

# Pacific Journal of Mathematics

## **INVERTING OPERATORS FOR SINGULAR BOUNDARY VALUE PROBLEMS**

CAULTON LEE IRWIN

## INVERTING OPERATORS FOR SINGULAR BOUNDARY VALUE PROBLEMS

C. L. IRWIN

Let  $S$  denote a Banach Space,  $B$  the bounded linear transformations on  $S$ , and let  $Q$  and  $A$  denote functions from  $[0, \infty)$  into  $B$  with  $Q$  continuous. The objective here is to derive a Green's function  $K_A$  and hence an integral inverting operator  $R_A$  for the singular boundary value problem

$$(1) \quad \begin{cases} Y' - QY = H \\ A(0)Y(0) + \lim_{n \rightarrow \infty} A(c_n)Y(c_n) = 0, \end{cases}$$

where  $\{c_n\}_{n=1}^\infty$  is a positive, increasing, unbounded number sequence and  $H$  is a continuous function from  $[0, \infty)$  into  $S$ .

The method here provides Green's functions for singular boundary value problems associated with nonself-adjoint, as well as self-adjoint, linear differential expressions. The asymptotic boundary conditions in (1) permit one to extend some of the regular two-point boundary value problem techniques suggested by [3] and [4] to the singular case without being restricted to the Hilbert Space  $L_2[0, \infty)$ . Similar, but different, asymptotic boundary conditions are used by Codrington and Levinson in [2, Chapter 10], and by Benzinger [1].

As noted in § 3 of [3] there exists a unique continuous function  $M$  from  $[0, \infty) \times [0, \infty)$  to  $B$  so that if each of  $x$ ,  $t$ , and  $u$  is in  $[0, \infty)$ ,

(i)  $M_1(x, t) = Q(x)M(x, t)$  and  $M(t, t) = I$

(ii)  $M(x, t)M(t, u) = M(x, u)$

(iii) if  $H$  is a continuous function from  $[0, \infty)$  to  $S$  and  $\alpha$  is in  $S$ , then the only function  $Y$  such that  $Y' - QY = H$  and  $Y(0) = \alpha$  is given by

$$Y(x) = M(x, 0)\alpha + \int_0^x dt M(x, t)H(t)$$

for all  $x$  in  $[0, \infty)$ .

**DEFINITION.**  $A$  is a determinate boundary condition function for  $Q$  on  $c_1, c_2, \dots$  means that if  $H$  is a continuous function on  $[0, \infty)$  and  $Y$  is a solution of the boundary value problem (1) for the non-homogeneous term  $H$ , then  $Y$  is unique.

**NOTATION.** If  $A$  is a boundary condition function for  $Q$  on  $c_1, c_2, \dots$  and  $n$  is a positive integer, let  $T_n$  denote the transformation

$$[A(0) + A(c_n)M(c_n, 0)].$$

**THEOREM 1.** *A is a determinate boundary condition function for Q on  $c_1, c_2, \dots$  if and only if the convergence of  $\{T_n\alpha\}_{n=1}^\infty$  to zero implies that  $\alpha$  is the zero of S.*

*Proof.* The proof follows from property (iii) of the M function and the linearity of the problem.

**NOTATION.** Let  $D_0$  denote the continuous functions with compact support on  $[0, \infty)$ .

**THEOREM 2.** *Suppose A is a determinate boundary condition function for Q on  $c_1, c_2, \dots$ ; the following two statements are equivalent.*

(i) *There is an integral inverting operator  $R_A$  with kernel  $K_A$  of the form*

$$K_A(x, t) = \begin{cases} M(x, 0)K_A(0, 0)M(0, t) & \text{if } 0 \leq t \leq x \\ M(x, 0)[K_A(0, 0) - I]M(0, t) & \text{if } 0 \leq x < t \end{cases}$$

*for boundary value problem (1) so that  $D_0$  is a subset of the domain of  $R_A$ .*

(ii) *There is a transformation  $\pi$  in B such that if  $\alpha$  is in S, then  $\{T_n(\pi\alpha)\}_{n=1}^\infty$  converges to  $A(0)\alpha$ .*

*Proof.* Assume (i) holds; if  $H$  is in  $D_0$  and  $U = R_A H$ , then  $U$  is a solution of boundary value problem (1) and so

$$\lim_{n \rightarrow \infty} [A(0)U(0) + A(c_n)U(c_n)] = 0.$$

Let  $b$  denote a positive number so that if  $x > b$ ,  $H(x) = 0$ ; then, if  $c_n > b$ ,

$$\begin{aligned} A(0)U(0) + A(c_n)U(c_n) &= A(0) \int_0^b dt [K_A(0, 0) - I]M(0, t)H(t) \\ &\quad + A(c_n) \int_0^b dt M(c_n, 0)K_A(0, 0)M(0, t)H(t) \\ &= T_n \left( K_A(0, 0) \int_0^b dt M(0, t)H(t) \right) \\ &\quad - A(0) \int_0^b dt M(0, t)H(t) \end{aligned}$$

and so

$$\lim_{n \rightarrow \infty} T_n \left( K_A(0, 0) \int_0^b dt M(0, t)H(t) \right) = A(0) \int_0^b dt M(0, t)H(t) .$$

