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ON THE HYPERPLANE SECTION THROUGH A RATIONAL POINT OF AN ALGEBRAIC VARIETY

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ON THE HYPERPLANE SECTION THROUGH A RATIONAL POINT OF AN ALGEBRAIC VARIETY

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Let V/k be an irreducible affine algebraic variety of dimension ≥ 3 defined over an infinite field k with \mathfrak{p} as its prime ideal in $k[X_1, \dots, X_n]$. Let P be a rational normal point on V/k. It is proved that (1) for a generic hyperplane H_u through P, (\mathfrak{p}, H_u) is a prime ideal and (\mathfrak{p}, H_u) is quasi-absolutely (absolutely irreducible) if \mathfrak{p} is quasi-absolutely (absolutely irreducible). (2) It is not true in general that $V \cap H_u$ is normal at P; however, $V \cap H_u$ is normal at P if the local ring of V/k at P is also Cohen-Macaulay (Theorem 8).

It is well known [11] that if V/k is a normal variety of dimension ≥ 2 , then for almost all hyperplanes H the section $V \cap H$ is again a normal variety. This research is motivated by this result to study the following problem: If V/k is normal at a rational point P on V, will hyperplane sections of V through P be normal at P? Section 1 localizes some of the results of [11]. Section 2 describes the ideal decomposition of the generic hyperplane section through a given rational point of an irreducible variety, and Section 3 gives a negative answer to the problem of normality. As a consequence the converse of [3; Lemma 4, p. 360] is invalid in general.

1. Generalities. In the following and the subsequent sections, a variety V/k shall mean an irreducible algebraic variety in the affine space A^n defined over a field k of arbitrary characteristic.

Recall the following definitions.

DEFINITION 1. Let V/k be a variety with $(\xi) = (\xi_1, \dots, \xi_n)$ as a generic point over k, and let P be a point on V. Let

$$k[\hat{\xi}]_p = \left\{ rac{f(\hat{\xi})}{g(\hat{\xi})} \, | \, f, \, g \in k[\hat{\xi}] \quad ext{and} \quad g(P)
eq 0
ight\}$$

be the local ring of V at P in the function field $k(\xi)$ of V over k. We say that P is k-normal on V if $k[\xi]_p$ is integrally closed in $k(\xi)$, that P is k-simple on V if $k[\xi]_p$ is a regular local ring, and that P is singular on V if P is not k-simple on V.

DEFINITION 2. Let V/k be a variety of dimension r, and let P be a point on V. We say that V/k is locally free of s-dimensional

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singularities at P if every s-dimensional subvariety of V containing P is k-simple on V.

DEFINITION 3. Let R be a finite integral domain $k[\xi_1, \dots, \xi_n]$ over a field k or a localization thereof relative to a prime ideal of $k[\xi_1, \dots, \xi_n]$. Let \mathfrak{p} be a prime ideal of R we define

 $ht \mathfrak{p} = \max$. (length of chains of prime ideals contained in \mathfrak{p}), depth $\mathfrak{p} = \max$. (length of chains of prime ideals containing \mathfrak{p}), dim $\mathfrak{p} =$ transcendence degree of the quotient field of R/\mathfrak{p} over k, dim R =transcendence degree of the quotient field of R over k.

It is well known that $ht \mathfrak{p} + \operatorname{depth} \mathfrak{p} = \dim R$ and $\dim \mathfrak{p} = \operatorname{depth} \mathfrak{p}$.

The following criterion for local normality is parallel to [11; Th. 3, p. 363] and is well known [8; (12.9), p. 41].

PROPOSITION 1. Let V/k be a variety of dimension r defined over a field k, and let P be a point of dimension s on V. P is k-normal on V if and only if (1) V/k is locally free of (r-1)-dimensional singularities at P, (2) every nonzero principal ideal $(a) \cdot k[\xi]_p$ is unmixed of dimension r-s-1.

PROPOSITION 2. Let V/k, (ξ) , and P be the same as those in Proposition 1, let $k[\xi]_p^*$ be the integral closure of $k[\xi]_p$, and let \mathbb{C}_p be the conductor of $k[\xi]_p$. If V is locally free of (r-1)-dimensional singularities at P and if $\mathbb{C}_p \neq (1)$, then every nonzero element of \mathbb{C}_p generates a mixed principal ideal.

Proof. Let $\alpha \in k[\xi]_p^*$ not in $k[\xi]_p$, and let $c \in \mathbb{G}_p$, whence $c\alpha \in k[\xi]_p$, say $c\alpha = b, b \in k[\xi]_p$. Then $(c) \cdot k[\xi]_p$ must be mixed. Indeed, if $(c) k[\xi]_p$ were unmixed, and let $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ be the associated prime ideals of $(c) k[\xi]_p$, then dim $\mathfrak{p}_i = r - s - 1$, for $i = 1, 2, \dots, t$. α is integral over $k[\xi]_p$, hence integral over $(k[\xi]_p)_{\mathfrak{p}_i}$ for $i = 1, 2, \dots, t$. By hypothesis $(k[\xi]_p)_{\mathfrak{p}_i}$ is a regular local ring of dimension 1, for $i = 1, 2, \dots, t$, therefore $(k[\xi]_p)_{\mathfrak{p}_i}$ is integrally closed for $i = 1, 2, \dots, t$. Hence $\alpha \in$ $\bigcap_{i=1}^t (k[\xi]_p)_{\mathfrak{p}_i}$ and $b \in (\bigcap_{i=1}^t (c)(k[\xi]_p)_{\mathfrak{p}_i}) \cap k[\xi]_p = \bigcap_{i=1}^t \mathfrak{q}_i$, where $\mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_t$ is a primary decomposition of $(c) k[\xi]_p$. Thus $b \in (c) b[\xi]_p$, i.e., $\alpha \in k[\xi]_p$, a contradiction.

Let V/k be a variety of dimension r defined over a field k with (ξ) as a generic point, and let P be a point on V. Let u be an indeterminate over $k(\xi)$, it is well known that V is a variety over k(u) with (ξ) as a generic point of V over the pure transcendental extension field k(u). Let $k(u)[\xi]_p = \{f(u; \xi) | f, g \in k(u)[\xi] \text{ and } g(u; p) \neq 0\}$

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be the local ring of V at P over k(u). We have, by [10, (d), p. 64], the following lemma.

LEMMA 1. $k[\xi]_p$ is integrally closed if and only if $k(u)[\xi]_p$ is integrally closed.

