A DECOMPOSITION THEOREM FOR TOPOLOGICAL GROUP EXTENSIONS

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G. W. Mackey has developed a characterization of the group of equivalence classes of extensions of a fixed group G by a fixed abelian group A restricting all groups to be locally compact second countable Hausdorff spaces. Calvin C. Moore incorporated his results into a cohomology theory of group extensions such that the second cohomology group, $H^2(G, A)$, coincides with Mackey's group of extension classes. The purpose of this paper is to consider the special case in which G and A are connected, A is a Lie group, G is locally arcwise connected and locally simply connected. Under these conditions G and A admit universal covering groups U_G and U_A. Allowing $\pi(G)$ and $\pi(A)$ to denote the fundamental groups of G and A respectively (with basepoint the identity) any extension of G by A determines an extension of U_G by U_A and an extension of $\pi(G)$ by $\pi(A)$ uniquely up to equivalence. Hence there is a map Φ, in fact a homomorphism, constructed from $H^2(G, A)$ to $H^2(U_G, U_A) \oplus H^2(\pi(G), \pi(A))$. In this paper $H^2(G, A)$ is determined as a direct sum of subgroups of $H^2(U_G, U_A)$ and $H^2(\pi(G), \pi(A))$, and of a third group which is computed.

Throughout all topological groups are assumed to be locally compact, Hausdorff and second countable.

Let G and A be topological groups, A abelian. A is called a G-module if there is given an action of G on A, that is, a continuous function from $G \times A$ into A, carrying (g, a) onto ga, such that (i) for any fixed $g \in G$, the mapping of A into A determined by $a \longrightarrow ga$ is an automorphism, (ii) the automorphism thus determined by the identity of G is the identity automorphism, and (iii) for any g and h in G and $a \in A$, $(gh)a = g(ha)$. If A is a G-module then an extension of G by A is an exact sequence

$$D: 1 \longrightarrow A \xrightarrow{\ i\ } E \xrightarrow{\ j\ } G \longrightarrow 1$$

such that E is a topological group, $i: A \longrightarrow i(A)$ is a homeomorphism and $E/i(A)$ is topologically isomorphic to G or equivalently, j is open, and finally, the action of G on A defined by $xa = i^{-1}(xi(a)x^{-1})$ coincides with the given action of G on A. Sometimes, when G and A are fixed, and when no confusion can arise, the extension D will be denoted by $D(E, i, j)$. Two extensions $D(E, i, j)$ and $D'(E', i', j')$, of G by A are called equivalent if there is a topological isomorphism ϕ of

E onto E' such that $\phi i = i'$ and $j'\phi = j$. Let Ext (G, A) denote the family of equivalence classes of group extensions of G by A. The problem of determining Ext (G, A) was essentially solved by G. W. Mackey in [7] by applying certain results on Borel groups proved in [6]. The solution involves the theory of analytic Borel spaces (c.f. the introduction of [1]). We shall assume the definitions and results of [1]. One useful result is that for the exact sequence

$$D: 1 \longrightarrow A \xrightarrow{i} E \xrightarrow{j} G \longrightarrow 1$$

to be an extension of G by the G-module A, the requirements on i and j can be weakened to only require that i and j be continuous. One proves that i is a homeomorphism of A onto $i(A)$ and j is open. In [10] C. C. Moore incorporates Mackey's solution to the extension problem into a cohomology theory of topological groups. In his theory the second cohomology group of G with respect to A, $H^2(G, A)$, can be put into a one-to-one correspondence with Ext (G, A). We shall draw upon the results of [10] and identify Ext (G, A) with $H^2(G, A)$.

1. Construction of Φ. We fix topological groups G and A so that A is a connected abelian Lie group with an action of G, and G is connected, locally arcwise connected, and locally simply connected. We fix an extension

$$D: 1 \longrightarrow A \xrightarrow{i} E \xrightarrow{j} G \longrightarrow 1$$

of G by A.

PROPOSITION 1.1.   *$j: E \to G$ is a locally trivial fibre space with fibre A.*

Proof. A is a product of a vector group R^m and an n-torus T^n, and hence A is a covering group for T^{m+n}. Hence the result easily follows from [3] and the fact that A acts on E as a group of transformations by $a(x) = i(a)x$.
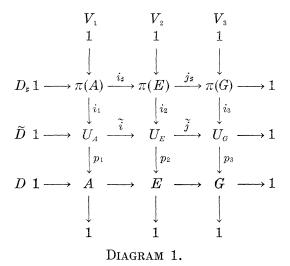
PROPOSITION 1.2.   *E is connected, locally arcwise connected and locally simply connected. Hence E as well as G and A admit universal covering groups.*

Proof. E is connected since A and G are. By Proposition 1.1, E is locally a product of A and G. Thus the other two properties carry over from A and G to E. Thus A, E and G each more than satisfy the conditions required to construct covering groups (e.g., see [2]).

Let U_A, U_E, and U_G denote the respective universal covering groups of A, E, and G, with respective projections p_1, p_2, and p_3. It is well known that the kernel of each p_i is a closed, countable, discrete, and central subgroup. Also the kernel of each p_i can be considered the fundamental group of A, E, and G respectively, and is therefore denoted by $\pi(A)$, $\pi(E)$, and $\pi(G)$. Let i_1, i_2, and i_3 denote the inclusion maps of $\pi(A)$ into U_A, etc. Thus we obtain three exact sequences

$$V_1(U_A, i_1, p_1), \quad V_2(U_E, i_2, p_2) \quad \text{and} \quad V_3(U_G, i_3, p_3) .$$

Each V_i is an extension of the appropriate group by the appropriate fundamental group, with trivial group action. Since U_A, U_E, and U_G are simply connected, the homomorphisms i and j lift to unique homomorphisms $\tilde{i}$ from U_A to U_E and $\tilde{j}$ from U_E to U_G such that $p_2\tilde{i} = ip_1$, and $p_3\tilde{j} = jp_2$. It is easily seen that $\tilde{i}$ maps the kernel of p_1 into the kernel of p_2 and that $\tilde{j}$ maps the kernel of p_2 into the kernel of p_3, yielding homomorphisms $i_\sharp$ and $j_\sharp$ with $i_2i_\sharp = \tilde{i}i_1$ and $i_3j_\sharp = \tilde{j}i_2$. Thus we obtain a 3 by 3 commutative diagram.

DIAGRAM 1.

Let $D_\sharp$ denote the top row and $\tilde{D}$, the middle row of Diagram 1. Of course D, V_1, V_2, and V_3 are exact sequences.

PROPOSITION 1.3. (1) *$D_\sharp$ and $\tilde{D}$ are exact sequences.*
(2) *The induced action of U_G on U_A is compatible with the given action of G on A. That is, for any $x \in U_G$ and $t \in U_A$,*

$$p_1(xt) = (p_3(x))(p_1(t)) .$$

Furthermore this action is the only action of U_G on U_A compatible with the action of G on A.

(3) $D_\sharp$ and D are extensions of U_G by U_A and $\pi(G)$ by $\pi(A)$ respectively, with the action of (2), of U_G on U_A, and trivial action of $\pi(G)$ on $\pi(A)$.

Proof. First note that, by the construction of the universal covering groups, $i_\sharp$ and $j_\sharp$ are actually the standard morphisms induced by the functor $\pi = \pi_1$, operating on i and j. By Proposition 1.1 j: $E \to G$ is a fibre space with fibre A and hence we obtain the exact homotopopy sequence of the fibering (see [5], p. 152):

$$\cdots \longrightarrow \pi_2(G) \xrightarrow{\ d_\sharp\ } \pi_1(A) \xrightarrow{\ i_\sharp\ } \pi_1(E) \xrightarrow{\ j_\sharp\ } \pi_1(G) \xrightarrow{\ d_\sharp\ } \pi_0(A)\ .$$

Since G is an inverse limit of Lie groups (e.g., see [9]) by [11], $\pi_2(G) = 0$. Since A is connected, $\pi_0(A) = 0$. Hence $D_\sharp$ is exact. Exactness of $\tilde{D}$ follows by standard diagram chasing (see [8], p. 51); the needed condition $\tilde{j}\tilde{i}$ is trivial holds, since $\tilde{j}\tilde{i}$ maps the connected group U_A into the discrete group $i_3(\pi(G))$.