Now, if  $\alpha$  is in  $S$ , define  $H$  as

$$H(t) = \begin{cases} M(t, 0)(2 - 2t)\alpha & \text{if } 0 \leq t \leq 1 \\ 0 & \text{if } t > 1 . \end{cases}$$

$H$  belongs to  $D_0$  and  $\int_0^b dt M(0, t)H(t) = \alpha$ , so (ii) holds with  $\pi = K_A(0, 0)$ .

Now, assume (ii) holds; since  $A$  is a determinate boundary condition function for  $Q$  on  $c_1, c_2, \dots$ ,  $\pi$  must be unique. Define  $K_A$  on  $[0, \infty) \times [0, \infty)$  as

$$(2) \quad K_A(x, t) = \begin{cases} M(x, 0)\pi M(0, t) & \text{if } 0 \leq t \leq x \\ M(x, 0)[\pi - I]M(0, t) & \text{if } 0 \leq x < t \end{cases}$$

and let  $R_A$  denote the integral operator with kernel  $K_A$ . Let  $H$  be in  $D_0$  and  $b$  denote a positive number so that if  $x > b$ ,  $H(x) = 0$ . Define  $U$  on  $[0, \infty)$  as

$$U(x) = \begin{cases} \int_0^x dt M(x, 0)\pi M(0, t)H(t) \\ \quad + \int_x^b dt M(x, 0)[\pi - I]M(0, t)H(t) & \text{if } 0 \leq x \leq b \\ \int_0^b dt M(x, 0)\pi M(0, t)H(t) & \text{if } x > b . \end{cases}$$

Differentiation yields that  $U'(x) - Q(x)U(x) = H(x)$  for each  $x$  in  $[0, \infty)$  and if  $c_n > b$ ,

$$\begin{aligned} & A(0)U(0) + A(c_n)U(c_n) \\ &= T_n \left( \pi \int_0^b dt M(0, t)H(t) \right) - A(0) \int_0^b dt M(0, t)H(t) . \end{aligned}$$

By the definition of  $\pi$ ,  $\lim_{n \rightarrow \infty} [A(0)U(0) + A(c_n)U(c_n)] = 0$  and so (i) holds.

For the remainder of the paper suppose that  $A$  is a determinate boundary condition function for  $Q$  on  $c_1, c_2, \dots$ , condition (ii) in Theorem 2 holds and  $K_A$  is defined on  $[0, \infty) \times [0, \infty)$  by (2). (Condition (ii) is implied, for example, in case the sequence  $\{T_n\}_{n=1}^\infty$  converges in norm to a regular element of  $B$ .) Let  $D$  denote the set of continuous functions  $H$  on  $[0, \infty)$  such that  $\int_0^\infty dt K_A(x, t)H(t)$  exists for each  $x$  and furthermore, if  $U$  is defined as

$$U(x) = \int_0^\infty dt K_A(x, t)H(t) \quad \text{for } x \text{ in } [0, \infty) ,$$

then  $U$  is a solution of boundary value problem (1) for the nonhomo-

geneous term  $H$ . Let  $R_A$  denote the integral operator with kernel  $K_A$  and domain  $D$ ; i.e., if  $H$  belongs to  $D$

$$(R_A H)(x) = \int_0^\infty dt K_A(x, t)H(t) .$$

Two aspects of the present development which differ from other treatments of Green's functions for singular boundary value problems are: (1) the Green's functions here are not necessarily square integrable in either place and (2) the domains of the associated integral inverting operators are not restricted to functions which are square integrable on  $[0, \infty)$ . However, the domain of  $R_A$  does depend upon the problem, i.e., upon the particular  $Q$  and  $A$  involved. This dependence is the subject of the following two theorems.

Two sets of continuous function on  $[0, \infty)$  which are relevant to the description of  $D$  are defined as follows. Let  $D_1$  denote the collection of continuous functions  $H$  on  $[0, \infty)$  for which there exists a solution of (1) for the nonhomogeneous term  $H$ . Let  $D_2$  denote the collection of continuous functions  $H$  on  $[0, \infty)$  such that  $\int_0^\infty dt(\pi - I)M(0, t)H(t)$  exists.