Recall the definition of the ground form of an unmixed r-dimensional ideal \mathfrak{A}' , [11; p. 373], as following: Let \mathfrak{A} be an unmixed r-dimensional ideal in the polynomial ring $k[X_1, \dots, X_n]$, we form r+1 linear forms in the X_i 's with indeterminates coefficients u_{ij} : $z_i = u_{i1}x_1 + \cdots + u_{in}X_n$, $i = 1, 2, \dots, r+1$, and consider the ideal $\mathfrak{A} \cdot k(u)[X] \cap k(u)[z_1, \dots, z_{r-1}]$, where $k(u)[X] = k(u_{11}, \cdots, u_{r+1n})[X_1, \cdots, X_n]$, which is a principal ideal $(E(z_1, \dots, z_{r+1}; u))$ in k(u)[X]. If E is normalized so as to be a polynomial in the u_{ij} and primitive in them, so that E is defined to within a factor in k, then E is the elementary divisor form or the ground form of \mathfrak{A} . The polynomial E is integral in any z_i over the other z_i 's and is a polynomial in z_1, \dots, z_{r+1} of least degree in z_{r+1} , which is in $\mathfrak{A} \cdot k(u)[X]$. If \mathfrak{A} is prime, then its ground form is irreducible, the converse is not true in general; but \mathfrak{A} is primary if and only if its ground form is a power of an irreducible polynomial [9; Th. 9, p. 252]. \mathfrak{A} is prime and absolutely irreducible if and only if (E) is prime and absolutely irreducible [9; Th. 15, p. 259]. If a is prime and quasiabsolutely irreducible, then (E) is prime and quasi-irreducible [11, p. 373].

PROPOSITION 3. Let V/k be an r-dimensional variety defined over a field k with \mathfrak{p} as its prime ideal in k[X] $(=k[X_1, \dots, X_n])$. Let p be a point on V and let E be the ground form of \mathfrak{p} . Then V is knormal at p if and only if $(\mathfrak{p}, \partial E/\partial z_{r+1}) \cdot k(u)[X]_p$ is unmixed.

Proof. By Lemma 1, V is k-normal at P if and only if V is k(u)-normal at P. By [13; Lemma 2, p. 132] V/k(u) is free of (r-1)-dimensional singularities at P. Let (ξ) be a generic point of V/k(u), and pass to $k(u)[\xi]$, we assert that $k(u)[\xi]_p$ is integrally closed if and only if $(\partial \overline{E}/\partial \overline{z}_{r+1}) \cdot k(u)[\xi]_p$ is unmixed, where the bar denotes residue. By the proof of [11; Th. 5, p. 365], we have $\partial \overline{E}/\partial \overline{z}_{r+1} \in \mathbb{C}$, the conductor of $k(u)[\xi]_p$ in its integral closure $k(u)[\xi]^*$. Let \mathbb{C}_p be the conductor of $k(u)[\xi]_p$ in its integral closure $k(u)[\xi]_p^*$. By [15; Lemma, p. 269], $\mathbb{C} \cdot k(u)[\xi]_p = \mathbb{C}_p$. Therefore $\partial \overline{E}/\partial \overline{z}_{r+1} \in \mathbb{C}_p$. By Proposition 2, we have that $k(u)[\xi]_p$ is integrally closed if and only if $(\partial \overline{E}/\partial \overline{z}_{r+1}) \cdot k(u)[\xi]_p$ is unmixed.

2. Irreducibility of generic hyperplane section through a normal point. Let V/k be a variety of dimension $r \ge 2$. Let $P \in V$ be a rational point. We are studying the generic hyperplane section

of V through P. Without loss of generality, we may assume once for all in the sequel that V passes through (0) the origin of the affine space and that P = (0). We shall denote the prime ideal of V/k by \mathfrak{p} in the sequel. Let u_1, \dots, u_n be n indeterminates over k, and let H_u be the generic hyperplane through (0) defined by $u_1X_1 + \cdots + u_nX_n = 0$. We shall use H_u in two senses whenever it is proper: (1) H_u means the linear polynomial $u_1X_1 + \cdots + u_nX_n$ in k(u)[X] $(=k(u_1, \cdots, u_n)$ $[X_1, \dots, X_n]$, (2) H_u stands for the hyperplane defined by $u_1X_1 + \dots + u_nX_n$ $u_n X_n = 0$. Let $k(u) = k(u_1, \dots, u_n)$, V is a variety over k(u) and $V \cap H_u$ is defined over k(u). Let $(\mathfrak{p}, H_u) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$ be an irredundant primary decomposition with $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ as the associated prime ideals. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s, s \leq t$, be the isolated prime ideals. Since $(0) \in V$, $(\mathfrak{p}, H_u) \subset (X_1, \dots, X_n) \cdot k(u)[X]$. Hence $(X_1, \dots, X_n) \cdot k(u)[X]$ must contain at least one of the $\mathfrak{p}_i, i \leq s$, say \mathfrak{p}_1 . Let us denote \mathfrak{p}_1 by \mathfrak{p}_u and let W_u be the variety over k(u) of $\mathfrak{p}_u \cdot W_u$ is of dimension r-1 as it is well known that any component of $V \cap H$, where H is a hypersurface, is of dimension r-1. Let (ξ) be a generic point of W_u over k(u). Since tr. $\deg_{k(\xi)} k(u; \xi) + \text{tr.} \deg_k k(\xi) = \text{tr.} \deg_k k(u; \xi) = \text{tr.} \deg_k k(u; \xi)$ $k(u) + \operatorname{tr.deg}_{k(u)} k(u; \xi) = n + r - 1 \text{ and } \operatorname{tr.deg}_{k(\xi)} k(u; \xi) \leq n - 1, \text{ we}$ have tr. $\deg_{k(\xi)} k(u; \xi) \ge r$. But $(\xi) \in V$, therefore tr. $\deg_k k(\xi) = r$. We thus have

LEMMA 2. If dim $V \ge 2$, a generic point of W_u over k(u) is also a generic point of V over k.

LEMMA 3. If $\xi_j \neq 0$, then $u_1, \dots u_{j-1}, u_{j+1}, \dots, u_n$ are algebraically independent over $k(\xi)$.

