ON THE HYPERPLANE SECTION THROUGH A RATIONAL POINT OF AN ALGEBRAIC VARIETY

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Let $V/k$ be an irreducible affine algebraic variety of dimension $\geq 3$ defined over an infinite field $k$ with $\mathfrak{p}$ as its prime ideal in $k[X_1, \ldots, X_n]$. Let $P$ be a rational normal point on $V/k$. It is proved that (1) for a generic hyperplane $H_u$ through $P$, $(\mathfrak{p}, H_u)$ is a prime ideal and $(\mathfrak{p}, H_u)$ is quasi-absolutely (absolutely irreducible) if $\mathfrak{p}$ is quasi-absolutely (absolutely irreducible). (2) It is not true in general that $V \cap H_u$ is normal at $P$; however, $V \cap H_u$ is normal at $P$ if the local ring of $V/k$ at $P$ is also Cohen-Macaulay (Theorem 8).

It is well known [11] that if $V/k$ is a normal variety of dimension $\geq 2$, then for almost all hyperplanes $H$ the section $V \cap H$ is again a normal variety. This research is motivated by this result to study the following problem: If $V/k$ is normal at a rational point $P$ on $V$, will hyperplane sections of $V$ through $P$ be normal at $P$? Section 1 localizes some of the results of [11]. Section 2 describes the ideal decomposition of the generic hyperplane section through a given rational point of an irreducible variety, and Section 3 gives a negative answer to the problem of normality. As a consequence the converse of [3; Lemma 4, p. 360] is invalid in general.

1. Generalities. In the following and the subsequent sections, a variety $V/k$ shall mean an irreducible algebraic variety in the affine space $A^n$ defined over a field $k$ of arbitrary characteristic.

Recall the following definitions.

**Definition 1.** Let $V/k$ be a variety with $(\xi) = (\xi_1, \ldots, \xi_n)$ as a generic point over $k$, and let $P$ be a point on $V$. Let

$$k[\xi]_P = \left\{ \frac{f(\xi)}{g(\xi)} \mid f, g \in k[\xi] \text{ and } g(P) \neq 0 \right\}$$

be the local ring of $V$ at $P$ in the function field $k(\xi)$ of $V$ over $k$. We say that $P$ is $k$-normal on $V$ if $k[\xi]_P$ is integrally closed in $k(\xi)$, that $P$ is $k$-simple on $V$ if $k[\xi]_P$ is a regular local ring, and that $P$ is singular on $V$ if $P$ is not $k$-simple on $V$.

**Definition 2.** Let $V/k$ be a variety of dimension $r$, and let $P$ be a point on $V$. We say that $V/k$ is locally free of $s$-dimensional
singularities at P if every s-dimensional subvariety of V containing P is k-simple on V.

**DEFINITION 3.** Let \( R \) be a finite integral domain \( k[\xi_1, \cdots, \xi_s] \) over a field \( k \) or a localization thereof relative to a prime ideal of \( k[\xi_1, \cdots, \xi_s] \). Let \( \mathfrak{p} \) be a prime ideal of \( R \) we define

\[
ht \mathfrak{p} = \text{max. (length of chains of prime ideals contained in } \mathfrak{p}),
\]
\[
depth \mathfrak{p} = \text{max. (length of chains of prime ideals containing } \mathfrak{p}),
\]
\[
dim \mathfrak{p} = \text{transcendence degree of the quotient field of } R/\mathfrak{p} \text{ over } k,
\]
\[
dim R = \text{transcendence degree of the quotient field of } R \text{ over } k.
\]

It is well known that \( ht \mathfrak{p} + depth \mathfrak{p} = dim R \) and \( dim \mathfrak{p} = depth \mathfrak{p} \).

The following criterion for local normality is parallel to [11; Th. 3, p. 363] and is well known [8; (12.9), p. 41].

**PROPOSITION 1.** Let \( V/k \) be a variety of dimension \( r \) defined over a field \( k \), and let \( P \) be a point of dimension \( s \) on \( V \). \( P \) is \( k \)-normal on \( V \) if and only if (1) \( V/k \) is locally free of \((r - 1)\)-dimensional singularities at \( P \), (2) every nonzero principal ideal \((a) \cdot k[\xi]_p\) is unmixed of dimension \( r - s - 1 \).

**PROPOSITION 2.** Let \( V/k, (\xi) \), and \( P \) be the same as those in Proposition 1, let \( k[\xi]^*_p \) be the integral closure of \( k[\xi]_p \), and let \( \mathfrak{C}_p \) be the conductor of \( k[\xi]_p \). If \( V \) is locally free of \((r - 1)\)-dimensional singularities at \( P \) and if \( \mathfrak{C}_p \neq (1) \), then every nonzero element of \( \mathfrak{C}_p \) generates a mixed principal ideal.

