MATRX RINGS OF FINITE DEGREE OF NILPOTENCY

Abrahim A. Klein
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The degree of nilpotency of a ring $R$ is defined to be the supremum of the orders of nilpotency of its nilpotent elements and it is denoted by $\nu(R)$. We consider the degree of nilpotency of the ring of $m \times m$ matrices $R_m$ over a ring $R$. We obtain given results concerning the degrees $\nu(R_m)$ for distinct $m$'s, in the case $R$ has no nonzero two-sided annihilators. It is shown that if $\nu(R_m) = m$ for some $m$, and if $R'$ is a ring containing $R$ as an ideal such that $R'$ has no nonzero two-sided annihilators of $R$, then $\nu(R'_m) = m$. An application of this result is given.

$R$ will always be a nonzero associative ring. If $a \in R$ is nilpotent, we denote its order of nilpotency by $\nu(a) = \min\{k | a^k = 0\}$, and if $a$ is not nilpotent we put: $\nu(a) = 0$. The degree of nilpotency $\nu(R)$ of $R$ is defined by

$$\nu(R) = \sup_{a \in R} \nu(a).$$

If $R$ is a ring without nonzero nilpotent elements then $\nu(R) = \nu(0) = 1$, and we shall soon see that the ring $R_m$ of $m \times m$ matrices over $R$ satisfies $\nu(R_m) \geq m$ (Lemma 1).

There exist rings $R$ satisfying $\nu(R_m) > m$ and in [3] was shown that such an $R$ may even be a (noncommutative) integral domain. The object of this paper is to deal with rings $R$ which satisfy $\nu(R_m) = m$ for some $m$. We denote this condition by $\mathcal{R}_m$. First we shall consider the degree of nilpotency of matrix rings over rings without nonzero two-sided annihilators. Then we give some conditions equivalent to $\mathcal{R}_m$. Our main result is: If a nonzero ideal in an integral domain $R$ satisfies $\mathcal{R}_m$ then $R$ itself satisfies $\mathcal{R}_m$. This implication resembles the following one: If a nonzero ideal in an integral domain $R$ is embeddable in a field then $R$ itself is embeddable in a field [1]. This result together with other results obtained in [4], lead us to the conjecture: "The conditions $\mathcal{R}_m$, $m = 1, 2, \ldots$, are sufficient for embedding an integral domain in a field.

Our result is applied to prove that a ring which has no nonzero two-sided annihilators and satisfies $\mathcal{R}_m$ is embeddable in a ring with an identity which satisfies $\mathcal{R}_m$.

I wish to thank G. M. Bergman for his suggestions and comments on this paper.

2. Rings without nonzero two-sided annihilators. The following notations will be used later.
If \( a \in R \) then we denote by \( aE_{ij} \) the matrix with \( a \) in its \((i, j)\) position and 0 elsewhere.

If \( A = (a_{i,j}) \in R^m \) and \( r \) is an integer \( \geq 1 \), we denote the \((i, j)\) entry of \( A^r \) by \( a_{i,j}^{(r)} \). Since \( A^rA^s = A^{r+s} \) we have:

\[
\sum_{k=1}^n a_{ik}^{(r)}a_{kj}^{(s)} = a_{ij}^{(r+s)}.
\]

**Lemma 1.** If \( R \) is not nilpotent then \( \nu(R_m) \geq m \) for each \( m \geq 1 \).

**Proof.** The result is trivial for \( m = 1 \), so let \( m \geq 2 \). Since \( R^{m-1} \neq 0 \), there exist \( a_1, \ldots, a_{m-1} \in R \) such that \( a_1 \cdots a_{m-1} \neq 0 \). Hence the matrix \( A = \sum_{i=1}^{m-1} a_iE_{i,i+1} \) satisfies \( A^{m-1} = a_1 \cdots a_{m-1}E_{1m} \neq 0 \) and \( A^m = 0 \). Thus, \( \nu(R_m) \geq \nu(A) = m \).

**Corollary.** For rings \( R \) without nonzero nilpotent elements, the condition \( S_{sl,m} \) is inherited by (nonzero) subrings.