Proof. Say

$$egin{aligned} &i=1,\, ext{tr.}\, ext{deg}_{k(u_2,\cdots,u_n)}\,k(u_1,\,\cdots,\,u_n;\,\xi)\ &+ ext{tr.}\, ext{deg}_k\,k(u_2,\,\cdots\,u_n)=n+\,r\,-\,1 \end{aligned}$$

Therefore tr. $\deg_{k(u_2,\ldots,u_n)}k(u_1,\ldots,u_n;\xi)=r$. Since

$$\frac{u_2\xi_2+\cdots+u_n\xi_n}{\xi_1}\in k(u_2,\cdots,u_n;\xi_1,\cdots,\xi_n),$$

we have $k(u_1, \dots, u_n; \xi) = k(u_2, \dots, u_n; \xi)$. Now

$$\mathrm{tr.} \deg_{k(\zeta)} k(u_2, \cdots, u_n; \zeta) + r = r + n - 1$$
.

Therefore tr. deg_{k(ξ)} $k(u_2, \dots, u_n; \xi) = n - 1$, i.e., u_2, \dots, u_n are algebraically independent over $k(\xi)$.

PROPOSITION 4. Let (ξ) , \mathfrak{p}_u and W_u be as above. Then (\mathfrak{p}, H_u) :

 $(X_1, \dots, X_u)^{\rho} = \mathfrak{p}_u$ for sufficiently large integers ρ , where $(X_1, \dots, X_n) = (X, \dots, X_n) \cdot k(u)[X]$.

Proof. Let $F(u_1, \dots, u_n; X) \in \mathfrak{p}_u$ be a polynomial, we may assume $F(u_1, \dots, u_n; X) \in k[u_1, \dots, u_n][X]$. If $\xi_1 \neq 0$, $F(u_1, \dots, u_n; \xi) = 0$ implies that $F(-(u_2\xi_2 + \dots + u_n\zeta_n/\xi_1), u_2, \dots, u_n; \xi) = 0$. Hence there exists a nonnegative integer σ such that X_1^{σ} .

$$F\left(-\frac{u_{2}X_{2}+\cdots+u_{n}X_{n}}{X_{1}}, u_{2}, \cdots, u_{n}; X\right) \in k(u_{2}, \cdots, u_{n})[X]$$

vanishes at (ξ) . By Lemma 3, the prime ideal determined by (ξ) in $k(u_2, \dots, u_n)[X]$ is $\mathfrak{p}k(u_2, \dots, u_n)[X]$. Thus

$$X_1^{\sigma}F\left(-\frac{u_2X_2+\cdots+u_nX_u}{X_1}, u_2, \cdots, u_n; X\right) \in \mathfrak{p} \cdot k(u_1, \cdots, u_n)[X]$$

for sufficiently large σ . But

$$X_1^{\sigma}F\left(-rac{u_2X_2+\cdots+u_nX_n}{X_1}, u_2, \cdots, u_n; X
ight) \ -X_1^{\sigma}F(u_1, \cdots, u_n; X) \equiv 0$$

 $mod (u_1X_1 + \cdots + u_nX_n) \cdot k(u)[X]$ for sufficiently large σ . We have $X_1^{\sigma}F(u_1, \dots, u_n; X) \in (\mathfrak{p}, H_u) \cdot k(u)[X]$ for sufficiently large σ . The above discussion is symmetric with respect to those $\xi_i \neq 0$. Therefore for any $\xi_i \neq 0$, we have $X_i^{\sigma_i} F(u_1, \dots, u_n; X) \in (\mathfrak{p}, H_u)$ for sufficiently large integer σ_i and for all $F \in \mathfrak{p}_u$. For any j such that $\xi j = 0, X_j \in \mathfrak{p}$. Thus $X_{j}^{\sigma_{j}}F \in (\mathfrak{p}, H_{u})$ for any positive integer σ_{j} and for all $F \in \mathfrak{p}_{u}$. Thus $(\mathfrak{p}, H_u): (X_1, \dots, X_n)^{\rho} \supset \mathfrak{p}_u$ for sufficiently large integer ρ . We now Let $g(u_1, \dots, u_n; X)$ be an element in show the other inclusion. (\mathfrak{p}, H_u) : $(X_1, \cdots, X_n)^{\rho}$. Then for any $h(u_1, \dots, u_n; X) \in (X_1, \dots, X_n)^{\rho}$, $h(u; X) \cdot g(u; X) \in (\mathfrak{p}, H_u)$. Therefore, there exists $m_i(u; X)$, $n(u; X) \in \mathcal{H}_u$ k(u)[X] such that $h(u;X)g(u;X) = \sum_{i=1}^{s} m_i(u;X) \cdot F_i(X) + n(u;X)H_u$ where $(F_1, \dots, F_s) \cdot k[X] = \mathfrak{p}$. Thus $h(u; \xi)g(u; \xi) = 0$. If $g(u; \xi) \neq 0$, then h(u; X) = 0 at (ξ) for all $h(u; X) \in (X_1, \dots, X_n)^{\rho}$, which implies that $(\xi) = (0)$, a contradiction. Thus g(u; X) = 0 at (ξ) and therefore $\mathfrak{p} \supset (p, H_u): (X_1, \cdots, X_n)^{\rho}.$

COROLLARY. (\mathfrak{p}, H_u) has only one isolated component.

Proof. Suppose \mathfrak{p}_2 is another isolated component, by Proposition 4, we have (\mathfrak{p}, H_u) : $(X_1, \dots, X_n)^{\rho'} = \mathfrak{p}_2$, for sufficiently large integer ρ' . Hence we have $\mathfrak{p}_2 = (\mathfrak{p}, H_u) = (X_1, \dots, X_n)^{\rho} = \mathfrak{p}_u$.

THEOREM 1. If V/k is of dimension $r \ge 2$, then $(\mathfrak{p}, H_u) \cdot k(u)[X]$

is either a prime ideal \mathfrak{p}_u or an intersection of the prime ideal \mathfrak{p}_u with a primary ideal of which $(X_1, \dots, X_n) \cdot k(u)[X]$ is its radical.

