

# Pacific Journal of Mathematics

**MATRIX RINGS OF FINITE DEGREE OF NILPOTENCY**

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The degree of nilpotency of a ring  $R$  is defined to be the supremum of the orders of nilpotency of its nilpotent elements and it is denoted by  $\nu(R)$ . We consider the degree of nilpotency of the ring of  $m \times m$  matrices  $R_m$  over a ring  $R$ . We obtain given results concerning the degrees  $\nu(R_m)$  for distinct  $m$ 's, in the case  $R$  has no nonzero two-sided annihilators. It is shown that if  $\nu(R_m) = m$  for some  $m$ , and if  $R'$  is a ring containing  $R$  as an ideal such that  $R'$  has no nonzero two-sided annihilators of  $R$ , then  $\nu(R'_m) = m$ . An application of this result is given.

$R$  will always be a nonzero associative ring. If  $a \in R$  is nilpotent, we denote its order of nilpotency by  $\nu(a) = \min \{k \mid a^k = 0\}$ , and if  $a$  is not nilpotent we put:  $\nu(a) = 0$ . The degree of nilpotency  $\nu(R)$  of  $R$  is defined by

$$\nu(R) = \sup_{a \in R} \nu(a) .$$

If  $R$  is a ring without nonzero nilpotent elements then  $\nu(R) = \nu(0) = 1$ , and we shall soon see that the ring  $R_m$  of  $m \times m$  matrices over  $R$  satisfies  $\nu(R_m) \geq m$  (Lemma 1).

There exist rings  $R$  satisfying  $\nu(R_m) > m$  and in [3] was shown that such an  $R$  may even be a (noncommutative) integral domain. The object of this paper is to deal with rings  $R$  which satisfy  $\nu(R_m) = m$  for some  $m$ . We denote this condition by  $\mathfrak{N}_m$ . First we shall consider the degree of nilpotency of matrix rings over rings without nonzero two-sided annihilators. Then we give some conditions equivalent to  $\mathfrak{N}_m$ . Our main result is: If a nonzero ideal in an integral domain  $R$  satisfies  $\mathfrak{N}_m$  then  $R$  itself satisfies  $\mathfrak{N}_m$ . This implication resembles the following one: If a nonzero ideal in an integral domain  $R$  is embeddable in a field then  $R$  itself is embeddable in a field [1]. This result together with other results obtained in [4], lead us to the conjecture: "The conditions  $\mathfrak{N}_m, m = 1, 2, \dots$ , are sufficient for embedding an integral domain in a field.

Our result is applied to prove that a ring which has no nonzero two-sided annihilators and satisfies  $\mathfrak{N}_m$  is embeddable in a ring with an identity which satisfies  $\mathfrak{N}_m$ .

I wish to thank G. M. Bergman for his suggestions and comments on this paper.

2. Rings without nonzero two-sided annihilators. The following notations will be used later.

If  $a \in R$  then we denote by  $aE_{ij}$  the matrix with  $a$  in its  $(i, j)$  position and 0 elsewhere.

If  $A = (a_{ij}) \in R_m$  and  $r$  is an integer  $\geq 1$ , we denote the  $(i, j)$  entry of  $A^r$  by  $a_{ij}^{(r)}$ . Since  $A^r A^s = A^{r+s}$  we have:

$$(1) \quad \sum_{k=1}^m a_{ik}^{(r)} a_{kj}^{(s)} = a_{ij}^{(r+s)}.$$

LEMMA 1. *If  $R$  is not nilpotent then  $\nu(R_m) \geq m$  for each  $m \geq 1$ .*

*Proof.* The result is trivial for  $m = 1$ , so let  $m \geq 2$ . Since  $R^{m-1} \neq 0$ , there exist  $a_1, \dots, a_{m-1} \in R$  such that  $a_1 \cdots a_{m-1} \neq 0$ . Hence the matrix  $A = \sum_{i=1}^{m-1} a_i E_{i, i+1}$  satisfies  $A^{m-1} = a_1 \cdots a_{m-1} E_{1m} \neq 0$  and  $A^m = 0$ . Thus,  $\nu(R_m) \geq \nu(A) = m$ .