The first part of (2) follows simply from the commutativity of diagram 1. For the second part let α and β be two actions of U_G on U_A compatible with the action of G on A. Then for any $x \in U_G$ and $t \in U_A$, $p_1(\alpha(x, t)) = p_1(\beta(x, t)) = (p_3(x))(p_1(t))$. Define a function F from $U_G \times U_A$ to U_A by $F(x, t) = \alpha(x, t)(\beta(x, t))^{-1}$. Then p_1F is trivial, and hence the range of F lies in $i_1(\pi(A))$, a discrete set. Connectedness of $U_G \times U_A$ and continuity of F imply that $\alpha = \beta$.

The proof of (3) is trivial once (1) and (2) are established.

Thus the extension D determines extensions $\tilde{D}$ and $D_\sharp$. Now $\tilde{D}$ and $D_\sharp$ determine equivalence classes say

$$[\tilde{D}] \in H^2(U_G, U_A) \quad \text{and} \quad [D_\sharp] \in H^2(\pi(G), \pi(A))\ .$$

PROPOSITION 1.4. *Let* $D = D(E, i, j)$ *and* $D' = D'(E', i', j')$ *be two extensions of* G *by* A. *Then if* D *is equivalent to* D', $[\tilde{D}] = [\tilde{D}']$ *and* $[D_\sharp] = [D_\sharp']$.

Proof. Let f be a topological isomorphism from E onto E' such that $fi = i'$ and $j'f = j$. Then there exists a unique lifting f of $\tilde{f}$ from U_E onto $U_{E'}$ such that $\tilde{f}$ is a topological isomorphism from U_E onto $U_{E'}$, $\tilde{f}\tilde{i} = \tilde{i}'$, and $\tilde{j}'\tilde{f} = \tilde{j}$. Hence $\tilde{f}$ establishes an equivalence between $\tilde{D}$ and $\tilde{D}'$. Similarly on the homotopy level, $f_\sharp$ is an isomorphism between $\pi(E)$ and $\pi(E')$, and $f_\sharp i_\sharp = i_\sharp'$ and $j_\sharp'f_\sharp = j_\sharp$. Hence $[D_\sharp] = [D_\sharp']$.

Thus as a result of Proposition 1.4 we define mappings $\varPhi_1$ and $\varPhi_2$ each from $H^2(G, A)$ with $\varPhi_1$ going into $H^2(U_G, U_A)$ where the action of U_G on U_A is the unique one compatible with the action of G on A, and $\varPhi_2$ going into $H^2(\pi(G), \pi(A))$ where the action of $\pi(G)$ on $\pi(A)$ is

trivial, and where $\Phi_1([D]) = [\tilde{D}]$ and $\Phi_2([D]) = [D_\sharp]$. Since $\pi(E)$ is always abelian, Φ_2 actually maps $H^2(G, A)$ into the subgroup of $H^2(\pi(G), \pi(A))$ of the classes of symmetric extensions which we denote by $H_s^2(\pi(G), \pi(A))$. Combining Φ_1 and Φ_2 we obtain a mapping

$$\Phi = \Phi_1 \oplus \Phi_2 \colon H^2(G, A) \longrightarrow H^2(U_G, U_A) \oplus H_s^2(\pi(G), \pi(A)) \ .$$

In § 2 we establish that Φ is a homomorphism. Before we do this we make an observation about the above construction. Note that we carefully choose the vertical extensions of Diagram 1 to be constructed in a specified manner (e.g., see [2]). This is important since there exist inequivalent extensions involving the universal covering group of a given group. For example, consider the following two inequivalent extensions of the 1-torus, T, by the integers Z. Letting R denote the real numbers we have:

$$(1) \qquad\qquad 1 \longrightarrow Z \overset{i}{\longrightarrow} R \overset{\exp}{\longrightarrow} T \longrightarrow 1 \quad \text{and}$$

$$(2) \qquad\qquad 1 \longrightarrow Z \overset{-i}{\longrightarrow} R \overset{\exp}{\longrightarrow} T \longrightarrow 1 \ .$$

Here exp is the exponential map of R into T, i is the identity injection of Z into R, and $-i$ sends an integer n into the real number $-n$. Extensions (1) and (2) are inequivalent. The first extension is the one obtained by following the procedures of [2]. One may ask whether the substitution of inequivalent vertical extensions in Diagram 1 will give rise to inequivalent horizontal extensions. There can be a certain ambiguity to the definition of Φ unless we specify how we are to choose the vertical extensions. We do have one result, however, which does simplify the problem and which will be useful in subsequent sections.

DEFINITION 1.5. Let E be a connected, locally arcwise connected, and semi-locally simply connected group. Then the extension $V(F, p, q)$ is called a *universal covering extension* of E if F is a universal covering group for E with projection q.

PROPOSITION 1.6. *Let $V_2'(F, p, q)$ be any universal covering extension of the group E in Diagram 1. Let K denote the domain of p. Then there exist unique continuous homomorphisms $r, s, h,$ and k such that $r \colon \pi(A) \to K$ $s \colon K \to \pi(G)$, $h \colon U_A \to F$, and $k \colon F \to U_G$, and such that the new diagram obtained from Diagram 1 is commutative. That is such that $pr = hi_1$, $qh = ip_1$, $i_3 s = kp$, and $jq = p_3 k$.*

Furthermore, defining

H_2: $\qquad\qquad 1 \longrightarrow \pi(A) \overset{r}{\longrightarrow} K \overset{s}{\longrightarrow} \pi(G) \longrightarrow 1$

and H_1: $$1 \longrightarrow U_A \xrightarrow{\ h\ } F \xrightarrow{\ k\ } U_G \longrightarrow 1$$

then H_1 and H_2 are extensions equivalent to $\tilde{D}$ and $D_\sharp$ respectively (see Diagram 1).

Proof. Let us substitute V_2' for V_2 in the construction of Diagram 1. Then repeating earlier arguments we see that i and j induce homomorphisms $h: U_A \to F$ and $k: F \to U_G$, $r: \pi(A) \to K$ and $s: K \to \pi(G)$. Thus we obtain a new 3×3 commutative diagram which is the required one for this proposition. H_1 and H_2 are seen automatically to be extensions. The uniqueness of h and k follow from the type of topological argument involving connectedness and continuity which we have already seen. The proofs that r and s are unique involve simple algebraic arguments and are omitted. Hence we only have left to prove the equivalence of H_1 to $\tilde{D}$ and H_2 to $D_\sharp$. Since F and U_E are simply connected there exists a unique topological isomorphism $\lambda: U_E \to F$ such that $p_2 = q\lambda$. Since $p_2\tilde{i} = ip_1 = qh$, we have that $q\lambda\tilde{i} = qh$. Hence for all $x \in U_A$, $q(\lambda\tilde{i}(x)h(x^{-1})) = 1$ and therefore $\lambda i(x)h(x^{-1}) \in p(K)$, a discrete set. Hence $\lambda\tilde{i} = h$. Similarly $\tilde{j} = k\lambda$. Hence H_1 is equivalent to $\tilde{D}$. Next observe that $\lambda i_2(\pi(E)) \subset p(K)$. Hence define $\bar{\lambda} = p^{-1}\lambda i_2$. Then by a simple algebraic argument it is seen that $\bar{\lambda}$ is an isomorphism from $\pi(E)$ onto K and that $\bar{\lambda}i_\sharp = r$ and $s\bar{\lambda} = j_\sharp$. Hence H_2 is equivalent to $D_\sharp$.