Proof. Let $\mathfrak{B} = (\mathfrak{p}, H_u)$ and let $\mathfrak{B} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$ be the irredundant primary representation of \mathfrak{B} with $\mathfrak{p}_1, \cdots, \mathfrak{p}_t$ as the associated prime ideals. By the corollary, there exists only one isolated prime component, say \mathfrak{q}_i , and denote \mathfrak{p}_1 by \mathfrak{p}_u . Let $\mathfrak{m} = (X_1, \cdots, X_n) \cdot k(u)[X]$. Since $\mathfrak{B}: \mathfrak{m}^{\rho} = \mathfrak{p}_u$ for sufficiently large ρ , we have $(\mathfrak{q}_i:\mathfrak{m}^{\rho}) = \mathfrak{p}_u$. There are two possibilities (I) no \mathfrak{p}_i contains \mathfrak{m}^{λ} for any nonnegative integer λ , or (II) some of \mathfrak{p}_i contains a power of \mathfrak{m} . (I) leads to $\mathfrak{B} = \mathfrak{p}_u$. In case of (II), say \mathfrak{p}_2 contains \mathfrak{m}^{λ} for some λ then $\mathfrak{m} = \mathfrak{p}_2$. We may assume that there is no other \mathfrak{p}_j to contain \mathfrak{m}^{λ} for any $0 \leq \lambda \in \mathbb{Z}$. Thus for $i = 1, 3, 4, \cdots r, \mathfrak{q}_i: \mathfrak{m}^2 = \mathfrak{q}_i$ for any $0 \leq \lambda \in \mathbb{Z}$. Since $\mathfrak{q}_2: \mathfrak{m}^{\rho} = k(u)[X]$ for large ρ , hence $\mathfrak{B}: \mathfrak{m}^{\rho} = (\mathfrak{q}_i: \mathfrak{m}^{\rho}) \cap (\mathfrak{q}_2: \mathfrak{m}^{\rho}) \cap \cdots \cap (\mathfrak{q}_{\gamma}: \mathfrak{m}^{\rho}) = \mathfrak{q}_1 \cap \mathfrak{q}_3 \cap$ $\mathfrak{q}_4 \cap \cdots \cap \mathfrak{q}_t$ and thus $\mathfrak{p}_u \cap \mathfrak{q}_2 = (\mathfrak{p}, H_u)$.

COROLLARY 1. If V is normal over k, then $(\mathfrak{p}, H_u) = \mathfrak{p}_u$.

Proof. Passing to the coordinate ring of $V, k(u)[\eta]$, we have that $(u_1\eta_1 + \cdots + u_n\eta_n) \cdot k(u)[\eta]$ is unmixed. Letting $\overline{\mathfrak{p}}_u = \mathfrak{p}_u/\mathfrak{p}, \overline{\mathfrak{q}}_2 = \mathfrak{q}_2/\mathfrak{p}$ we have $(\sum u_i\eta_i) = \overline{\mathfrak{p}}_u \cap \overline{\mathfrak{q}}_2$ or $(\sum u_i\eta_i) = \overline{\mathfrak{p}}_u$, by Theorem 1. The unmixedness implies that $(\sum u_i\eta_i) = \overline{\mathfrak{p}}_u$, i.e., $(\mathfrak{p}, H_u) = \mathfrak{p}_u$.

COROLLARY 2. If V is k-normal at (0), then $(\mathfrak{p}, H_u) = \mathfrak{p}_u$ i.e., (\mathfrak{p}, H_u) is a prime ideal.

Proof. By Theorem 1, $(\mathfrak{p}, H_u) = \mathfrak{p}_u$ or $(\mathfrak{p}, H_u) = \mathfrak{p}_u \cap \mathfrak{q}_2$. Passing to the local ring $k(u)[\eta]_{(0)'}$ of V at (0), we have $(\sum u_i\eta_i)k(u)[\eta]_{(0)} = \overline{\mathfrak{p}}_u^e$ or $\overline{\mathfrak{p}}_u^e \cap \overline{\mathfrak{q}}_2^e$ where $\overline{\mathfrak{p}}_u = \mathfrak{p}_u/\mathfrak{p}, \overline{\mathfrak{q}}_2 = \mathfrak{q}_2/\mathfrak{p}\overline{\mathfrak{p}}_u^e$ and $\overline{\mathfrak{q}}_2^e$, are extensions of $\overline{\mathfrak{p}}_u$ and $\overline{\mathfrak{q}}_2$ in $k(u)[\eta]_{(0)}$ respectively. Since $k(u)[\eta]_{(0)}$ is integrally closed, the unmixedness of $(\sum u_i\eta_i) \cdot k(u)[\eta]_{(0)}$ implies that $(\sum u_i\eta_i)k(u)[\eta] = \overline{\mathfrak{p}}_u$ and $(\mathfrak{p}, H_u) = \mathfrak{p}_u$.

Recall that V/k is a quasi-absolutely irreducible variety if k is quasi-algebraically closed in the field $k(\xi_1, \dots, \xi_n)$ of rational functions on V/k; a prime ideal \mathfrak{A} in $k[X_1, \dots, X_n]$ is quasi-absolutely irreducible if $\overline{k}[X_1, \dots, X_n]\mathfrak{A}$ is primary, where \overline{k} is the algebraic closure of k. By [11; Th. 10, p. 371], \mathfrak{p} is quasi-absolutely irreducible if and only if V/k is quasi-absolutely irreducible. V/k is absolutely irreducible if kis algebraically closed in $k(\xi)$ and $k(\xi)$ is separable over k. A prime ideal \mathfrak{A} in $k[X_1, \dots, X_n]$ is absolutely irreducible if $\overline{k}[X_1, \dots, X_n]$. \mathfrak{A} is a prime ideal. It is well known that the prime ideal \mathfrak{P} of V/k is absolutely irreducible if and only if V/k is.

THEOREM 2. If V/k is quasi-absolutely irreducible of dimension

 $r \geq 3$ and if k is infinite, then $V \cap H_u/k(u)$ is quasi-absolutely irreducible.

Proof. Let (η) be a generic point of $V \cap H_u$ over

$$k(u) = k(u_1, \cdots, u_n)$$
.

By Lemma 2, (η) is a generic point of V over k. Let η_1, η_2 , and η_n be algebraically independent over k. By Lemma 3, (η) is a generic point of V over $k(u_2, \dots, u_n)$. By [11; Lemma 5, p. 368], $k(u_2, \dots, u_n)$ is quasi-algebraically closed in $k(u_2, \dots, u_n)(\eta)$. Let $\Sigma = k(u_2, \dots, u_{n-1})$ $(\eta), u_n$ is algebraically independent over Σ . Viewing $k(u_2, \dots, u_{n-1})$ as the field k and u_n as the u in [11; corollary, p. 369], we have $\Sigma(u_n) =$ $k(u_2, \dots, u_{n-1})(u_n)(\eta) = k(u)(\xi)$. Let ξ_1 and ξ_2 in [11; corollary, p. 369] be replaced by $-(u_2\eta_2 + \dots + u_{n-1}\eta_{n-1})/\eta_1$ and $-\eta_{n/n_1}$ respectively, one sees that $-(u_2\eta_2 + \dots + u_{n-1}\eta_{n-1})/\eta_1$ and η_n/η_1 are algebraically independent over $k(u_2, \dots, u_{n-1})$. Hence by the same corollary we have that

$$k(u_2, \dots, u_{n-1})(u_n)(-(u_2\eta_2 + \dots + u_{n-1})/\eta_1 - u_n\eta_n/\eta_1)$$

= $k(u_2, \dots, u_{n-1})(u_n)(u_1) = k(u)$

quasi-algebraically closed in $\Sigma(u_n) = k(u)(\eta)$.