Indeed, if \( R' \) is a subring of \( R \) then \( \nu(R'_m) \geq m \) since \( R' \) is not nilpotent. If \( R \) satisfies \( S_{sl,m} \) then since \( R'_m \) is a subring of \( R_m \) we have \( \nu(R'_m) \leq \nu(R_m) = m \).

If \( S \) is a nonempty subset of \( R \), we denote its right (left) annihilator in \( R \) by \( r_R(S)(l_R(S)) \). Clearly \( r_R(S) \cap l_R(S) \) is the set of two-sided annihilators of \( S \) in \( R \).

Note that if \( R \) is a (nonzero) ring such that \( r_R(R) \cap l_R(R) = \{0\} \) then \( R \) is not nilpotent.

The proof of our next result is similar to that of [4, Lemma 9].

**Lemma 2.** If \( r_R(R) \cap l_R(R) = \{0\} \) and \( A \in R_m \) is nilpotent of order \( h \), then there exist a matrix \( B \in R_{m+1} \) which is nilpotent of order \( h + 1 \).

**Proof.** If \( h = 1 \) then \( A = 0 \) and the result is trivial. If \( h \geq 2 \) then \( A^{h-1} \neq 0 \) and there exist \( p \) and \( q, 1 \leq p, q, \leq m \), such that \( a_{pq}^{(h-1)} \neq 0 \). Since \( r_R(R) \cap l_R(R) = \{0\} \), there exists an element \( b \in R \) such that either \( ba_{pq}^{(h-1)} \neq 0 \) or \( a_{pq}^{(h-1)}b \neq 0 \). Assume that we have \( a_{pq}^{(h-1)}b \neq 0 \) (the other case is treated similarly). Let \( A_1 \) be the matrix of \( R_{m+1} \) obtained from \( A \) by adjoining a row and a column of zeros and let \( B = A_1 + bE_{q,m+1} \). The powers of \( B \) are given by

\[
B^k = A_1^k + \sum_{i=1}^m a_{iq}^{(h-1)}bE_{i,m+1}, \quad k \geq 2.
\]

Since \( A_1^k = 0 \) and \( a_{pq}^{(h-1)}b \neq 0 \) we obtain \( B^k \neq 0 \) and \( B^{k+1} = 0 \).

This immediately yields:

**Theorem 3.** Let \( R \) be a ring such that \( r_R(R) \cap l_R(R) = \{0\} \). If \( \nu(R_m) \geq h \) then \( \nu(R_{m+r}) \geq h + r \) for each \( r \geq 1 \), and if \( \nu(R_m) \leq h \) then \( \nu(R_{m-r}) \leq h - r \) for each \( r = 1, 2, \ldots, m - 1 \).
THEOREM 4. If \( r_\mathcal{R}(R) \cap l_\mathcal{R}(R) = \{0\} \) and \( R \) satisfies \( \mathcal{R}_m \) for some \( m \), then it also satisfies \( \mathcal{R}_k \) for \( k = 1, 2, \ldots, m - 1 \). In particular it follows that \( R \) has no nonzero nilpotent elements.

3. Conditions equivalent to \( \mathcal{R}_m \).

THEOREM 5. Let \( m \) be a fixed integer \( > 1 \). The following conditions are equivalent for rings \( R \) without nonzero nilpotent elements.

(i) \( \mathcal{R}_m : \nu(R_m) = m \)

(ii) For all \( C \in R_m \), \( C^{m+1} = 0 \) implies \( C^m = 0 \).

(iii) For all \( A, B \in R_m \), \((AB)^m = 0 \) implies \((BA)^m = 0 \).

Proof. It is clear that (i) implies (ii). If (ii) holds and \((AB)^m = 0 \) then \((BA)^{m+1} = B(AB)^mA = 0 \), hence \((BA)^m = 0 \) and (iii) holds.