It is clear that  $D$  is a subset of the intersection of  $D_1$  and  $D_2$ ; not so obvious is the extent to which  $D_1 \cap D_2$  is contained in  $D$ .

LEMMA. Suppose  $H$  belongs to  $D_1 \cap D_2$ ; let  $Y$  denote the solution of (1) for  $H$  and let  $X(x) = \int_0^\infty dt K_A(x, t)H(t)$  for all  $x$  in  $[0, \infty)$ , then

$$\begin{aligned} T_n [Y(0) - X(0)] &= [A(0)Y(0) + A(c_n)Y(c_n)] \\ &\quad + [A(0) - T_n\pi] \int_0^{c_n} dt M(0, t)H(t) \\ &\quad - T_n \int_{c_n}^\infty dt(\pi - I)M(0, t)H(t) \end{aligned}$$

for each positive integer  $n$ .

*Proof.* Let  $n$  denote a positive integer; property (iii) of the  $M$  function provides that

$$Y(c_n) = M(c_n, 0)Y(0) + M(c_n, 0) \int_0^{c_n} dt M(0, t)H(t) ,$$

so

$$\begin{aligned} T_n Y(0) &= A(0)Y(0) + A(c_n)M(c_n, 0)Y(0) \\ &= A(0)Y(0) + A(c_n)Y(c_n) - A(c_n)M(c_n, 0) \int_0^{c_n} dt M(0, t)H(t) . \end{aligned}$$

Also,

$$T_n X(0) = [A(0) + A(c_n)M(c_n, 0)] \int_0^\infty dt(\pi - I)M(0, t)H(t).$$

A straightforward computation provides the result of the lemma.

The domain  $D$  of the inverting operator  $R_A$  may be studied for the following three cases.

Case 1. There is an increasing sequence of positive integers  $n_1, n_2, \dots$  such that  $T_{n_i}^{-1}$  exists for all  $i$  and the transformation sequence  $\{T_{n_i}^{-1}\}_{i=1}^\infty$  is uniformly norm bounded.

Case 2. There is an increasing sequence of positive integers  $n_1, n_2, \dots$  such that  $T_{n_i}^{-1}$  exists for all  $i$ , but no subsequence of inverses is uniformly norm bounded.

Case 3. There is a positive integer  $N$  such that if  $n > N$ , then  $T_n^{-1}$  does not exist.

*Note.* Case 1 above is a sufficient condition for a function  $A$  from  $[0, \infty)$  into  $B$  to be a determinate boundary condition function for  $Q$  on  $c_1, c_2, \dots$ .

**THEOREM 3.** *Suppose Case 1 above holds; if  $H$  is in  $D_1 \cap D_2$  then  $H$  is in  $D$  if*

$$\lim_{i \rightarrow \infty} [A(0) - T_{n_i} \pi] \int_0^{c_{n_i}} dt M(0, t)H(t) = 0.$$

*Proof.* By the lemma and existence of  $T_{n_i}^{-1}$  for all  $i$ , we obtain in the notation of the lemma that

$$\begin{aligned} Y(0) - X(0) &= T_{n_i}^{-1} [A(0)Y(0) + A(c_{n_i})Y(c_{n_i})] \\ &\quad + T_{n_i}^{-1} [A(0) - T_{n_i} \pi] \int_0^{c_{n_i}} dt M(0, t)H(t) \\ &\quad - \int_{c_{n_i}}^\infty dt(\pi - I)M(0, t)H(t) \quad \text{for each } i. \end{aligned}$$

$H$  in  $D_2$  provides that  $\lim_{i \rightarrow \infty} \int_{c_{n_i}}^\infty dt(\pi - I)M(0, t)H(t) = 0$  and  $Y$  satisfies the asymptotic boundary condition so

$$Y(0) - X(0) = \lim_{i \rightarrow \infty} T_{n_i}^{-1} [A(0) - T_{n_i} \pi] \int_0^{c_{n_i}} dt M(0, t)H(t).$$

Now, if  $\lim_{i \rightarrow \infty} [A(0) - T_{n_i} \pi] \int_0^{c_{n_i}} dt M(0, t)H(t) = 0$ , then  $Y(0) - X(0) = 0$

and so  $Y = X$ , i.e.,  $X$  is the unique solution of (1) for  $H$  and so  $H$  is in the domain of  $R_A$ .

A subcase of Cases 1, 2, and 3 above is that the transformation sequence  $\{T_n\}_{n=1}^\infty$  be uniformly norm bounded, which occurs, for example, with  $S = E_2$  and  $T_n = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$   $n$  odd,  $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$  if  $n$  is even.