LEMMA 4. Let K be a regular finitely generated extension of an infinite field k with tr. deg_k $K \ge 3$. Let x, y, z be three elements of K algebraically independent over k, and $z/x \notin K^p k$, where p is the characteristic of k. Then for all but a finite number of constants $c \in k$, K is a regular extension of k(y + cz/x). Moreover, let τ be an indeterminate $K(\tau)$ is regular over $k(\tau)(y + \tau z/x)$.

Proof. [5; Lemma 3].

THEOREM 3. If V/k is an absolutely irreducible variety of dimension $r \ge 3$ defined over an infinite field k, then $V \cap H_u/k(u)$ is an absolutely irreducible variety.

Proof. $V \cap H_u/k(u)$ is irreducible. Let (ξ) be a generic point of $V \cap H_u$ over k(u). By Lemma 3, (ξ) is a generic point of V over k, hence tr. deg_k $k(\xi) \geq 3$ and $k(\xi)$ is a regular extension over k by [12; Proposition 1, p. 69]. Let ξ_1, ξ_2 and ξ_n be three elements in a separable transendental basis of $k(\xi)$ over k. Let $K = k(u_2, \dots, u_{n-1})(\xi), u_n$ is algebraically independent over K. Viewing $k(u_2, \dots, u_{n-1})$ as the field k and u_n as the τ in Lemma 4, we have $K(u_n) = k(u)(\xi)$. Let $y = -(u_2\xi_2 + \dots + u_{n-1}\xi_{n-1}), z = \xi_n$ and $x = \xi_1$, then x, y and z are

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algebraically over $k(u_2, \dots, u_{n-1})$. By [6, Proposition 1, p. 185] and [6; corollary to Proposition 2, p. 186], $z/x = -\xi_n/\xi_1 \notin K^p k(u_2, \dots, u_{n-1})$, we have that $K(u_n)$ is a regular extension over

$$k(u_2, \cdots, u_{n-1})(u_n)\left(\frac{y-u_nz}{k}\right) = k(u)$$
.

Therefore $k(u)(\xi)$ is a regular extension over k(u), hence $V \cap H_n/k(u)$ is an absolutely irreducible variety.

Let $\{F_1, \dots, F_s\}$ be a set of generators of \mathfrak{P} in k[x]. Let P be a point on V. According to [14], P is k-simple on V if and only if the mixed Jacobian of $\{F_1, \dots, F_s\}$ is of rank n - r at P. When k(P) is separable over k, P is k-simple on V if and only if the classical Jacobian of $\{F_1, \dots, F_s\}$ is of rank n - r at P.

Following Theorem 1, we denote \mathfrak{p}_u as the sole isolated component of (\mathfrak{p}, H_u) and $W_u/k(u)$ as its variety in the sequel.

THEOREM 4. Let V/k be of dimension $r \ge 2$. Then $P \in W_u$ is k(u)-simple if and only if P is k-simple on V.

Proof. Let $P \in W_u$ be k-simple on V. By Theorem 1, $(\mathfrak{p}, H_u) = \mathfrak{p}_u \cap \mathfrak{A}$, where \mathfrak{A} is the embedded component with (X_1, \dots, X_n) as radical. Let (η) be a generic point of V over k(u), and let (\mathfrak{F}) be a generic point of W_u over k(u). Let $k(u)[\eta]_p$ and $k(u)[\mathfrak{F}]_p$ be the local rings of V and W_u at P respectively. $k(u)[\eta]_p$ is regular and

$$k(u)[\xi]_p \cong k(u)[\eta]_p/\overline{\mathfrak{p}}_u \cdot k(u)[\eta]_p$$

where $\overline{\mathfrak{p}}_u$ is the residue of \mathfrak{p}_u modulo \mathfrak{p} . If $P \neq (0)^1$, let \mathfrak{A} be the residue of \mathfrak{A} modulo \mathfrak{p} and let \mathfrak{m}_p be the maximal ideal of $k(u)[\eta]_p$, then $\mathfrak{A}k(u)[\eta] \not\subset \mathfrak{m}_p$. For otherwise $(\eta_1, \dots, \eta_n)^{\rho} \subset \mathfrak{m}_p$ for some integer $\rho > 0$, as $(X_1, \dots, X_n)^{\rho} \subset \mathfrak{A}$. Thus P = (0), a contradiction. Therefore, when $P \neq (0), (\Sigma u_i \eta_i) \cdot k(u)[\eta]_p = \overline{\mathfrak{p}}_u \cdot k(u)[\eta]_p$, and $k(u)[\xi]_p \cong k(u)[\eta]_p/$ $(\Sigma u_i \eta_i) k(u) [\eta]_p$. By [16; Th. 26, p. 303], to show that $k(u) [\xi]_p$ is regular it is sufficient to show that $\sum u_i \eta_i \notin \mathfrak{m}_p^2$. But this is the case, for if $\sum u_i \eta_i \in \mathfrak{m}_p^2$, taking partial derivatives with respect to u_i for i =1, 2, \cdots , n, we have $\eta_i \in \mathfrak{m}_p$ for $i = 1, 2, \cdots, n$, i.e., P = (0) a con-Therefore $k(u)[\xi]_p$ is regular. If P = (0), then (0) is ktradiction. normal on V. By Corollary 2 to Theorem 1, $(\mathfrak{p}, H_u) = \mathfrak{p}_u$. In viewing [14, Th. 7, p. 28], we let F_1, \dots, F_s be a basis of \mathfrak{p} , and let F_i 's and X_i 's be so arranged that $(\det (\partial F_i / \partial X_j))_{(0)} \neq 0$, where $i, j = 1, 2, \dots$, n-r, and the subscript (0) means that we replace (X) by (0) after the determinant of the Jacobian is formed, as the rank of

¹ If $P \neq 0$, and if P is k-simple on V, then P remains simple on $W_u/k(u)$ follows also from [13; the theorem of Bertini, p. 138].

$$J(F_1, \dots, F_s, X_1, \dots, X_n)_{(0)} = n - r$$
.