Assume (iii) holds and we proceed to prove (i). Since \( R \) has no nonzero nilpotent elements \( r_\mathcal{R}(R) = l_\mathcal{R}(R) = 0 \), so \( \nu(R_m) \geq m \). Let \( C = (c_{ij}) \in R_m \), we have to prove that \( \nu(C) \leq m \). Assume \( \nu(C) = h > m \) and let \( c_{pq}^{(h+1)} \neq 0 \). We define two matrices \( A = (a_{ij}) \in R_m \) and \( B = (b_{ij}) \in R_m \) as follows:

\[
a_{ij} = \begin{cases} c_{pq}^{(i)}, & i = 1, \ldots, m - 1 \\ c_{pq}^{(h+1)}, & i = m \\ c_{pq}^{(h-1)}, & j = 1, \ldots, m \\ c_{pq}^{(h-1)}, & i, j = 1, \ldots, m 
\end{cases}
\]

Using (1) we obtain for \( j = 1, \ldots, m \)

\[
\sum_{k=1}^{m} a_{ik}b_{kj} = c_{pq}^{(h+i-j)}, \quad i = 1, \ldots, m - 1.
\]

Since \( C^h = 0 \), it follows that \( C^{h+r} = 0 \) and \( c_{pq}^{(h+r)} = 0 \) for each \( r \geq 0 \). Hence the \((i, j)\) entry of \( AB \) is 0 for \( i \geq j \), and it is \( c_{pq}^{(h-1)} \) for \( j = i + 1, i = 1, \ldots, m - 1 \). This implies that \((AB)^{m-1} = (c_{pq}^{(h-1)})^{m-1}E_{1m} \) and \((AB)^m = 0 \). Since (iii) holds we have \((BA)^m = 0 \). But

\[
(BA)^m = B(AB)^{m-1}A
\]

and its \((i, j)\) entry is \( b_{ij}(c_{pq}^{(h-1)})^{m-1}a_{mj} = 0 \). Taking \( i = p \) and \( j = q \) we obtain \((c_{pq}^{(h-1)})^{m-1} = 0 \) and since \( R \) has no nonzero nilpotent elements, it follows that \( c_{pq}^{(h-1)} = 0 \), a contradiction. Hence \( h \leq m \) and \( R \) satisfies (i).

4. The main result. If \( T \neq 0 \) is an ideal in \( R \) and \( T \) as a ring satisfies \( \mathcal{R}_m \), then it does not follow that \( R \) satisfies \( \mathcal{R}_m \), even if \( R \) has no nonzero nilpotent elements. Indeed, \( R \) may be a direct sum of \( T \) and a ring \( R' \) such that \( \nu(R'_m) > m \) and it is possible to choose
T and $R'$ without nonzero nilpotent elements. Clearly, here the two-sided annihilator of $T$ in $R$ is not 0. On the other hand we have:

**Theorem 6.** If $T$ is an ideal in $R$ such that $r_R(T) \cap l_R(T) = \{0\}$ and $\nu(T_m) = m$, then $\nu(R_m) = m$.

**Proof.** We have $r_T(T) \cap l_T(T) \subseteq r_R(T) \cap l_R(T) = 0$ and $\nu(T_m) = m$, hence it follows by Theorem 4 that $T$ has no nonzero nilpotent elements. Since $R_m$ contains $T_m$ we have $\nu(R_m) \geq m$. Let $C \in R_m$, we have to prove that $\nu(C) \leq m$. As in the proof of Theorem 5, assume $\nu(C) = h > m$ and $c_{pq}^{(-1)} \neq 0$. Construct the same matrices $A$ and $B$ and take arbitrary elements $a, b \in T$. Then $A_i = aA$ and $B_i = Bb$ belong to $T_w$. We have $A_iB_i = a(AB)b$, hence the $(i, j)$ entry of $A_iB_i$ is 0 for $i \geq j$ and it is $ac_{pq}^{(-1)}b$ for $j = i + 1, i = 1, \ldots, m - 1$. From this it follows that $(A_iB_i)^{m-1} = (ac_{pq}^{(-1)}b)^{m-1}E_{1m}$ and $(A_iB_i)^m = 0$. Since $A_i, B_i \in T_m$ and $\nu(T_m) = m$ it follows that $(B_iA_i)^m = 0$. As in the proof of Theorem 5 we obtain that the $(p, q)$ entry of $(A_iB_i)^{-1}A_i = 0$ is

$$c_{pq}^{(-1)}b(ac_{pq}^{(-1)}b)^{m-1}a(c_{pq}^{(-1)}) = 0.$$ 

This implies that

$$(bac_{pq}^{(-1)})^{m-1} = 0, \quad (ac_{pq}^{(-1)}b)^{m-1} = 0, \quad (c_{pq}^{(-1)}ba)^{m-1} = 0.$$ 