**THEOREM 4.** *Suppose  $\{T_n\}_{n=1}^\infty$  is uniformly norm bounded; if  $H$  is in  $D_1 \cap D_2$ , then  $H$  is in  $D$  if and only if*

$$\lim_{n \rightarrow \infty} [A(0) - T_n \pi] \int_0^{c_n} dt M(0, t) H(t) = 0.$$

*Proof.* Let  $H$  denote a function in  $D_1 \cap D_2$ ; by the lemma

$$\begin{aligned} T_n [Y(0) - X(0)] &= [A(0)Y(0) + A(c_n)Y(c_n)] \\ &\quad + [A(0) - T_n \pi] \int_0^c dt M(0, t) H(t) \\ &\quad - T_n \int_{c_n}^\infty dt (\pi - I) M(0, t) H(t) \end{aligned}$$

for each positive integer  $n$ . Where  $Y$  denotes the solution of (1) for  $H$  and  $X$  is defined on  $(0, \infty)$  by

$$X(x) = \int_0^\infty dt K_A(x, t) H(t) \quad x \text{ in } [0, \infty).$$

$Y$  satisfies the asymptotic boundary condition so

$$A(0)Y(0) + \lim_{n \rightarrow \infty} A(c_n)Y(c_n) = 0.$$

The transformation sequence  $\{T_n\}_{n=1}^\infty$  is uniformly norm bounded and  $H$  is in  $D_2$  so

$$\lim_{n \rightarrow \infty} T_n \int_{c_n}^\infty dt (\pi - I) M(0, t) H(t) = 0.$$

So

$$\lim_{n \rightarrow \infty} T_n [Y(0) - X(0)] = \lim_{n \rightarrow \infty} [A(0) - T_n \pi] \int_0^{c_n} dt M(0, t) H(t).$$

The result of the theorem follows from  $A$  being a determinate boundary condition function.

The following example illustrates the subcase for a Case 1 problem and shows that the domain of  $R_A$  may be a proper subset of  $D_1 \cap D_2$ .

EXAMPLE. Let  $c_1, c_2, \dots$  denote a positive, increasing, unbounded number sequence. Consider the singular boundary value problem associated with the differential expression  $Ly = y''$  and the boundary condition function  $A$  defined as

$$A(x) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} & \text{if } x = 0, \\ \begin{bmatrix} -1/[1 + \log(1 + x)] & x/[1 + \log(1 + x)] \\ 0 & -1 \end{bmatrix} & \text{if } x > 0, \end{cases}$$

We have  $Q(x) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  for  $x \geq 0$ ,  $M(x, t) = \begin{bmatrix} 1 & x - t \\ 0 & 1 \end{bmatrix}$  for all numbers  $x$  and  $t$  and if  $n$  is a positive integer,

$$T_n = \begin{bmatrix} \log(1 + c_n)/[1 + \log(1 + c_n)] & 0 \\ 0 & 1 \end{bmatrix}.$$

So,  $\lim_{n \rightarrow \infty} T_n = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and  $\pi = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ .  $A$  is a determinate boundary condition function for  $Q$  on  $c_1, c_2, \dots$  and  $K_A$  is calculated by equation (2).

Let  $H(x) = \begin{bmatrix} 0 \\ 1/(1 + x^2) \end{bmatrix}$   $x \geq 0$ .  $H$  is in  $D_1$  since the function  $Y$  defined by

$$Y(x) = \begin{bmatrix} x \arctan x - \log + (1x^2)^{1/2} + (\pi/2)x - 1 \\ \arctan x + \pi/2 \end{bmatrix}$$

for  $x \geq 0$  is a solution of the singular boundary value problem with nonhomogeneous term  $H$ . Also,  $\int_0^\infty dt(\pi - I)M(0, t)H(t)$  exists so  $H$  is in  $D_2$  and  $\int_0^\infty dt K_A(x, t)H(t)$  exists for each  $x \geq 0$ . The function  $X$  defined by  $X(x) = \int_0^\infty dt K_A(x, t)H(t)$  for  $x \geq 0$  does not satisfy the asymptotic boundary condition and so  $H$  is in  $D_1 \cap D_2$  but not in the domain of  $R_A$ .

It remains to more completely describe how the domain of  $R_A$  depends upon the problem and to investigate the complex numbers  $\lambda$  for which one obtains an inverting operator  $R(A, \theta, \lambda)$  for the singular boundary value problem

$$\begin{cases} Y' - (Q + \lambda\theta)Y = H \\ A(0)Y(0) + \lim_{n \rightarrow \infty} A(c_n)Y(c_n) = 0 \end{cases}$$

where  $\theta$  denotes a function from  $[0, \infty)$  into  $B$ .



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