Consider

$$\Delta_{j} = \det \begin{bmatrix} \frac{\partial F_{1}}{\partial X_{1}} & \cdots & \frac{\partial F_{1}}{\partial X_{n-r}} & \frac{\partial F_{1}}{\partial X_{j}} \\ \vdots \\ \frac{\partial F_{n-r}}{\partial X_{1}} & \cdots & \frac{\partial F_{n-r}}{\partial X_{n-r}} & \frac{\partial F_{n-r}}{\partial X_{j}} \\ u_{1} & \cdots & u_{n-r} & u_{j} \end{bmatrix}_{(0)}$$

where $\eta - r + 1 \leq j < \eta$. If $\underline{\beta} = 0$ for some j then u_1, \dots, u_{n-r}, u_j are algebraically dependent over k. This is a contradiction, hence (0) is k-simple on W_u . Conversely, assume that $P \in W_u$ is k(u)-simple on W_u . If $P \neq (0)$, we have $k(u)[\underline{\beta}]_p \cong k(u)[\underline{\eta}]_p/(\underline{\Sigma}u_i\eta_i) \cdot k(u)[\underline{\eta}]_p$ from the above. If P = (0), then P is k(u)-normal on W_u . By Theorem 6 in the following V/k is normal at (0), therefore $(\mathfrak{p}, H_u) = \mathfrak{p}_u$ and $k(u)[\underline{\beta}]_{(0)} \cong$ $k(u)[\underline{\eta}]_{(0)}/(\underline{\Sigma}u_i\eta_i) \cdot k(u)[\underline{\eta}]_{(0)}$. Therefore $k(u)[\underline{\beta}]_p \cong k(u)[\underline{\eta}]_p/(\underline{\Sigma}u_i\eta_i) \cdot k(u)[\underline{\eta}]_p$ if P is k(u)-simple on W_u . Since $ht((\underline{\Sigma}u_i\eta_i) \cdot k(u)[\underline{\eta}]_p) = 1$, it follows from [8; (9; 11), p. 28] that $k(u)[\underline{\eta}]_p$ is a regular local ring. Hence Pis k-simple on V.

By an argument similar to the proof of Lemma 2, we have the following.

COROLLARY. If V/k is of dimension $r \ge 3$ and if V/k is locally free of (r-1)-dimensional singularities, then $V \cap H_u/k(u)$ is locally free of (r-2)-dimensional singularities.

Note. If r = 2, the corollary is clearly false as one sees by taking V to be a cone with vertex at (0).

THEOREM 5. If V/k is a complete intersection of dimension ≥ 3 and if V is k-normal at (0), then the generic hyperplane section $V \cap H_u$ is also k(u)-normal at (0).

Proof. V/k(u) is k(u)-normal at (0), by Lemma 1. By corollary to Theorem 1, $(\mathfrak{p}, H_u) = \mathfrak{p}_u$ is prime. For any polynomial $F \neq 0$ in k(u)[X], by [7; Th. p. 49] or [16; Th. 26, p. 203], $(\mathfrak{p}_u, F) = (\mathfrak{p}, H_u, F)$ is unmixed. Hence, passing to the quotient modulo \mathfrak{p}_u , we have that every nonzero principal ideal in the coordinate ring $k(u)[\mathfrak{f}]$ of $V \cap H_u$ is unmixed. It follows that every nonzero principal ideal in the local ring of $V \cap H_u$ at (0), $k(u)[\mathfrak{f}]_{(0)}$, is also unmixed. Since V/k is k-normal at (0), therefore V/k is locally free of (r-1)-dimensional singularities at (0). By the above corollary, $V \cap H_u$ is locally free of (r-2)-dimensional singularities at (0). It follows from Proposition 1 that $V \cap H_u$ is k(u)-normal at (0).

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THEOREM 6. If $V \cap H_u$ is k(u)-normal at (0), then V/k is normal at (0).

Proof. This theorem is really a consequence of [3; Lemma 4, p. 360] ([8; (36.9), p. 134]). Indeed, let (η) be a generic point of V over k(u). Passing to $k(u)[\eta]$, by Theorem 1, we have $(u_1\eta_1 + \cdots + u_n\eta_n) \cdot k(u)[\eta] = \bar{\mathfrak{p}}_u \cap \bar{\mathfrak{q}}$, where $\bar{\mathfrak{p}}_u$ and $\bar{\mathfrak{q}}$ are residues of \mathfrak{p}_u and \mathfrak{q} modulo \mathfrak{p} respectively. It is clear that (1) $(u_1\eta_1 + \cdots + u_n\eta_n) \cdot k(u)[\eta]_{(0)} = \bar{\mathfrak{p}}_u \cdot k(u)[\eta]_{(0)}, (2) \quad (u_1\eta_1 + \cdots + u_n\eta_n) \cdot (k(u)[\eta]_{(0)})_{\bar{\mathfrak{p}}_u} = \bar{\mathfrak{p}}_u \cdot (k(u)[\eta]_{(0)})_{\bar{\mathfrak{p}}_u}$, and (3) let (ξ) be a generic point of $V \cap H_u$ over k(u), then

$$\frac{k(u)[\eta]_{(0)}}{[\bar{\mathfrak{p}}_{u}k(u)[\eta]_{(0)}} \cong k(u)[\xi]_{(0)} ,$$

which is integrally closed as $V \cap H_u$ is k(u)-normal at (0). Moreover, let $k(u)[\eta]_{(0)}^*$ be the integral closure of $k(u)[\eta]_{(0)}$ in $k(u)(\eta)$, and let \mathfrak{p}' be a minimal prime divisor of $(u_1\eta_1 + \cdots + u_n\eta_n) \cdot k(u)[\eta]_{(0)}^*$. It follows from [2; Th. 2, p. 253] and [2; Th. 3; p. 254] that $ht(\mathfrak{p}' \cap k(u)[\eta]_{(0)}) =$ $ht\mathfrak{p} = 1$. Therefore $\mathfrak{p}' \cap k(u)[\eta]_{(0)} = \overline{\mathfrak{p}}_u$, i.e., every minimal prime divisor of $(u_1\eta_1 + \cdots + u_n\eta_n) \cdot k(u)[\eta]_{(0)}^*$ lies over \mathfrak{p}_u . The above verify the conditions of [3; Lemma 4, p. 360], therefore $k(u)[\eta]_{(0)}$ is integrally closed.

3. The local normal problem. Throughout this section let V/k be a variety of dimension $r \ge 3$, passing through (0) with (ξ) as a generic point over k and let $H_u: u_1X_1 + \cdots + u_nX_n = 0$ be a generic hyperplane through (0). If V/k is normal at (0), is it true that $H_u \cap V$ k(u)-normal at (0)? If V/k is a complete intersection then by Theorem 5, the answer to the question is yes. However we shall prove the answer to the question is negative in general.