Since $T$ has no nonzero nilpotent elements it follows that

$$bac_{pq}^{(-1)} = 0, \quad ac_{pq}^{(-1)}b = 0, \quad c_{pq}^{(-1)}ba = 0.$$ 

This is true for all $a, b \in T$, hence $ac_{pq}^{(-1)} \in r_T(T) \cap l_T(T) = \{0\}$ and $c_{pq}^{(-1)}b \in r_T(T) \cap l_T(T) = \{0\}$ and this implies that $c_{pq}^{(-1)} \in r_R(T) \cap l_R(T) = \{0\}$; a contradiction. Hence $h \leq m$ and $\nu(R_m) = m$.

If $R$ is an integral domain and $T$ a nonzero ideal in $R$, then it is clear that $r_R(T) = l_R(T) = \{0\}$, hence we obtain our main result which is:

**Theorem 7.** If $R$ is an integral domain and $T \neq \emptyset$ an ideal in $R$ which satisfies $R_m$, then $R$ also satisfies $R_m$.

5. Embedding. Let $R$ be a ring without nonzero nilpotent elements. Embed $R$ in a ring $R'$ with 1 in the usual way [2, p. 86]: $R' = R + I, R \cap I = 0$, where $I$ is the ring of integers. $R$ is an ideal in $R'$ and since $r_R(R) = l_R(R) = \{0\}$ it follows that $r_{R'}(R) \cap R = l_R(R) \cap R = \{0\}$. Thus, $R$ is embeddable in $R'/r_{R'}(R) = R''$. One shows easily that $r_{R'}(R) = l_{R'}(R)$. If we identify $R$ with its image in $R''$ we obtain that $R$ is an ideal in $R''$ and $r_{R''}(R) = \{0\}$. Hence by Theorem 6 we obtain:
**Theorem 8.** If $R$ is a ring without nonzero nilpotent elements and satisfies $\mathcal{R}_m$, then $R$ is embeddable in a ring with 1 which satisfies $\mathcal{R}_m$.

If $R$ is an integral domain then the ring $R''$ obtained above is also an integral domain. Thus, we have:

**Corollary.** If $R$ is an integral domain which satisfies $\mathcal{R}_m$ then $R$ is embeddable in an integral domain with 1 which satisfies $\mathcal{R}_m$.

Note that this result enables us to simplify the proof in [4, Theorem 7] taking $t = 1$.

Now, if $R$ is a ring with 1 and satisfies $\mathcal{R}_m$ then $R$ has no nonzero nilpotent elements since $r_n(R) = \{0\}$. Let $C$ be the center of $R$ and assume that the nonzero elements of $C$ are regular in $R$. Thus, we may embed $R$ in the ring $R' = \{ac^{-1} \mid a \in R, \; 0 \neq c \in C\}$ whose center is the quotient field of the commutative integral domain $C$. If $B = (b_{ij}) \in R'_m$ then it is possible to write its entries with a common denominator: $b_{ij} = a_{ij}c^{-1}, \; a_{ij} \in R, \; 0 \neq c \in C, \; 1 \leq i, \; j \leq m$. Let $A = (a_{ij}) \in R_m$ then $Bc = A$. If $B$ is nilpotent then $A$ is also nilpotent and since $R$ satisfies $\mathcal{R}_m$ we have $A^m = 0$. It follows that $B^m c^m = (Bc)^m = 0$ and so $B^m = 0$ since $c^m$ is a unit in $R'$. We have proved:

**Theorem 9.** If $R$ is a ring with 1 which satisfies $\mathcal{R}_m$ and all the elements of its center $C$ are regular, then $R$ is embeddable in a central $K$-algebra which satisfies $\mathcal{R}_m$, $K$ the field of fractions of $C$.

**References**


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