

Pacific Journal of Mathematics

INTERSECTIONS OF NILPOTENT HALL SUBGROUPS

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A family \mathcal{H} of subgroups of a finite group G is said to satisfy (property) B^* if whenever $U = H_1 \cap \cdots \cap H_r$ is a representation of U as intersection of elements of \mathcal{H} of minimal length r , then $r \leq 2$. The aim of this paper is to prove

THEOREM 1. Let H be a nilpotent Hall π -subgroup of a group G and assume that if $H_1, H_2 \in S_\pi(G)$ then $H_1 \cap H_2 \triangleleft H_1$. Then $S_\pi(G)$ satisfies B^* .

All groups in this work are finite. A family \mathcal{H} of subgroups of a group G will be said to satisfy (property) B if there exist H_1 and H_2 in \mathcal{H} such that

$$H_1 \cap H_2 = \bigcap \{H \mid H \in \mathcal{H}\}.$$

We will denote by $S_p(G)$ the family of Sylow p -subgroups of G and the (possibly empty) family of Hall π -subgroups of G will be denoted by $S_\pi(G)$. It was shown by Brodkey [1] that if G possesses an Abelian Sylow p -subgroup, then $S_p(G)$ satisfies B . Itô has shown in [3] that if G is of odd order, hence solvable by [2], then $S_p(G)$ satisfies B for all primes. He has also shown that if G is solvable, then $S_p(G)$ satisfies B in several other cases.

As indicated above, we will consider here a more restrictive condition B^* on families of subgroups of G . It follows from our main result, Theorem 1, that even the property B^* is satisfied by $S_\pi(G)$, whenever G possesses an Abelian or Hamiltonian (i.e., Dedekind) Hall π -subgroup. Theorem 1 yields the following

COROLLARY 1. Let H be a nilpotent Hall subgroup of the group G and suppose that the index $[H: H \cap H^x]$ is prime for all $x \in G - N_G(H)$. Then either $H \triangleleft G$ or for all $x, y \in G$ such that $xy^{-1} \notin N_G(H)$ we have

$$H^x \cap H^y = B \triangleleft G$$

and $[H: B] = p$, a prime. B is independent of x and y .

2. Generalizations. As a matter of fact, we will prove a more general result than Theorem 1. We will say that a group N satisfies

(property) D_π , where π is a set of primes, if N contains at least one Hall π -subgroup, any two Hall π -subgroups of N are conjugate and each π -subgroup of N is contained in a Hall π -subgroup of N . The set of all maximal π -subgroups of a group G will be denoted by $\text{Syl}_\pi(G)$. Theorem 1 follows from

THEOREM 2. *Let the group G satisfy the following conditions.*

(i) *If $H_1, H_2 \in \text{Syl}_\pi(G)$ then $C = H_1 \cap H_2 \triangleleft H_1$ and*

(ii) *If $C = H_1 \cap H_2 \triangleleft G$, then $N_G(C)$ satisfies D_π . Then $\text{Syl}_\pi(G)$ satisfies B^* .*

A subgroup N of the group G will be called a π -local subgroup of G if $N = N_G(H)$, where H is a nontrivial π -subgroup of G . An N_π -group is a group all of whose π -local subgroups are solvable. Theorem 1 yields the following

THEOREM 3. *Let G be an N_π -group and suppose that all its maximal π -subgroups are Dedekind groups. Then $\text{Syl}_\pi(G)$ satisfies B^* .*

As a consequence, we have

COROLLARY 2. *Let G be a nonsolvable N_π -group and suppose that each $H \in \text{Syl}_\pi(G)$ is a Dedekind group. Then there exist $H_1, H_2 \in \text{Syl}_\pi(G)$ such that:*

$$o(G) \geq o(H_1)o(H_2).$$

3. Proofs. We begin with a proof of Theorem 2. Let $U = H_1 \cap \cdots \cap H_r$ be a representation of U as intersection of elements of $\text{Syl}_\pi(G)$ of minimal length and suppose that $r > 2$. It follows from assumption (i) and the minimality of r that

$$(1) \quad U \subseteq H_1 \cap H_2 = C \triangleleft H_1, H_2.$$

The minimality of r also implies that $C \not\triangleleft H_3$ and consequently $C \triangleleft G$. By assumption (ii) $N = N_G(C)$ has the D_π -property and by (1) $H_1, H_2 \in S_\pi(N)$. There exists $Q \in S_\pi(N)$ containing $H_3 \cap \cdots \cap H_r \cap N$ and since the elements of $S_\pi(N)$ are conjugate in N , there exists $R \in S_\pi(N)$ such that $Q \cap R = C$. Since Q and R are conjugates of H_1 , they belong to $\text{Syl}_\pi(G)$. However,

$$\begin{aligned} U &= Q \cap R \cap H_3 \cap \cdots \cap H_r \supset R \cap N \cap H_3 \cap \cdots \cap H_r \\ &= R \cap H_3 \cap \cdots \cap H_r \supset U \end{aligned}$$

in contradiction to the minimality of r . The proof of Theorem 2 is complete.

Theorem 1 follows immediately from Theorem 2 and Wielandt's Theorem [4] which states that if G contains a nilpotent Hall π -subgroup then G satisfies D_π . Theorem 3 also follows immediately from Theorem 2.

Corollary 1 follows from Theorem 1. Suppose that $H \triangleleft G$, and let $H_1, H_2 \in S_\pi(G)$, $H_1 \neq H_2$. Since by our assumptions and the above mentioned Theorem of Wielandt $[H_1: H_1 \cap H_2]$ is prime, it follows that $H_1 \cap H_2 \triangleleft H_1$. Let $B = \bigcap \{H^x \mid x \in G\}$; obviously $B \triangleleft G$ and since $H \triangleleft G$, $B \subseteq H$. By Theorem 1 there exist $u, v \in G$ such that for all $x, y \in G$ we have:

$$H^u \cap H^v = B \subset H^x \cap H^y.$$

Suppose that $xy^{-1} \notin N_G(H)$; then $H^x \neq H^y$ and

$$p = [H^u: B] = [H^x: B] = [H^x: H^x \cap H^y] [H^x \cap H^y: B].$$

Since $[H^x: H^x \cap H^y] > 1$, it follows that $H^x \cap H^y = B$. The proof of Corollary 1 is complete.

Finally, Corollary 2 follows from Theorem 3. Since G is a non-solvable N_π -group, $\bigcap \{H \mid H \in \text{Syl}_\pi(G)\} = \{1\}$. Consequently, by Theorem 3, there exist $H_1, H_2 \in \text{Syl}_\pi(G)$ such that $H_1 \cap H_2 = \{1\}$. Obviously $o(G) \leq o(H_1)o(H_2)$ and the proof is complete.

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