DEFINITION 4. (a) Let R be a Noetherian ring. Subset $\{a_1, \dots, a_q\}$ of R is a prime sequence if for each $i = 1, 2, \dots, q, a_i$ is not a zero divisor in the ring $R/(a_1, \dots, a_{i-1}) \cdot R$.

(b) Let R be a local ring, the number of elements of a maximal prime sequence in R is called the homological co-dimension of R, and is denoted by $\operatorname{cod} h(A)$. If $\operatorname{cod} h(A) = \dim A$, we say that A is a Cohen-Macaulay ring.

For a general commutative ring R and a multiplicative system Swhich does not contain 0, it is well known [15, p. 219] that $(\mathfrak{A}: \mathfrak{B})^{e} \subset$ $\mathfrak{A}^{e}: \mathfrak{B}^{e}$ and $(\mathfrak{X}: \mathfrak{Y})^{e} \subset \mathfrak{X}^{e}: \mathfrak{Y}^{e}$, where $(^{*})^{e} = (^{*}) \cdot R_{s}$, $(^{*})^{e} = f^{-1}(^{*})$, f is the canonical homomorphism of R into R_{s} and where $\mathfrak{A}, \mathfrak{B}$ are two ideals in R, and $\mathfrak{X}, \mathfrak{Y}$ are two ideals in R_{s} .

PROPOSITION 5. Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{X}$ and \mathfrak{Y} be the same as above. Then (a) $(\mathfrak{A}: \mathfrak{B})^e = \mathfrak{A}^e: \mathfrak{B}^e$; if $\mathfrak{A} \supset \operatorname{Ker} f$ and \mathfrak{B} is finitely generated, also (b) $(\mathfrak{X}: \mathfrak{Y})^e = \mathfrak{X}^e: \mathfrak{Y}^e$ if \mathfrak{Y} is finitely generated.

Proof. Let $\mathfrak{B} = (b_1, \dots, b_i)R$, we have $\mathfrak{B}^e = (f(b_1), \dots, f(b_i)) \cdot R_s$. Let $x \in \mathfrak{A}^e : \mathfrak{B}^e$. Then $x\mathfrak{B}^e \subset \mathfrak{A}^e$ and $xf(b_i) = f(a_i)/f(s_i)$ for some $a_i \in \mathfrak{A}$ and $s_i \in S$. Therefore $f(\pi_i s_i)xf(b_i) \in f(\mathfrak{A})$. For each $b \in f(\mathfrak{B})$, $b = \sum_j f(r_j)f(b_j)$ for some $r_j \in R$. Now $f(\pi_i s_i)xb = \sum_j f(\pi_i s_i)xf(r_j)f(b_j) \in f(\mathfrak{A})$, which implies that $f(\pi_i s_i)x \in f(\mathfrak{A}): f(\mathfrak{B})$. Hence $x \in (f(\mathfrak{A}): f(\mathfrak{B}))R_s$. Since $\mathfrak{A} \supset \operatorname{Ker} f$, by [15; (15), p. 148], $f(\mathfrak{A}): f(\mathfrak{B}) = f(\mathfrak{A}:\mathfrak{B})$. Therefore $x \in (\mathfrak{A}:\mathfrak{B})^e$ and $\mathfrak{A}^e : \mathfrak{B}^e = (\mathfrak{A}:\mathfrak{B})^e$. The proof of (b) is similar.

LEMMA 5. $k(u)[\xi]_{(0)}$ is Cohen-Macaulay if and only if $k[\xi]_{(0)}$ is Cohen-Macaulay, where $k[\xi]$ is the coordinate ring of V/k, and u is an indeterminate over $k(\xi)$.

Proof. If $k[\xi]_{(0)}$ is Cohen-Macaulay, then there exist ζ_1, \dots, ζ_7 such that $\{\zeta_1, \dots, \zeta_r\}$ forms a maximal prime sequence, where $r = \dim V$. Thus $(\zeta_1, \dots, \zeta_i)k[\xi]_{(0)}: (\zeta_{i+1}) \cdot k[\xi]_{(0)} = (\zeta_1, \dots, \zeta_i) \cdot k[\xi]_{(0)}$ for $i = 1, 2, \dots, r$. By [15; (1), p. 227], [15; (15), (21), p. 148] Proposition 5 and [16; (3), p. 221] one has $(\zeta_1, \dots, \zeta_i)k(u)[\xi]_{(0)}: (\zeta_{i+1})k(u)[\xi]_{(0)} = (\zeta_1, \dots, \zeta_i)k(u)[\xi]_{(0)}$, for $i = 1, 2, \dots, r$. Therefore $\{\zeta_1, \dots, \zeta_r\}$ remains as a maximal prime sequence of $k(u)[\xi]_{(0)}$. Thus $k(u)[\xi]_{(0)}$ is Cohen-Macaulay.

Conversely, let $k(u)[\xi]_{(0)}$ be Cohen-Macaulay, let $\{ \ell_i(u;\xi), \dots, \ell_r(u;\xi) \}$ be a maximal prime sequence of $k(u)[\xi]_{(0)}$. Then, for $i = 1, 2, \dots, r$, we have $(\ell_i(u;\xi), \dots, \ell_i(u;\xi)) \cdot k(u)[\xi]_{(0)} : (\ell_{i+1}(u;\xi)) \cdot k(u)[\xi]_{(0)} = (\ell_i(u;\xi), \dots, \ell_i(u;\xi)) \cdot k(u)[\xi]_{(0)}$. By [15; (21), p. 148], going back to the polynomial ring k(u)[x], we have $(\ell_i(u;x), \dots, \ell_i(u;x), \mathfrak{p})k(u)[x]_{(0)} : (\ell_{i+1}(u;x), \mathfrak{p})k(u)[x]_{(0)} = (\ell_i(u;x), \dots, \ell_i(u;x), \mathfrak{p})k(u)[x]_{(0)}$. In viewing [4; Satz 3, p. 59], one sees that

$$\overline{(\mathscr{C}_1(u;x),\cdots,\mathscr{C}_i(u;x),\mathfrak{p})k(u)[x]_{\scriptscriptstyle(0)}}$$
: $\overline{(\mathscr{C}_{i+1}(u;x),\mathfrak{p})k(u)[x]_{\scriptscriptstyle(0)}} = (\mathscr{C}_1(u;x),\cdots,\mathscr{C}_i(u;x),\mathfrak{p})k(u)[x]_{\scriptscriptstyle(0)}$