2. Φ is a homomorphism. The proof that Φ is a homomorphism necessitates the examination of an equivalent method of summing elements of $H^2(G, A)$ called the Baer sum. It involves the direct construction of a new group extension from two given extensions. Extension equivalence is a congruence with respect to this binary operation and hence there is defined a binary operation on $H^2(G, A)$, the Baer sum. This operation coincides with the group operation on $H^2(G, A)$. Their equivalence is shown in [8] for the nontopological case. For topological group extensions the same proof carries over, but with the appropriate measure theoretical arguments, applying the theory of Borel spaces. Since it is an integral part of our proof, we shall briefly outline the construction. Let $D_1(E_1, f_1, g_1)$ and $D_2(E_2, f_2, g_2)$ be two extensions of G by A. Define

$$\bar{E} = \{(x, y) \in E_1 \times E_2: g_1(x) = g_2(y)\} .$$

Define a map $\bar{j}: E \to G$ by $\bar{j}(x, y) = g_1(x)$. Then letting $\bar{i}$ denote $f_1 \times f$ we obtain an extension

$\bar{D}$ $$1 \longrightarrow A \times A \xrightarrow{\ \bar{i}\ } \bar{E} \xrightarrow{\ \bar{j}\ } G \longrightarrow 1$$

of G by $A \times A$, where G acts on $A \times A$ pointwise. Next define $\varDelta A \subset A \times A$ by $\varDelta A = \{(a, a^{-1}): a \in A\}$. Then $\varDelta A$ is invariant under the action of G. Hence we may 'divide' the extension by A obtaining a new extension

$$1 \longrightarrow A \times A/\varDelta A \xrightarrow{i'} \bar{E}/\bar{i}(\varDelta A) \xrightarrow{j} G \longrightarrow 1$$

where i' and j are the cannonical maps obtained from $\bar{i}$ and $\bar{j}$ as a result of taking quotients. Finally let $\theta_A: A \to A \times A/\varDelta A$ be the isomorphism defined by taking the element a into the coset $(a, 1)\varDelta A$. Note that $\theta_A^{-1}((a, b)\varDelta A) = ab$. Now define $i = i'\theta_A$. Then we obtain an extension

$$1 \longrightarrow A \xrightarrow{i} \bar{E}/\bar{i}(\varDelta A) \xrightarrow{j} G \longrightarrow 1$$

of G by A. This extension is the required one whose equivalence class is the sum of $[D_1]$ and $[D_2]$. Naturally the same method of summing may be applied to compute sums for $H^2(U_G, U_A)$ and $H^2(\pi(G), \pi(A))$.

Our plan is to apply the method of construction simultaneously to the sum of two extensions of G by A and to the sum of each pair if images under $\varPhi_1$ and $\varPhi_2$. Consider the extensions D_1 and D_2 above. For each of these extensions we obtain a 3×3 commutative diagram as in Diagram 1. If we take the product of these two diagrams we obtain a new diagram, Diagram 2.

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 \longrightarrow & \pi(A) \times \pi(A) & \xrightarrow{f_{1\sharp} \times f_{2\sharp}} & \pi(E_1) \times \pi(E_2) & \xrightarrow{g_{1\sharp} \times g_{2\sharp}} & \pi(G) \times \pi(G) & \longrightarrow 1 \\
 & \downarrow{\scriptstyle i_1 \times i_1} & & \downarrow{\scriptstyle i_2' \times i_2''} & & \downarrow{\scriptstyle i_3 \times i_3} & \\
1 \longrightarrow & U_A \times U_A & \xrightarrow{\tilde{f}_1 \times \tilde{f}_2} & U_{E_1} \times U_{E_1} & \xrightarrow{\tilde{g}_1 \times \tilde{g}_2} & U_G \times U_G & \longrightarrow 1 \\
 & \downarrow{\scriptstyle p_1 \times p_1} & & \downarrow{\scriptstyle p_2' \times p_2''} & & \downarrow{\scriptstyle p_3 \times p_3} & \\
1 \longrightarrow & A \times A & \xrightarrow{f_1 \times f_2} & E_1 \times E_2 & \xrightarrow{g_1 \times g_2} & G \times G & \longrightarrow 1 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & & 1 & & 1 & & 1 & &
\end{array}
$$

DIAGRAM 2.

If we now continue the process of simultaneously constructing the Baer sum in each of the three rows we obtain groups $\overline{\pi(E)}$, $\overline{U_E}$ and $\bar{E}$ defined by

$$\overline{\pi(E)} = \{(a, b) \in \pi(E_1) \times \pi(E_2): g_{1\sharp}(a) = g_{2\sharp}(b)\} \,,$$
$$\overline{U_E} = \{(x, y) \in U_{E_1} \times U_{E_2}: \tilde{g}_1(x) = \tilde{g}_2(y)\} \,,$$

and $\bar{E}$, defined as in the beginning of this section. Then also as in the beginning of this section we obtain maps, $\bar{i}$, $\tilde{\bar{i}}$, and $\bar{i}_\sharp$, $\bar{j}$, $\tilde{\bar{j}}$, and $\bar{j}_\sharp$. Then it is easily seen that, referring to Diagram 2,

$$p_2' \times p_2''(\overline{U_E}) \subset \bar{E}$$

and $i_2' \times i_2''(\overline{\pi(E)}) \subset \bar{U}_E$. In fact it is easily seen that

$$W: \qquad 1 \longrightarrow \overline{\pi(E)} \xrightarrow{\;i_2' \times i_2''\;} \overline{U_E} \xrightarrow{\;p_2' \times p_2''\;} \bar{E} \longrightarrow 1$$

is an exact sequence and hence an extension. We thus obtain a commutative 3×3 diagram with the above sequence, W, as the middle vertical sequence, V_3 (see Diagram 1) as the right most vertical sequence, and the left vertical sequence coinciding with the left most vertical sequence of diagram 2. Next observe that $i_1 \times i_1(\varDelta\pi(A)) \subset \varDelta U_A$ and $p_1 \times p_1(\varDelta U_A) \subset \varDelta A$. Hence $i_1 \times i_1$ induces a homomorphism i_1'' from $\pi(A) \times \pi(A)/\varDelta\pi(A)$ to $U_A \times U_A/\varDelta U_A$. Also note that $i_1'\theta_{\pi(A)} = \theta_{U_A}i_1$. Similarly we obtain a homomorphism p_1' such that $\theta_A p_1 = p_1'\theta_{U_A}$. Now if we devide the horizontal sequences by the appropriate $\varDelta$'s and substitute the extension V_1 of diagram 1 for the left most vertical extension we obtain a 3×3 diagram

$$
\begin{array}{ccccccccc}
& & 1 & & 1 & & 1 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 \longrightarrow & \pi(A) & \xrightarrow{i_\sharp} & \overline{\pi(E)}/i_\sharp(\varDelta\pi(A)) & \xrightarrow{j_\sharp} & \pi(G) & \longrightarrow & 1 \\
& \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle i_3} & & \\
1 \longrightarrow & U_A & \xrightarrow{\tilde{i}} & \overline{U_E}/\tilde{i}(\varDelta U_A) & \xrightarrow{\tilde{j}} & U_G & \longrightarrow & 1 \\
& \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p_3} & & \\
1 \longrightarrow & A & \xrightarrow{i} & \bar{E}/i(\varDelta A) & \xrightarrow{j} & G & \longrightarrow & 1 \\
& \downarrow & & \downarrow & & \downarrow & & \\
& & 1 & & 1 & & 1 & &
\end{array}
$$

<div align="center">DIAGRAM 3.</div>

where p and q are induced from $i_1' \times i_2'$ and $p_1' \times p_2'$ by taking quotients of $\overline{\pi(E)}$ by $i_\sharp(\varDelta\pi(A))$ and $\bar{U}_E$ by $\tilde{i}(\varDelta U_A)$. It should be verified that the middle vertical sequence of diagram 3 is exact and that the entire diagram is commutative. We now need a few facts concerning topological properties of the middle vertical sequence.

PROPOSITION 2.1. $\bar{U}_E/\tilde{i}(\varDelta U_A)$ is simply connected and p embeds $\overline{\pi(E)}/i_\sharp(\varDelta\pi(A))$ as a discrete subgroup. Hence the middle vertical sequence of Diagram 3 is a universal covering sequence for $\bar{E}/i(\varDelta A)$.