COROLLARY. *For rings  $R$  without nonzero nilpotent elements, the condition  $\mathfrak{N}_m$  is inherited by (nonzero) subrings.*

Indeed, if  $R'$  is a subring of  $R$  then  $\nu(R'_m) \geq m$  since  $R'$  is not nilpotent. If  $R$  satisfies  $\mathfrak{N}_m$  then since  $R'_m$  is a subring of  $R_m$  we have  $\nu(R'_m) \leq \nu(R_m) = m$ .

If  $S$  is a nonempty subset of  $R$ , we denote its right (left) annihilator in  $R$  by  $r_R(S)$  ( $l_R(S)$ ). Clearly  $r_R(S) \cap l_R(S)$  is the set of two-sided annihilators of  $S$  in  $R$ .

Note that if  $R$  is a (nonzero) ring such that  $r_R(R) \cap l_R(R) = \{0\}$  then  $R$  is not nilpotent.

The proof of our next result is similar to that of [4, Lemma 9].

LEMMA 2. *If  $r_R(R) \cap l_R(R) = \{0\}$  and  $A \in R_m$  is nilpotent of order  $h$ , then there exist a matrix  $B \in R_{m+1}$  which is nilpotent of order  $h + 1$ .*

*Proof.* If  $h = 1$  then  $A = 0$  and the result is trivial. If  $h \geq 2$  then  $A^{h-1} \neq 0$  and there exist  $p$  and  $q$ ,  $1 \leq p, q, \leq m$ , such that  $a_{pq}^{(h-1)} \neq 0$ . Since  $r_R(R) \cap l_R(R) = \{0\}$ , there exists an element  $b \in R$  such that either  $ba_{pq}^{(h-1)} \neq 0$  or  $a_{pq}^{(h-1)}b \neq 0$ . Assume that we have  $a_{pq}^{(h-1)}b \neq 0$  (the other case is treated similarly). Let  $A_1$  be the matrix of  $R_{m+1}$  obtained from  $A$  by adjoining a row and a column of zeros and let  $B = A_1 + bE_{q, m+1}$ . The powers of  $B$  are given by

$$B^k = A_1^k + \sum_{i=1}^m a_{iq}^{(k-1)} b E_{i, m+1}, \quad k \geq 2.$$

Since  $A_1^h = 0$  and  $a_{pq}^{(h-1)}b \neq 0$  we obtain  $B^h \neq 0$  and  $B^{h+1} = 0$ .

This immediately yields:

THEOREM 3. *Let  $R$  be a ring such that  $r_R(R) \cap l_R(R) = \{0\}$ . If  $\nu(R_m) \geq h$  then  $\nu(R_{m+r}) \geq h + r$  for each  $r \geq 1$ , and if  $\nu(R_m) \leq h$  then  $\nu(R_{m-r}) \leq h - r$  for each  $r = 1, 2, \dots, m - 1$ .*

**THEOREM 4.** *If  $r_R(R) \cap l_R(R) = \{0\}$  and  $R$  satisfies  $\mathfrak{N}_m$  for some  $m$ , then it also satisfies  $\mathfrak{N}_k$  for  $k = 1, 2, \dots, m - 1$ . In particular it follows that  $R$  has no nonzero nilpotent elements.*

**3. Conditions equivalent to  $\mathfrak{N}_m$ .**

**THEOREM 5.** *Let  $m$  be a fixed integer  $> 1$ . The following conditions are equivalent for rings  $R$  without nonzero nilpotent elements.*

- (i)  $\mathfrak{N}_m: \nu(R_m) = m$
- (ii) For all  $C \in R_m, C^{m+1} = 0$  implies  $C^m = 0$ .
- (iii) For all  $A, B \in R_m, (AB)^m = 0$  implies  $(BA)^m = 0$ .

*Proof.* It is clear that (i) implies (ii). If (ii) holds and  $(AB)^m = 0$  then  $(BA)^{m+1} = B(AB)^m A = 0$ , hence  $(BA)^m = 0$  and (iii) holds.