almost always for $i = 1, 2, \dots, r$, where the bar means specialization of u to elements in k. Passing to the local ring of V/k(u) at (0), by [15; (15), p. 148], we have $\overline{\zeta_i(u;\xi,\dots,\zeta_i(u;\xi))k(u)[\xi]_{(0)}:\zeta_{i+1}(u;\xi)k(u)[\xi]_{(0)}} = \overline{(\zeta_i(u;\xi),\dots,\zeta_i(u;\xi))k(u)[\xi]_{(0)}}$ almost always for $i = 1, 2, \dots, r$. Let $a \in k$ be such that the above holds and $\zeta_i(a;\xi) \neq 0$, for $i = 1, 2, \dots, r$, then $(\zeta_i(a;\xi),\dots,\zeta_i(a;\xi))k[\xi]_{(0)}:(\zeta_{i+1}(a;\xi))k[\xi]_{(0)} = (\zeta_i(a;\xi),\dots,\zeta_i(a;\xi)) \cdot k[\xi]_{(0)}$ for $i = 1, 2, \dots, r$. Therefore $\{\zeta_i(a,\xi),\dots,\zeta_r(a,\xi)\}$ forms a system of prime sequence of $k[\xi]_{(0)}$. Hence $k[\xi]_{(0)}$ is Cohen-Macaulay.

THEOREM 7. Let V/k and H_u be the same as the above. It is not

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true in general that if V/k is k-normal at (0), then $V \cap H_u/k(u)$ is k(u)-normal at (0).

Proof. Suppose that if V/k is k-normal at (0), then $V \cap H_u/k(u)$ is k(u)-normal at (0). Let (ξ) be a generic point of V over k and let (η) be that of $V \cap H_u$ over k(u). Applying the supposition to $V \cap H_u/k(u)$, we get $(V \cap H_u) \cap H_{u(2)}k(u, u(2))$ -normal at (0), where

 $H_{u(2)}: u_{21}X_1 + \cdots + u_{2n}X_n = 0$

is a generic hyperplane through (0) on

$$V \cap H_u/k(u)$$
 and $u(2) = \{u_{21}, \dots, u_{2n}\}$

are algebraically independent over $k(u)(\xi, \eta)$. Repeating the supposition and Corollary 2 to Theorem 1 in this way until dimension r of V is cut down to 2, we have then

 $V \cap H_u \cap H_{u(2)} \cap \cdots \cap H_{u(r-2)}k(u, u(2), \cdots, u(\gamma - 2))$ -normal

at (0), where $u(i) = \{u_{i1}, \dots, u_{in}\}$, and $\{u_{i1}, \dots, u_{in}\}$ are indeterminates over $k(u, u(2), \dots, u(i-1)(\xi, \eta, \eta_2, \eta_{i-1})$ with $\eta_j = (\eta_{j_1}, \dots, \eta_{j_n})$ being a generic point of $V \cap H_u \cap H_{u^{(2)}} \cap \cdots \cap H_{u^{(j)}}$ over $k(u, u(2), \cdots u(j))$. Let $U = \{u, u(2), \dots, u(\gamma - 2)\}$, then $k(U) = k(u, u(2), \dots, u(\gamma - 2))$. Consider V/k(U), (ξ) is a generic point of V over k(U). Correspondingly in the coordinate ring $k(U)[\xi]$ of V over k(U) we have then r-2quantities $\ell_i = u_{i1}\xi_1 + \cdots + u_{in}\xi_n$, $i = 1, 2, \cdots r - 2$, such that (ℓ_1, \cdots, ℓ_i) is a prime ideal in $k(U)[\xi]_{(0)}$ and $\ell_{i+1} \notin (\ell_1, \dots, \ell_i)k(U)[\xi]_{(0)}$. Thus $\{\ell_1, \dots, \ell_{r-2}\}$ is a prime sequence in the local ring $k(U)[\xi]_{(0)}$. Let R be $k(U)[\xi]_{(0)}/(\ell_1, \dots, \ell_{r-2}) \cdot k(U)[\xi]_{(0)}$, then R is integrally closed of dimension 2. By [16; (3), p. 397], R is Cohen-Macaulay. Let $a, b \in k(U)[\xi]_{(0)}$ be such that their residues modulo $(\ell_1, \dots, \ell_{r-2}) \cdot k(U)[\xi]_{(0)}$ form a maximal prime sequence of R, then $\{\ell_1, \dots, \ell_{r-2}, a, b\}$ is a prime sequence of $k(U)[\xi]_{(0)}$. Therefore dim $k(U)[\xi]_{(0)} = \operatorname{cod} hk(U)[\xi]_{(0)}$ and hence $k(U)[\xi]_{(0)}$, is a Cohen-Macaulay ring. It follows from Lemma 5 that $k[\xi]_{(0)}$ is a Cohen-Macaulay ring. So under the supposition, we conclude that $k[\xi]_{(0)}$ is integrally closed implies that $k[\xi]_{(0)}$ is Cohen-Macaulay. But on the other hand, [1; Proposition, p. 655] and [1; Th. 5, p. 653] yield an example of a local ring of an algebraic variety at a rational point which is a factorial local ring (hence normal), but not a Cohen-Macaulay local ring. Hence the above supposition yields a contradiction.

THEOREM 8. If V/k is normal at (0), and the local ring $k[\xi]_{(0)}$ is a Cohen-Macaulay ring, then $V \cap H_u/k(u)$ is normal at (0).

Proof. By the corollary to Theorem 4, (\mathfrak{p}, H_u) is free of $(\gamma - 2)$ -

dimensional singularities. By Lemma 5, $k(u)[\xi]_{(0)}$ is Cohen-Macaulay. For any nonzero $a(u; \xi)$ in $k(u)[\xi]_{(0)}$ not in the prime ideal

$$(u_1\xi_1 + \cdots + u_n\xi_n) \cdot k(u)[\xi]_{(0)}, \{a(u,\xi), u_1\xi_1 + \cdots + u_n\xi_n\}$$

forms a prime sequence of $k(u)[\xi]_{(0)}$, therefore by [16; Lemma 5, p. 401], $(a(u, \xi), u_1\xi_1 + \cdots + u_n\xi_n) \cdot k(u)[\xi]_{(0)}$, is unmixed. Hence every nonzero principal ideal of $k(u)[\xi]_{(0)}/(u_1\xi_1 + \cdots + u_n\xi_n) \cdot k(u)[\xi]_{(0)}$, is unmixed. It follows from Proposition 1 that $V \cap H_u$ is k(u)-normal at (0).

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