Proof. By a repetition of the argument of Proposition 1.1 it is easily seen that $\tilde{j}\colon \bar{U}_E/\tilde{i}(\Delta U_A) \to U_G$ is a fibre space with fibre U_A. Hence applying the homotopy sequence of a fibre space [5] and the fact that U_A and U_G are simply connected we obtain the first result. The second assertion follows from the fact that $\overline{\pi(E)}/i_*(\Delta\pi(A))$ is itself discrete and p embeds this discrete group homeomorphically into $\overline{U_E}/\tilde{i}(\Delta U_A)$. Hence the final result follows.

We are now prepared for the main result.

THEOREM 2.2. Φ_1, Φ_2 *and hence* Φ *are homomorphisms.*

Proof. Given the extensions D_1 and D_2 above, follow the construction outlined above to obtain Diagram 3. Then we obtain three horizontal extensions, the lowest belonging to the class $[D_1] + [D_2]$, the middle belonging to class $\Phi_1([D_1]) + \Phi_1([D_2])$, and the top belonging to the class $\Phi_2([D_1]) + \Phi_2([D_2])$. However by Propositions 1.6 and 2.1 the middle horizontal extension belongs to $\Phi_1([D_1] + [D_2])$ and the top horizontal extension belongs to class

$$\Phi_2([D_1] + [D_2]) .$$

Hence Φ_1, Φ_2, and Φ are homomorphisms.

3. **The range of Φ.** We have seen previously the range of Φ_2 is contained in $H_s^2(\pi(G), \pi(A))$. We now attempt to find the range of Φ_1. To begin with, observe that since U_G acts on U_A we may define an action of $\pi(G)$ on U_A by mapping (g, t) into $i_3(g)t$ for $g \in \pi(G)$ and $t \in U_A$.

LEMMA 3.1. *The action of* $\pi(G)$ *on* U_A *defined above is trivial.*

Proof. Let $D(E, i, j)$ be any extension of G by A (e.g., the trivial semidirect product). Then since the action of U_G on U_A is independent of the extension D it suffices to show that $\pi(G)$ acts trivially on U_A for this extension. Repeat the construction of § 1 and obtain Diagram 1. Now let $a \in \pi(G)$. Choose a $b \in \pi(E)$ such that $j_*(b) = a$. Then it is clear by commutativity that for $t \in U_A$,

$$i_3(a)t = \tilde{i}^{-1}(i_2(b)\tilde{i}(t)i_2(b^{-1})) = t$$

since $i_2(\pi(E))$ lies in the center of U_E. Hence the result follows.

In the following lemmas we make use of the notation of Diagram 1.

LEMMA 3.2. *Let D be an extension of G by A. Let $\alpha\colon \pi(G) \to \pi(E)$ be any cross section of $j_\sharp$ carrying the identity into the identity. Then there exists a Borel cross section $\beta\colon U_G \to U_E$ of $\tilde{j}$ such that $i_2\alpha = \beta i_3$.*

Proof. First choose any Borel cross section β' of $\tilde{j}$. Then define $\beta\colon U_G \to U_E$ by

   (1)   for $x \in i_3(\pi(G))$ $\beta(x) = i_2\alpha i_3^{-1}(x)$

   (2)   otherwise define $\beta(x) = \beta'(x)$.

Then it is clear that β is Borel and satisfies the conditions of the lemma.

DEFINITION 3.3. (1) Let $Z_c^2(i_3,\ U_G,\ U_A)$ denote the subgroup of $Z^2(U_G,\ U_A)$ consisting of all 2-cocycles σ such that

$$\sigma(i_3(g),\ x) = \sigma(x,\ i_3(g))$$

for all $g \in \pi(G)$ and $x \in U_G$.

   (2) Let $H_c^2(i_3,\ U_G,\ U_A)$ be the subgroup of $H^2(U_G,\ U_A)$ obtained as the image of $Z_c^2(i_3,\ U_G,\ U_A)$ under the quotient homomorphism, mapping $Z^2(U_G,\ U_A)$ onto $Z^2(U_G,\ U_A)/B^2(U_G,\ U_A) = H^2(U_G,\ U_A)$.

LEMMA 3.4. *Let D be an extension of G by A. Let $\tilde{D}$ be the resulting extension of U_G by U_A obtained in Diagram 1. Then there exists a Borel cross section $\beta\colon U_G \to U_E$ of $\tilde{j}$ such that the 2-cocycle obtained from β, ν, defined by $\nu(x, y) = \tilde{i}^{-1}(\beta(x)\beta(y)\beta(xy)^{-1})$ lies in $Z_c^2(i_3,\ U_G,\ U_A)$.*

Proof. Let $D_\sharp$ be the resulting extension of $\pi(G)$ by $\pi(A)$ and let α be a cross section of $j_\sharp$ mapping the identity into the identity. Then by Lemma 3.2 there exists a Borel cross section $\beta\colon U_G \to U_E$ of $\tilde{j}$ such that $i_2\alpha = \beta i_3$. Then observe that since $i_2(\pi(E))$ lies in the center of U_E that

$$\beta i_3(g)\beta(x) = i_2\alpha(g)\beta(x) = \beta(x)i_2\alpha(g) = \beta(x)\beta i_3(g)$$

for all $x \in U_G$ and $g \in \pi(G)$. Hence since $i_3(\pi(G))$ lies in the center of U_G it is easily seen that $\nu(i_3(g), x) = \nu(x, i_3(g))$ for all $x \in U_G$ and $g \in \pi(G)$. This proves the result.

COROLLARY 3.5. $\Phi_1(H^2(G, A)) \subset H_c^2(i_3,\ U_G,\ U_A)$ *and hence*

$$\Phi(H^2(G, A)) \subset H_c^2(i_3,\ U_G,\ U_A) \oplus H_s^2(\pi(G), \pi(A)) \ .$$

We reserve the remainder of § 3 for the argument asserting the converse of Corollary 3.5. We fix for the rest of the section extensions

H_1 and H_2 such that $[H_1] \in H_c^2(i_3, U_G, U_A)$ and $[H_2] \in H_s^2(\pi(G), \pi(A))$. Suppose $H_1 = H_1(F, h, k)$ and $H_2 = H_2(K, r, s)$. Since

$$[H_1] \in H_c^2(i_3, U_G, U_A)$$

we can and do choose a Borel cross section β of k so that the cocycle ν determined by β lies in $Z_c^2(i_3, U_G, U_A)$. Next we choose any cross section α of s carrying the identity into the identity and let

$$\mu \in Z_s^2(\pi(G), \pi(A))$$

be the cocycle determined by α. Then $r\mu(a, b) = \alpha(a)\alpha(b)\alpha(ab)^{-1}$ for a and b in $\pi(G)$ and $h\nu(x, y) = \beta(x)\beta(y)\beta(xy)^{-1}$ for x and y in U_G. We wish to put the extensions H_1 and H_2 together to construct an extension of G by A. Our approach is to construct a homomorphism $p: K \to F$ which embeds K into F as a discrete central subgroup. By taking quotients we then obtain the required extension. The homomorphism p is constructed in considerably more generality than is necessary. This is to allow for certain applications in the § 4.

Let $\operatorname{Hom}(\pi(G), U_A)$ denote the set of all group homomorphisms from $\pi(G)$ into U_A.

THEOREM 3.6. *Let* $\theta \in \operatorname{Hom}(\pi(G), U_A)$. *Define* $p: K \to F$ *by*

$$p(x) = h(i_1 r^{-1}(x(\alpha s(x))^{-1})\theta s(x))\beta i_3 s(x) .$$

Then
(1) $pr = hi_1, kp = i_3 s$,
(2) $p\alpha(a) = h\theta(a)\beta i_3(a)$ *for* $a \in \pi(G)$,
(3) $i_1\mu(a, b) = \nu(i_3(a), i_3(b))$ *for* a *and* b *in* $\pi(G)$,
(4) p *is an injective homomorphism, and*
(5) $p(K)$ *is a central subgroup of* F *if and only if the action of* U_G *on* $\theta(\pi(G))$ *is trivial.*

Proof. (1) and (2) follow by direct computation. A proof of (3) follows easily from (2) and (1) and the fact that each element of $\beta i_3(\pi(G))$ commutes with each element of $h(U_A)$ (see Lemma 3.1). To prove (4) observe by computation (we omit the details), applying Lemma 3.1 when required, that

$$p(xy) = h(i_1\mu(s(x), s(y))\nu(i_3 s(x), i_3 s(y))^{-1})p(x)p(y) .$$

That p is a homomorphism then follows from the above equation and (3). To show that p is injective let x lie in the kernel of p. Then $kp(x) = i_3 s(x) = 1$. Hence since i_3 is injective $s(x) = 1$. Hence for some $a \in \pi(A)$, $r(a) = x$. Then $1 = pr(a) = hi_1(a)$. Since h and i_1 are each injective the result follows. For the proof of (5) we first establish

the following two lemmas.

