is a nondecreasing function of  $n$ ; this is because  $1 = b_1 \in T_i$  and thus every partition of  $n - 1$  into parts taken from  $T_i$  may be transformed into a partition of  $n$  merely by adjoining a 1.

Now  $\rho(T_1; n) = 1$  for all  $n$  since  $T_1 = \{1\}$ . Since  $S_1 = \{a_1\}$

$$\rho(S_1; n) = \begin{cases} 1 & \text{if } a_1 | n \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\rho(T_1; n) \geq \rho(S_1; n).$$

Now assume that  $\rho(T_{i-1}; n) \geq \rho(S_{i-1}; n)$  for all  $n$ . Hence if we define  $\rho(T_i; 0) = \rho(S_i; 0) = 1$ ,

$$\begin{aligned} & \sum_{n=0}^{\infty} (\rho(T_i; n) - \rho(S_i; n))q^n \\ &= \prod_{j=1}^i \frac{1}{1 - q^{b_j}} - \prod_{j=1}^i \frac{1}{1 - q^{a_j}} \\ &= \left( \prod_{j=1}^{i-1} \frac{1}{1 - q^{b_j}} \right) \left( \frac{1}{1 - q^{a_i}} + \frac{q^{b_i} - q^{a_i}}{(1 - q^{a_i})(1 - q^{b_i})} \right) - \prod_{j=1}^i \frac{1}{1 - q^{a_j}} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{1 - q^{a_i}} \left( \prod_{j=1}^{i-1} \frac{1}{1 - q^{b_j}} - \prod_{j=1}^{i-1} \frac{1}{1 - q^{a_j}} \right) + \frac{q^{b_i} - q^{a_i}}{(1 - q^{a_i})} \prod_{j=1}^i \frac{1}{1 - q^{b_j}} \\
 &= \frac{1}{1 - q^{a_i}} \left( \sum_{n=0}^{\infty} (\rho(T_{i-1}; n) - \rho(S_{i-1}; n)) q^n \right. \\
 &\quad \left. + \sum_{n=0}^{\infty} (\rho(T_i; n - b_i) - \rho(T_i; n - a_i)) q^n \right).
 \end{aligned}$$

Now the coefficients of these two infinite series are nonnegative: the first by the induction hypothesis, and the second by the fact that  $\rho(T_i; n)$  is a nondecreasing sequence. Since  $(1 - q^{a_i})^{-1} = \sum_{j=0}^{\infty} q^{ja_i}$ , we see that all coefficients in the power series expansion of our last expression must be nonnegative. Hence

$$\rho(T_i; n) \geq \rho(S_i; n),$$

and Theorem 3 is proved.

5. **Proof of Theorem 4.** Since  $d = 2^s - 1$ , we see that the  $\nu$  of Theorem 1 is just  $s - 1$ . Now

$$\begin{aligned}
 \sum_{n=0}^{\infty} \mathcal{L}_d(n) q^n &= \prod_{j=0}^{\infty} (1 + q^{dj+1})(1 + q^{dj+2}) \cdots (1 + q^{dj+2^{\nu}}) \\
 &= \prod_{j=0}^{\infty} \frac{1}{(1 - q^{2dj+1})(1 - q^{2dj+d+2})(1 - q^{2dj+d+4}) \cdots (1 - q^{2dj+d+2^{\nu}})}.
 \end{aligned}$$

Thus  $\mathcal{L}_d(n) = \rho(T; n)$  where  $T = \{m \mid m \equiv 1, d + 2, d + 4, \dots, \text{or } d + 2^{s-1} \pmod{2d}\}$ . Clearly,  $1 \in T$ . We now show that for  $s \geq 4$  the  $i^{\text{th}}$  element of  $T$  (arranged in increasing magnitude) is no larger than the  $i^{\text{th}}$  element of  $S$  where  $S = \{m \mid m \equiv 1, d + 2 \pmod{d + 3}\}$ . Since  $s \geq 4$ , the first four elements of  $T$  are

$$1, d + 2, d + 4, d + 8 \quad (2d + 5 > d + 8 \text{ since } d \geq 15).$$

Thus the first four elements of  $T$  are less than or equal the first four elements of  $S$  respectively. In general the  $(4m + 1) - st$  element of  $T$  is  $\leq 2dm + 1$  while the  $(4m + 1) - st$  element of  $S$  is  $2m(d + 3) + 1$ ; for  $2 \leq j \leq 4$  the  $(4m + j) - \text{th}$  element of  $T$  is  $\leq 2dm + d + 2^{j-1}$  while the  $(4m + j) - \text{th}$  element of  $S$  is  $\geq 2m(d + 3) + d + 2$  and for  $2 \leq j \leq 4, m \geq 1, 2dm + d + 2^{j-1} \leq 2dm + d + 8 \leq 2dm + d + 6 + 2 \leq 2m(d + 3) + d + 2$ . Hence, the conditions of Theorem 3 are met, and therefore

$$q_d(n) \geq \mathcal{L}_d(n) = \rho(T; n) \geq \rho(S; n) = Q_d(n).$$

Thus Theorem 4 is established.

6. **Conclusion.** By modification of the results in [3], it appears possible to apply the techniques of §4 to prove that  $\Delta_d(n) \geq 0$  for

any  $d \geq 15$  which is a difference of powers of 2; however, since this approach does not yield a complete answer to Alder's problem it seems hardly worth undertaking.

Lengthier versions of the following table indicate that Alder's problem may be extended as follows.

*Conjecture.*  $\Delta_d(n) > 0$  for  $n \geq d + 6$  if  $d \geq 8$ .

$n$	$\Delta_3(n)$	$\Delta_4(n)$	$\Delta_5(n)$	$\Delta_6(n)$	$\Delta_7(n)$	$\Delta_8(n)$
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	0	0	0	0	0	0
6	0	0	0	0	0	0
7	0	0	0	0	0	0
8	0	0	0	0	0	0
9	1	0	0	0	0	0
10	0	1	0	0	0	0
11	0	1	1	0	0	0
12	0	1	1	1	0	0
13	0	0	2	1	1	0
14	0	0	1	2	1	1
15	1	0	1	2	2	1
16	1	0	0	2	2	2
17	1	1	0	1	3	2
18	1	2	0	1	2	3
19	1	2	1	0	2	3
20	1	2	2	0	1	3
21	2	2	3	1	1	2
22	2	2	3	2	0	2
23	2	3	3	3	1	1
24	2	4	3	4	2	1

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