Assume (iii) holds and we proceed to prove (i). Since  $R$  has no nonzero nilpotent elements  $r_R(R) = l_R(R) = 0$ , so  $\nu(R_m) \geq m$ . Let  $C = (c_{ij}) \in R_m$ , we have to prove that  $\nu(C) \leq m$ . Assume  $\nu(C) = h > m$  and let  $c_{pq}^{(h-1)} \neq 0$ . We define two matrices  $A = (a_{ij}) \in R_m$  and  $B = (b_{ij}) \in R_m$  as follows:

$$a_{ij} = \begin{cases} c_{pj}^{(i)}, & i = 1, \dots, m - 1 \\ c_{pj}^{(h-1)}, & i = m \end{cases}, \quad j = 1, \dots, m.$$

$$b_{ij} = c_{iq}^{(h-j)}, \quad i, j = 1, \dots, m.$$

Using (1) we obtain for  $j = 1, \dots, m$

$$\sum_{k=1}^m a_{ik} b_{kj} = c_{pq}^{(h+i-j)}, \quad i = 1, \dots, m - 1.$$

$$\sum_{k=1}^m a_{mk} b_{kj} = c_{pq}^{(h+h-1-j)}.$$

Since  $C^h = 0$ , it follows that  $C^{h+r} = 0$  and  $c_{pq}^{(h+r)} = 0$  for each  $r \geq 0$ . Hence the  $(i, j)$  entry of  $AB$  is 0 for  $i \geq j$ , and it is  $c_{pq}^{(h-1)}$  for  $j = i + 1, i = 1, \dots, m - 1$ . This implies that  $(AB)^{m-1} = (c_{pq}^{(h-1)})^{m-1} E_{1m}$  and  $(AB)^m = 0$ . Since (iii) holds we have  $(BA)^m = 0$ . But

$$(BA)^m = B(AB)^{m-1}A$$

and its  $(i, j)$  entry is  $b_{i1}(c_{pq}^{(h-1)})^{m-1}a_{mj} = 0$ . Taking  $i = p$  and  $j = q$  we obtain  $(c_{pq}^{(h-1)})^{m+1} = 0$  and since  $R$  has no nonzero nilpotent elements, it follows that  $c_{pq}^{(h-1)} = 0$ , a contradiction. Hence  $h \leq m$  and  $R$  satisfies (i).

**4. The main result.** If  $T \neq 0$  is an ideal in  $R$  and  $T$  as a ring satisfies  $\mathfrak{N}_m$ , then it does not follow that  $R$  satisfies  $\mathfrak{N}_m$ , even if  $R$  has no nonzero nilpotent elements. Indeed,  $R$  may be a direct sum of  $T$  and a ring  $R'$  such that  $\nu(R'_m) > m$  and it is possible to choose

$T$  and  $R'$  without nonzero nilpotent elements. Clearly, here the two-sided annihilator of  $T$  in  $R$  is not 0. On the other hand we have:

**THEOREM 6.** *If  $T$  is an ideal in  $R$  such that  $r_R(T) \cap l_R(T) = \{0\}$  and  $\nu(T_m) = m$ , then  $\nu(R_m) = m$ .*

*Proof.* We have  $r_T(T) \cap l_T(T) \subseteq r_R(T) \cap l_R(T) = 0$  and  $\nu(T_m) = m$ , hence it follows by Theorem 4 that  $T$  has no nonzero nilpotent elements. Since  $R_m$  contains  $T_m$  we have  $\nu(R_m) \geq m$ . Let  $C \in R_m$ , we have to prove that  $\nu(C) \leq m$ . As in the proof of Theorem 5, assume  $\nu(C) = h > m$  and  $c_{pq}^{(h-1)} \neq 0$ . Construct the same matrices  $A$  and  $B$  and take arbitrary elements  $a, b \in T$ . Then  $A_1 = aA$  and  $B_1 = Bb$  belong to  $T_m$ . We have  $A_1B_1 = a(AB)b$ , hence the  $(i, j)$  entry of  $A_1B_1$  is 0 for  $i \geq j$  and it is  $ac_{pq}^{(h-1)}b$  for  $j = i + 1, i = 1, \dots, m - 1$ . From this it follows that  $(A_1B_1)^{m-1} = (ac_{pq}^{(h-1)}b)^{m-1}E_{1m}$  and  $(A_1B_1)^m = 0$ . Since  $A_1, B_1 \in T_m$  and  $\nu(T_m) = m$  it follows that  $(B_1A_1)^m = 0$ . As in the proof of Theorem 5 we obtain that the  $(p, q)$  entry of  $B_1(A_1B_1)^{m-1}A_1 = 0$  is

$$c_{pq}^{(h-1)}b(ac_{pq}^{(h-1)}b)^{m-1}ac_{pq}^{(h-1)} = 0.$$

This implies that

$$(bac_{pq}^{(h-1)})^{m+1} = 0, (ac_{pq}^{(h-1)}b)^{m+1} = 0, (c_{pq}^{(h-1)}ba)^{m+1} = 0.$$