LEMMA 1. $\beta i_3(\pi(G))$ *lies in the center of* F.

Proof. First let $x \in U_G$ and $a \in \pi(G)$. Then by hypothesis

$$\nu(i_3(a), x) = \nu(x, i_3(a)) .$$

Hence $\beta i_3(a)\beta(x) = \beta(x)\beta i_3(a)$. Now let $t \in F$. Then $t(\beta i(t))^{-1} \in h(U_A)$ and hence by Lemma 3.1 commutes with $\beta i_3(a)$ for any $a \in \pi(G)$. Finally

$$\beta i_3(a)t = \beta i_3(a)(t(\beta k(t))^{-1})\beta k(t) = (t(\beta k(t))^{-1})\beta i_3(a)\beta k(t)$$
$$= (t(\beta k(t))^{-1})(\beta k(t))\beta i_3(a) = t\beta i_3(a) .$$

LEMMA 2. *The action of* U_G *on* U_A *is trivial on* $i_1(\pi(A))$. *Hence* $hi_1(\pi(A))$ *lies in the center of* F.

Proof. First observe that $i_1(\pi(A))$ is invariant with respect to this action. For let $g \in U_G$ and $a \in i_1(\pi(A))$. Then

$$p_1(ga) = p_3(g)p_1(a) = 1 .$$

Next fix any $a \in i_1(\pi(A))$. Then $g \to ga$ is a continuous function of connected set U_G into discrete set $i_1(\pi(A))$. Since a lies in the range of this function the result follows. The second statement then follows.

The proof of (5) now follows from Lemmas 1 and 2 and the fact that the action of U_G on $\theta(\pi(G))$ is trivial if and only if $h\theta(\pi(G))$ lie in the center of F.

We are now prepared to put the extensions H_1 and H_2 together to obtain the required extension. In what follows we choose

$$\theta \in \text{Hom} (\pi(G), U_A)$$

so that U_G acts trivially on $\theta(\pi(G))$. (For example let θ be the trivial homomorphism.) Then by Theorem 3.6 we obtain an injective homomorphism p of K into the center of F such that $pr = hi_1$, and $kp = i_3s$. Let $q: F \to F/p(K)$ be the natural quotient homomorphism. As a quotient space $F/p(K)$ inherits both a topological and a Borel structure from F, and q is both continuous and Borel. Now since

$$hi_1(\pi(A)) \subset p(K) \quad \text{and} \quad kp(K) \subset i_3(\pi(G)) ,$$

p and k induce unique homomorphisms i and j from A to $F/p(K)$ and from $F/p(K)$ to G respectively such that $ip_1 = qh$ and $jq = p_3k$. It is also clear that i and j are continuous and Borel. Thus we obtain a sequence

D: $1 \longrightarrow A \xrightarrow{i} F/p(K) \xrightarrow{j} G \longrightarrow 1$.

We are now prepared to formulate the main result of this section.

THEOREM 3.7. (1) *D is an extension of G by A.*
(2) $\Phi_1([D]) = [H_1]$
(3) $\Phi_2([D]) = [H_2]$.
Hence Φ *maps* $H^2(G, A)$ *onto* $H^2_c(i_3, U_G, U_A) \oplus H^2_s(\pi(G), \pi(A))$.

Proof. By the above we obtain a 3×3 commutative diagram whose top row is H_2, whose middle row is H_1, and whose bottom row is D. It is a simple matter, by chasing around this diagram, to verify that D is exact. Since i and j are continuous to show that D is an extension of G by A it suffices to show that $F/p(K)$ is a locally compact second countable Hausdorff space. This is true if and only if $p(K)$ is a closed subgroup of F. Thus it suffices to show that $F/p(K)$ is an analytic Borel space (see [1, p. 16]). (Note that the action of U_G on U_A will uniquely determine the action of G on A and hence we are guaranteed the correct action automatically). Since each of A and G are analytic Borel spaces it suffices to produce a bijective Borel map of $A \times G$ onto $F/p(K)$ whose inverse is also Borel. Let χ be a Borel cross section of p_3 carrying the identity into the identity. Define $\gamma: G \to F/q(K)$ by $\gamma = q\beta\chi$. Then it is easily verified that γ is a Borel cross section of j carrying the identity into the identity. Now define $\lambda: A \times G \to F/p(K)$ by $\lambda(a, g) = i(a)\gamma(g)$. Then λ is clearly Borel. It is also clear that λ is bijective and in fact

$$\lambda^{-1}(x) = (i^{-1}(x(\gamma j(x))^{-1})^{-1}), j(x)) .$$

Clearly λ^{-1} is also Borel. Hence we conclude that D is an extension of G by A. Next note that since F is simply connected and K is discrete, the sequence

$$1 \longrightarrow K \longrightarrow F \longrightarrow F/p(K) \longrightarrow 1$$

is a universal extension of $F/p(K)$. Finally by the above, making use of Proposition 1.6 we conclude the theorem.

4. **The kernel of** Φ. It is not always the case that Φ has trivial kernel. Let $SL_2(R)$ be the special linear group on R^2. Then it can be shown that $\Phi(H^2(SL_2(R), T) = 0$ where the action is trivial. In fact $H^2(SL_2(R), T) \cong T$. Our task is to compute the kernel of Φ in general. Consider any extension D of G by A such that $\Phi([D]) = 0$. Now apply the techniques of §1 to D to obtain Diagram I. Then the extensions $\tilde{D}$ and $D_{\#}$ split. Hence there exist Borel cross sections

α and β of $j_\#$ and $\tilde{j}$ such that α and β are each homomorphisms. Now for any $x \in \pi(G)$, $i_2\alpha(x)(\beta i_3(x))^{-1}$ lies in the kernel of $\tilde{j}$. Hence define $\theta(x) = i^{-1}(i_2\alpha(x)(\beta i_3(x))^{-1})$. It is clear that $\theta \in \mathrm{Hom}(\pi(G), U_A)$. Now by an argument similar to that used in Lemma 1 within the proof of Theorem 3.6 it is easily seen that $\beta i_3(\pi(G))$ lies in the center of U_E. Also $i_2(\pi(E))$ lies in the center of U_E. Hence the range of $\tilde{i}\theta$ lies in the center of U_E. Hence the action of U_G on the range of θ is trivial.

DEFINITION 4.1. Let $\mathrm{Hom}\,(\pi(G), U_A)_{U_G}$ be the subgroup of $\mathrm{Hom}\,(\pi(G), U_A)$ consisting of all elements θ such that the action of U_G on $\theta(x)$ is trivial for each $x \in \pi(G)$.

DEFINITION 4.2. Let $\theta \in \mathrm{Hom}\,(\pi(G), U_A)_{U_G}$. Define $\rho(\theta) \in$ kernel Φ according to the following construction in the course of which we make use of the notation and results of Theorems 3.6 and 3.7. Let H_1 and H_2 be trivial extensions of U_G by U_A and $\pi(G)$ by $\pi(A)$. Let α and β be splitting homomorphisms associated with s and k. Then by Theorems 3.6 and 3.7 θ determines an extension D such that

$$\Phi_1([D]) = [H_1] = 0 \quad \text{and} \quad \Phi_2([D]) = [H_2] = 0 .$$

Thus $[D] \in$ kernel Φ. Then define $\rho(\theta) = [D]$.

PROPOSITION 4.3. (1) *The definition of $\rho(\theta)$ is independent of the choice of splitting homomorphisms α and β.*
(2) *ρ is a homomorphism of $\mathrm{Hom}\,(\pi(G), U_A)_{U_G}$ onto the kernel of Φ.*

Proof. Let $\chi: G \to U_G$ be a Borel cross section of p_3 carrying the identity into the identity. Let $\eta \in Z^2(G, \pi(G))$ be the cocycle associated with this cross section, i.e., $\eta(x, y) = i_3^{-1}(\chi(x)\chi(y)\chi(xy)^{-1})$ for all x and y in G. We first establish the following lemma.

LEMMA. *Let $\sigma = p_1\theta\eta^{-1}$. Then $\sigma \in Z^2(G, A)$ and in fact σ belongs to the cohomology class determined by $\rho(\theta)$.*

Proof. We continue to use the notation of Theorems 3.6 and 3.7. Define $\gamma: G \to F/p(K)$ by $\gamma = q\beta\chi$. Then γ is a Borel cross section of j carrying the identity into the identity. Now define

$$\sigma(x, y) = i^{-1}(\gamma(x)\gamma(y)\gamma(xy)^{-1}) .$$

Then clearly $\sigma \in Z^2(G, A)$ and belongs to the cohomology class determined by $\rho(\theta)$. We shall show that σ coincides with the expression stated in the lemma. For x and y in G,

$$i\sigma(x, y) = \gamma(x)\gamma(y)\gamma(xy)^{-1} = q\beta i_3 \eta(x, y)$$
$$= q(h\theta(\eta^{-1}(x, y))p\alpha\eta(x, y)) \text{ by Theorem 3.6—(2)}$$
$$= qh\theta\eta^{-1}(x, y) = ip_1\theta\eta^{-1}(x, y) .$$

Hence $\sigma(x, y) = p_1\theta\eta^{-1}(x, y)$.

We now proceed with the proof of the proposition. Define a map $\bar{\rho}$: Hom $(\pi(G),\ U_A)_{U_G} \to Z^2(G, A)$ by $\bar{\rho}(\theta) = \sigma$ where σ is as in the above lemma. Then by the result of the lemma it is easily seen that $\bar{\rho}$ is a homomorphism. Let $\nu: Z^2(G, A) \to H^2(G, A)$ be the natural quotient map. Then the composition $\nu\bar{\rho}$ is a homomorphism and is identical to ρ. Also by the equation of the lemma it is clear that ρ is independent of α and β. Finally the proof that ρ is onto follows easily from the introductory remarks of this section. For suppose $[D] \in$ kernel Φ. Repeating the arguments at the beginning of this section we obtain trivial extensions $\tilde{D}$ and $D_\sharp$ and splitting homomorphisms α and β, and $\theta \in$ Hom $(\pi(G),\ U_A)_{U_G}$ such that $\tilde{i}\theta(x)\beta i_3(x) = i_2\alpha(x)$ for all $x \in \pi(G)$. Next it is easily seen that if we construct the function $p: \pi(E) \to U_E$ using α, β and θ and the definition of p given in Theorem 3.6 that $p = i_2$. Hence $\rho(\theta) = [D]$. This shows that ρ is onto.

Hom $(\pi(G),\ U_A)_{U_G}$ is a vector group and hence is divisible. The homomorphic image of a divisable group is divisible and hence the kernel of Φ is divisible. Thus we obtain

COROLLARY 4.4. $H^2(G, A) \cong H^2_c(i_3,\ U_G,\ U_A) \oplus H^2_s(\pi(G),\ \pi(A)) \oplus$ kernel Φ.

We now compute the kernel of ρ. Consider the extensions V_1 and V_3 in Diagram 1. Applying [10] we obtain a commutative diagram with rows and columns exact, Diagram 4.

$$\text{Hom } (\pi(G),\ \pi(A))^G$$
$$\downarrow i_{1*}$$

$$H^1(U_G,\ U_A) \xrightarrow{\text{res} = i_{3*}} \text{Hom } (\pi(G),\ U_A)^G \xrightarrow{d_2} H^2(G,\ U_A)$$
$$\downarrow p_{1*} \qquad\qquad\qquad \downarrow p_{1*} \qquad\qquad\qquad \downarrow p_{1*}$$
$$H^1(U_G,\ A) \xrightarrow{\text{res} = i_{3*}} \text{Hom } (\pi(G),\ A)^G \xrightarrow{d_2} H^2(G,\ A)$$
$$\downarrow \partial_1$$
$$H^2(U_G,\ \pi(A)) = 0$$

DIAGRAM 4.

Since $\pi(G)$ acts trivially on $\pi(A)$, U_A, and A the functor H^1 reduces to Hom in the appropriate column. Next observe that since U_G is simply connected and $\pi(A)$ is discrete, $H^2(U_G, \pi(A)) = 0$. Now

let χ be as before, a Borel cross section of p_3 in extension V_3, such that χ carries the identity into the identity. Let $\eta \in Z^2(G, \pi(G))$ be the cocyle determined by χ. Then as a 2-cocycle η determines the extension V_3. Then by [10, p. 53] we have an explicit representation of $d_2 \colon \mathrm{Hom}\,(\pi(G),\, U_A)^G \to H^2(G,\, U_A)$ in terms of cocycles. For

$$\theta \in \mathrm{Hom}\,(\pi(G),\, U_A)^G$$

and x and y in G, the cocycle taking (x, y) into $\theta(\eta(x, y)^{-1})$ belongs to the cohomology class $d_2(\theta)$. Consequently, by the lemma in the proof of Proposition 4.3 we have immediately (referring to Diagram 4)

PROPOSITION 4.5. $\rho = p_{1*}d_2 = d_2 p_{1*}$ on $\mathrm{Hom}\,(\pi(G),\, U_A)_{U_G}$.

We now establish a few more preliminaries before obtaining our final result.

PROPOSITION 4.6.   *Referring to Diagram* 4.

$$\mathrm{Hom}\,(\pi(G),\, U_A)^G = \mathrm{Hom}\,(\pi(G),\, U_A)_{U_G} \ .$$

Proof. First note that since $\pi(G)$ acts trivially on U_A, the action of U_G on U_A determines an action of G on U_A. For $\theta \in \mathrm{Hom}\,(\pi(G),\, U_A)$ and $g \in G$ g acts on θ in the following manner. For $a \in \pi(G)$ $(g\theta)(a) = g(\theta(g^{-1}a))$. But $\pi(G)$ is central in U_G and hence the action of G on $\pi(G)$ is trivial. Hence $(g\theta)(a) = g(\theta(a))$. Hence $g\theta = \theta$ for all $g \in G$ if and only if x acts trivially on the range of θ for all $x \in U_G$. The result follows.

THEOREM 4.7.   *The kernel of* ρ *is*

$$i_{1*}(\mathrm{Hom}\,(\pi(G),\, \pi(A))) \oplus i_3^*(H^1(U_G,\, U_A)) \ .$$

Hence

$$\mathrm{kernel}\ \Phi \cong \frac{\mathrm{Hom}\,(\pi(G),\, U_A)^G}{i_{1*}(\mathrm{Hom}\,(\pi(G),\, \pi(A))) \oplus i_3^*(H^1(U_G,\, U_A))}$$

Proof. First note that since U_G acts trivially on $\pi(A)$ so does G. Hence since G acts trivially on $\pi(G)$,

$$\mathrm{Hom}\,(\pi(G),\, \pi(A))^G = \mathrm{Hom}\,(\pi(G),\, \pi(A)) \ .$$

Hence by exactness and commutativity of Diagram 4 and by Lemma 4.6 it is clear that the kernel of ρ contains the sum. Now let $a \in$ kernel ρ. Then by Proposition 4.5 $d_2 p_{1*}(a) = 0$. Then by a suitable argument involving chasing around Diagram 4 it is a simple matter

to show that a lies in this sum. This completes the proof.

5. **Special cases.** We now proceed to compute $H^2(G, A)$ in the light of the above structure theory under certain circumstances.

PROPOSITION 5.1. *If G is compact or if $\pi(G)$ is a torsion group the kernel of Φ is 0.*

Proof. If G is compact $H^2(G, U_A) = 0$, since U_A is a vector group (see Moore [10]). Hence by Proposition 4.5, ρ is trivial. Since ρ is surjective onto the kernel of Φ the result is assured.