Since  $T$  has no nonzero nilpotent elements it follows that

$$ba c_{pq}^{(h-1)} = 0, ac_{pq}^{(h-1)}b = 0, c_{pq}^{(h-1)}ba = 0.$$

This is true for all  $a, b \in T$ , hence  $ac_{pq}^{(h-1)} \in r_T(T) \cap l_T(T) = \{0\}$  and  $c_{pq}^{(h-1)}b \in r_T(T) \cap l_T(T) = \{0\}$  and this implies that  $c_{pq}^{(h-1)} \in r_R(T) \cap l_R(T) = \{0\}$ ; a contradiction. Hence  $h \leq m$  and  $\nu(R_m) = m$ .

If  $R$  is an integral domain and  $T$  a nonzero ideal in  $R$ , then it is clear that  $r_R(T) = l_R(T) = \{0\}$ , hence we obtain our main result which is:

**THEOREM 7.** *If  $R$  is an integral domain and  $T \neq \emptyset$  an ideal in  $R$  which satisfies  $\mathfrak{N}_m$ , then  $R$  also satisfies  $\mathfrak{N}_m$ .*

**5. Embedding.** Let  $R$  be a ring without nonzero nilpotent elements. Embed  $R$  in a ring  $R'$  with 1 in the usual way [2, p. 86]:  $R' = R + I, R \cap I = 0$ , where  $I$  is the ring of integers.  $R$  is an ideal in  $R'$  and since  $r_R(R) = l_R(R) = \{0\}$  it follows that  $r_{R'}(R) \cap R = l_{R'}(R) \cap R = \{0\}$ . Thus,  $R$  is embeddable in  $R'/r_{R'}(R) = R''$ . One shows easily that  $r_{R'}(R) = l_{R'}(R)$ . If we identify  $R$  with its image in  $R''$  we obtain that  $R$  is an ideal in  $R''$  and  $r_{R''}(R) = \{0\}$ . Hence by Theorem 6 we obtain:

**THEOREM 8.** *If  $R$  is a ring without nonzero nilpotent elements and satisfies  $\mathfrak{N}_m$ , then  $R$  is embeddable in a ring with 1 which satisfies  $\mathfrak{N}_m$ .*

If  $R$  is an integral domain then the ring  $R''$  obtained above is also an integral domain. Thus, we have:

**COROLLARY.** *If  $R$  is an integral domain which satisfies  $\mathfrak{N}_m$  then  $R$  is embeddable in an integral domain with 1 which satisfies  $\mathfrak{N}_m$ .*

Note that this result enables us to simplify the proof in [4, Theorem 7] taking  $t = 1$ .

Now, if  $R$  is a ring with 1 and satisfies  $\mathfrak{N}_m$  then  $R$  has no nonzero nilpotent elements since  $r_R(R) = \{0\}$ . Let  $C$  be the center of  $R$  and assume that the nonzero elements of  $C$  are regular in  $R$ . Thus, we may embed  $R$  in the ring  $R' = \{ac^{-1} | a \in R, 0 \neq c \in C\}$  whose center is the quotient field of the commutative integral domain  $C$ . If  $B = (b_{ij}) \in R'_m$  then it is possible to write its entries with a common denominator:  $b_{ij} = a_{ij}c^{-1}$ ,  $a_{ij} \in R$ ,  $0 \neq c \in C$ ,  $1 \leq i, j \leq m$ . Let  $A = (a_{ij}) \in R_m$  then  $Bc = A$ . If  $B$  is nilpotent then  $A$  is also nilpotent and since  $R$  satisfies  $\mathfrak{N}_m$  we have  $A^m = 0$ . It follows that  $B^m c^m = (Bc)^m = 0$  and so  $B^m = 0$  since  $c^m$  is a unit in  $R'$ . We have proved:

**THEOREM 9.** *If  $R$  is a ring with 1 which satisfies  $\mathfrak{N}_m$  and all the elements of its center  $C$  are regular, then  $R$  is embeddable in a central  $K$ -algebra which satisfies  $\mathfrak{N}_m$ ,  $K$  the field of fractions of  $C$ .*

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