If $\pi(G)$ is torsion then $\text{Hom}\,(\pi(G), U_A) = 0$, since U_A is a vector group. Hence again by Proposition 4.5 ρ is trivial and the result follows.

The following generalizes Theorem A of [10].

PROPOSITION 5.2. *If G is simply connected then*

$$p_{1*}: H^2(G, U_A) \longrightarrow H^2(G, A)$$

is an isomorphism.

Proof. By Proposition 5.1 the kernel of Φ is 0. Clearly

$$H_s^2(\pi(G), \pi(A)) = 0 .$$

Since i_3 is trivial, $H_c^2(i_3, U_G, U_A) = H^2(U_G, U_A)$. Hence Φ is an isomorphism from $H^2(G, A)$ onto $H^2(U_G, U_A)$. Now since G is simply connected p_3 is an isomorphism, and hence $p_3^*: H^2(G, U_A) \to H^2(U_G, U_A)$ is an isomorphism. Hence the composite $p_3^{*-1}\Phi$ is an isomorphism. We show that the inverse of this isomorphism is p_{1*}. Let $D(E, i, j)$ be any extension of G by A. Then $\Phi([D]) = [\tilde{D}]$ (using the usual notation of Diagram 1). Let $\beta: U_G \to U_E$ be a Borel cross section of $\tilde{j}$ carrying the identity into the identity. Then define $\gamma: G \to E$ by $\gamma = p_2\beta p_3^{-1}$. Then γ is a Borel cross section of j carrying the identity into the identity. Define σ and ν by the equations

$$\sigma(x, y) = i^{-1}(\gamma(x)\gamma(y)\gamma(xy)^{-1})$$
$$\nu(x, y) = i^{-1}(\beta(x)\beta(y)\beta(xy)^{-1}) .$$

Then σ and ν are the cocycles which determine the extensions D and $\tilde{D}$ respectively. It is easily verified that $\sigma(p_3(x), p_3(y)) = p_1\nu(x, y)$ for all x and y in U_G. Hence on the cohomology level, $p_3^*([\sigma]) = p_{1*}([\nu])$. But $[\nu] = \Phi([\sigma])$. Hence $p_{1*}\Phi = p_3^*$. Hence we obtain Diagram 5, with lower right part commutative, as is the entire diagram excluding Φ.

$$H^2(G,\ U_A) \xrightarrow{\quad p_{1*} \quad} H^2(G,\ A)$$
$$p_3^*\Big\downarrow \qquad\qquad \overset{\Phi}{\swarrow} \qquad\qquad p_3^*\Big\downarrow$$
$$H^2(U_G,\ U_A) \xrightarrow{\quad p_{1*} \quad} H^2(U_G,\ A)\ .$$

<div align="center">DIAGRAM 5.</div>

Now let $\lambda \in H^2(G,\ U_A)$. Then $p_{1*}\Phi p_{1*}(\lambda) = p_3^* p_{1*}(\lambda) = p_{1*}p_3^*(\lambda)$. Hence $\Phi p_{1*}(\lambda) - p_3^*(\lambda)$ lies in the kernel of $p_{1*} : H^2(U_G,\ U_A) \to H^2(U_G,\ A)$. Since U_G is simply connected and $\pi(A)$ is discrete, $H^2(U_G,\ \pi(A)) = 0$ and hence by [10] p_{1*} has trivial kernel. Hence the upper left-hand part of Diagram 5 is commutative and result follows.

PROPOSITION 5.3. *If G is compact $H_c^2(i_3,\ U_G,\ U_A) = 0$ and hence $\Phi: H^2(G,\ A) \cong H_s^2(\pi(G),\ \pi(A))$.*

Proof. Since G is compact and $H^1(\pi(G),\ U_A)$ is a vector group, $H^1(G,\ H^1(\pi(G),\ U_A)) = 0$ by [10]. Hence again by [10, p. 47] the sequence

$$H^2(G,\ U_A) \xrightarrow{\ p_{3*}\ } H^2(U_G,\ U_A) \xrightarrow{\ i_{3*}\ } H^2(\pi(G),\ U_A)^G$$

is exact. Again by [10] since G is compact $H^2(G,\ U_A) = 0$. Hence i_3^* is injective. In particular the subgroup $H_c^2(i_3,\ U_G,\ U_A)$ injects by i_3^* into $H^2(\pi(G),\ U_A)$. However observe that the image under i_3^* of an element of $H_c^2(i_3,\ U_G,\ U_A)$ actually lies in $H_s^2(\pi(G),\ U_A)$, the family of classes of symmetric extensions of $\pi(G)$ by U_A. This is so since the action of $\pi(G)$ on U_A is trivial. But U_A is divisable and hence any abelian extension of $\pi(G)$ by U_A splits. Since $\pi(G)$ is discrete any such extension will automatically be topological. Hence $H_s^2(\pi(G),\ U_A) = 0$. We conclude that $H_c^2(i_3,\ U_G,\ U_A) = 0$. The result now follows from Proposition 5.1.

We now further investigate the relationship between $H_c'(i_3,\ U_G,\ U_A)$ and $H^2(U_G,\ U_A)$.

DEFINITION 5.4. Fix $\nu \in Z^2(U_G,\ U_A)$. For $a \in \pi(G)$ and $x \in U_G$ define $L(\nu)_a(x) = \nu(i_3(a),\ x)\nu(x,\ i_3(a))^{-1}$.

THEOREM 5.5. (1) *For each $a \in \pi(G)$ and $\nu \in Z^2(U_G,\ U_A)$, $L(\nu)_a \in Z^1(U_G,\ U_A)$.*
  (2) *Define $\bar{L}(\nu)_a$ to be the cohomology class to which $L(\nu)_a$ belongs. Then $\bar{L}(\nu)$ is a homomorphism of $\pi(G)$ into $H^1(U_G,\ U_A)$.*
  (3) *$\bar{L}$ is a homomorphism from $Z^2(U_G,\ U_A)$ into*

$$\mathrm{Hom}\,(\pi(G),\ H^1(U_G,\ U_A))\ ,$$

and $B^2(U_G, U_A)$ lies in the kernel of $\bar{L}$.

(4) *Hence $\bar{L}$ determines a homomorphism T of $H^2(U_G, U_A)$ into* Hom $(\pi(G), H^1(U_G, U_A))$.

(5) *The kernel of T is precisely $H^2_c(i_3, U_G, U_A)$.*

Proof. We omit most of the computational details necessary to establish the theorem. The reader is referred to the identity (2) of p. 171 of G. W. Mackey [7] and the fact that $\pi(G)$ acts trivially on U_A. From these facts the following identities can easily be derived: For $\nu \in Z^2(U_G, U_A)$, $a \in \pi(G)$ and x and y in U_G:

(1) $L(\nu)_a(xy) = (L(\nu)_a(x))(xL(\nu)_a(y))$

(2) $L(\nu)_{ab}(x) = L(\nu)_a(x)L(\nu)_b(x)(x\nu(i_3(a), i_3(b)))\nu(i_3(a), i_3(b))^{-1}$.

Part 1 then follows immediately from equation (1) above. Since the mapping taking $x \in U_G$ into $x\nu(i_3(a), i_3(b))\nu(i_3(a), i_3(b))^{-1}$ belongs to $B^1(U_G, U_A)$ it easily follows from identity (2) above that $\bar{L}(\nu)$ is a homomorphism. This proves 2. It is a trivial matter that $\bar{L}$ is a homomorphism. Next let θ be any Borel mapping from U_G to U_A carrying the identity into the identity. Then θ determines a coboundary η_0 satisfying equation (3) of [7, p. 172]. Then for $a \in \pi(G)$ and $x \in U_G$ it is easily seen that $L(\eta_0)_a(x) = \theta(i_3(a))(x\theta(i_3(a)))^{-1}$ and hence $L(\eta_0)_a$ lies in $B^1(U_G, U_A)$. Hence $\bar{L}(\eta_0) = 0$. This proves 3. The proof of 4 is now immediate, T being the obvious induced mapping on the quotient.

We now establish 5. Let $\lambda \in H^2_c(i_3, U_G, U_A)$. Then there exists an element $\nu \in Z^2_c(i_3, U_G, U_A)$ such that $\nu \in \lambda$. Hence

$$\nu(i_3(a), x)\nu(x, i_3(a))^{-1} = 1$$

for all $a \in \pi(G)$ and $x \in U_G$. Hence $T(\lambda) = 0$. Conversely suppose $\lambda \in$ kernel T. Choose a cocycle ν in λ. Then $L(\nu): \pi(G) \to B^1(U_G, U_A)$. Hence for each $a \in \pi(G)$ there exists a $t_a \in U_A$ such that

$$L(\nu)_a(x) = (xt_a)t_a^{-1}$$

for all $x \in U_G$. Thus $a \to t_a$ determines a map from $\pi(G)$ into U_G, and it can be taken so that $t_1 = 1$. This map can then be 'extended' to a Borel function ϕ from U_G into U_A satisfying the condition that $\phi(i_3(a)) = t_a$ for all $a \in \pi(G)$. Then ϕ determines a coboundary $\delta\phi \in B^2(U_G, U_A)$. Let $\nu' = \nu + \delta\phi \in Z^2(U_G, U_A)$. Then

$$L(\nu')_a(x) = L(\nu)_a(x)\phi(i_3(a))(x\phi(i_3(a)))^{-1} = xt_a(t_a)^{-1}t_a(xt_a)^{-1} = 1 .$$

Hence $\nu'(i_3(a), x) = \nu(x, i_3(a))$. Hence $\nu' \in Z^2_c(i_3, U_G, U_A)$. Hence

$$[\nu'] = [\nu] = \lambda \in H^2_c(i_3, U_G, U_A) .$$

COROLLARY 5.6. *If $\pi(G)$ is a torsion group then $H^2_c(i_3, U_G, U_A) =$*

$H^2(U_G, U_A)$.

DEFINITION 5.7. An abelian group H is *almost torsion* if the quotient group $H/$torsion H is cyclic.

LEMMA 5.8. *If* $\pi(G)$ *is almost torsion then* $L(\nu)_a(i_3(b)) = 1$ *for all* $\nu \in Z^2(U_G, U_A)$ *and* a *and* b *in* $\pi(G)$.

Proof. Fix a $\nu \in Z^2(U_G, U_A)$ and a and b in $\pi(G)$. Then there exist elements r, s, and t in $\pi(G)$ such that s and t are torsion elements and there exist integers m and n such that $a = r^m s$ and $b = r^n t$. Then $\bar{L}(\nu)_a = \bar{L}(\nu)_{r^m s} = m\bar{L}(\nu)_r + \bar{L}(\nu)_s = m\bar{L}(\nu)_r$ since s is torsion and $H^1(U_G, U_A)$ is torsion free.

Now i_3 induces a map on the cohomology level $i_3^*: H^1(U_G, U_A) \rightarrow H^1(\pi(G), U_A)$. Now since $\pi(G)$ acts trivially on U_A, $B^1(\pi(G), U_A) = 0$ and $Z^1(\pi(G), U_A)$ is $\text{Hom}(\pi(G), U_A)$, the group of homomorphisms. Consequently the composition $L(\nu)_r i_3$ is a homomorphism. Hence

$$L(\nu)_a(i_3(b)) = (mL(\nu)_r)(i_3(b)) = (mL(\nu)_r)(i_3(r^n t))$$
$$= (mL(\nu)_r)(i_3(r^n)) = (mnL(\nu)_r)(i_3(r)) .$$

But $(mnL(\nu)_r)i_3(r) = (\nu(i_3(r), i_3(r))\nu(i_3(r), i_3(r))^{-1})^{mn} = 1$. Hence the result.

THEOREM 5.9. *If* $\pi(G)$ *is almost torsion and* G *is compact* $T = 0$ *and hence* $H_c^2(i_3, U_G, U_A) = H^2(U_G, U_A) = 0$.

Proof. Fix $\nu \in Z^2(U_G, U_A)$. Then by Lemma 5.8, $i_3^* \bar{L}(\nu)$ is trivial. Now by [10] the sequence

$$0 \longrightarrow H^1(G, U_A) \xrightarrow{p_{3*}} H^1(U_G, U_A) \xrightarrow{i_{3*}} H^1(\pi(G), U_A)^G$$

is exact and hence $\bar{L}(\nu)$ lies in the image of p_3^*. But since G is compact and U_A is a vector group $H^1(G, U_A) = 0$ by [10]. Hence $\bar{L}(\nu)$ is trivial. We conclude that T is trivial and hence the result follows.

The following theorem which we state for central extensions does not seem to generalize for arbitrary topological extensions.

THEOREM 5.10. *Suppose the action of* G *on* A *is trivial. Then if* G *is semisimple*

$$H_c^2(i_3, U_G, U_A) = H^2(U_G, U_A) .$$

Proof. Let $\nu \in Z^2(U_G, U_A)$. Then since the action of U_G on U_A is trivial $L(\nu)_a$ is a homomorphism of U_G into U_A for each $a \in \pi(G)$. Now since G is semisimple so is U_G. Hence $L(\nu)_a$ is trivial for each $\nu \in Z^2(U_G, U_A)$ and for each $a \in \pi(G)$. Hence T is trivial and the result follows.

Recall by Proposition 5.3 that if G is compact

$$\Phi: H^2(G, A) \cong H_s^2(\pi(G), \pi(A)) .$$

Now $\pi(A)$ is a finite product of copies of the integers, Z and hence $H_s^2(\pi(G), \pi(A))$ is a finite product of copies of $H_s^2(\pi(G), Z)$.

PROPOSITION 5.11. *If $\pi(G)$ is finitely generated then*

$$H^2(\pi(G), Z) = H_s^2(\pi(G), Z) \cong (torsion\ \pi(G))^\frown ,$$

the Pontryagin dual of the torsion subgroup of $\pi(G)$.

Proof. Since $\pi(G)$ is finitely generated $\pi(G) \cong F \oplus K$ where F is free and K is the torsion part of $\pi(G)$. Then by [8] $H^2(F, Z) = 0$ and $H^2(K, Z) \cong \hat{K}$. From the exactness of the sequence,

$$1 \longrightarrow F \overset{i}{\longrightarrow} \pi(G) \overset{j}{\longrightarrow} K \longrightarrow 1$$

we obtain, assuming that all group actions are trivial,

$$\longrightarrow \operatorname{Hom}(\pi(G), Z) \overset{i^*}{\longrightarrow} \operatorname{Hom}(F, Z) \overset{d_2}{\longrightarrow} H^2(K, Z)$$
$$\overset{j^*}{\longrightarrow} H^2(\pi(G), Z) \longrightarrow 0 .$$

Next note that since K is torsion i^* is an isomorphism. Thus we obtain the exactness of the sequence

$$0 \overset{d_2}{\longrightarrow} H^2(K, Z) \overset{j^*}{\longrightarrow} H^2(\pi(G), Z) \longrightarrow 0 .$$

Hence j^* is an isomorphism and we conclude that $H^2(\pi(\pi(G), Z) \cong \hat{K}$.

Next observe that $H^2(K, Z) = H_s^2(K, Z)$. For let

$$1 \longrightarrow Z \overset{i}{\longrightarrow} B \overset{j}{\longrightarrow} K \longrightarrow 1$$

be any central extension K by Z. Let $\sigma: K \times K \to Z$ be any cocycle associated with this extension. Fix $a \in K$. Then define

$$L_a(x) = \sigma(a, x)\sigma(x, a)^{-1}$$

for $x \in K$. Then $L_a: K \to Z$ is a homomorphism. Hence since K is torsion and Z is free, L_a is trivial for all $a \in K$. Hence σ is symmetric and therefore $H^2(K, Z) = H_s^2(K, Z)$.

Finally j^* maps symmetric cohomology classes onto symmetric classes and hence since j^* is an isomorphism,

$$H^2(\pi(G), Z) = H_s^2(\pi(G), Z) \ .$$

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