**Proof.** Let \( \alpha \in k[\xi]^*_p \) not in \( k[\xi]_p \), and let \( c \in \mathfrak{C}_p \), whence \( c \alpha \in k[\xi]_p \), say \( c \alpha = b, b \in k[\xi]_p \). Then \((c) \cdot k[\xi]_p\) must be mixed. Indeed, if \((c) k[\xi]_p\) were unmixed, and let \( \mathfrak{p}_1, \cdots, \mathfrak{p}_t \) be the associated prime ideals of \((c) k[\xi]_p\), then \( \dim \mathfrak{p}_i = r - s - 1 \), for \( i = 1, 2, \cdots, t \). \( \alpha \) is integral over \( k[\xi]_p \), hence integral over \((k[\xi]_p)_{\mathfrak{p}_i} \) for \( i = 1, 2, \cdots, t \). By hypothesis \((k[\xi]_p)_{\mathfrak{p}_i} \) is a regular local ring of dimension 1, for \( i = 1, 2, \cdots, t \), therefore \((k[\xi]_p)_{\mathfrak{p}_i} \) is integrally closed for \( i = 1, 2, \cdots, t \). Hence \( \alpha \in \bigcap_{i=1}^t (k[\xi]_p)_{\mathfrak{p}_i} \) and \( b \in (\bigcap_{i=1}^t (c)(k[\xi]_p)_{\mathfrak{p}_i}) \cap k[\xi]_p = \bigcap_{i=1}^t q_i \), where \( q_1 \cap \cdots \cap q_t \) is a primary decomposition of \((c) k[\xi]_p\). Thus \( b \in (c) b[\xi]_p \), i.e., \( \alpha \in k[\xi]_p \), a contradiction.

Let \( V/k \) be a variety of dimension \( r \) defined over a field \( k \) with \((\xi)\) as a generic point, and let \( P \) be a point on \( V \). Let \( u \) be an indeterminate over \( k(\xi) \), it is well known that \( V \) is a variety over \( k(u) \) with \((\xi)\) as a generic point of \( V \) over the pure transcendental extension field \( k(u) \). Let \( k(u)[\xi]_p = \{f(u; \xi)/g(u; \xi) \mid f, g \in k(u)[\xi] \text{ and } g(u; p) \neq 0\} \)
be the local ring of $V$ at $P$ over $k(u)$. We have, by [10, (d), p. 64], the following lemma.

**Lemma 1.** $k[z]_p$ is integrally closed if and only if $k(u)[z]_p$ is integrally closed.

Recall the definition of the ground form of an unmixed $r$-dimensional ideal $\mathfrak{A}$, [11; p. 373], as following: Let $\mathfrak{A}$ be an unmixed $r$-dimensional ideal in the polynomial ring $k[X_1, \cdots, X_n]$, we form $r + 1$ linear forms in the $X_i$'s with indeterminates coefficients $u_{ij}: z_i = u_{ij}x_1 + \cdots + u_{in}X_n$, $i = 1, 2, \cdots, r + 1$, and consider the ideal $\mathfrak{A} \cdot k(u)[X] \cap k(u)[z_1, \cdots, z_{r+1}]$, where $k(u)[X] = k(u_1, \cdots, u_{r+1})[X_1, \cdots, X_n]$, which is a principal ideal $(E(z_1, \cdots, z_{r+1}; u))$ in $k(u)[X]$. If $E$ is normalized so as to be a polynomial in the $u_i$s and primitive in them, so that $E$ is defined to within a factor in $k$, then $E$ is the elementary divisor form or the ground form of $\mathfrak{A}$. The polynomial $E$ is integral in any $z_i$ over the other $z_i$'s and is a polynomial in $z_1, \cdots, z_{r+1}$ of least degree in $z_{r+1}$, which is in $\mathfrak{A} \cdot k(u)[X]$. If $\mathfrak{A}$ is prime, then its ground form is irreducible, the converse is not true in general; but $\mathfrak{A}$ is primary if and only if its ground form is a power of an irreducible polynomial [9; Th. 9, p. 252]. $\mathfrak{A}$ is prime and absolutely irreducible if and only if $(E)$ is prime and absolutely irreducible [9; Th. 15, p. 259]. If $\mathfrak{A}$ is prime and quasi-absolutely irreducible, then $(E)$ is prime and quasi-irreducible [11, p. 373].

**Proposition 3.** Let $V/k$ be an $r$-dimensional variety defined over a field $k$ with $p$ as its prime ideal in $k[X] (=k[X_1, \cdots, X_n])$. Let $p$ be a point on $V$ and let $E$ be the ground form of $p$. Then $V$ is $k$-normal at $p$ if and only if $(p, \partial E/\partial z_{r+1}) \cdot k(u)[X]_p$ is unmixed.

**Proof.** By Lemma 1, $V$ is $k$-normal at $P$ if and only if $V$ is $k(u)$-normal at $P$. By [13; Lemma 2, p. 132] $V/k(u)$ is free of $(r - 1)$-dimensional singularities at $P$. Let $(\xi)$ be a generic point of $V/k(u)$, and pass to $k(u)[\xi]$, we assert that $k(u)[\xi]_p$ is integrally closed if and only if $(\partial E/\partial z_{r+1}) \cdot k(u)[\xi]_p$ is unmixed, where the bar denotes residue. By the proof of [11; Th. 5, p. 365], we have $\partial E/\partial z_{r+1} \in \mathbb{C}$, the conductor of $k(u)[\xi]$ in its integral closure $k(u)[\xi]^*$. Let $\mathbb{C}_p$ be the conductor of $k(u)[\xi]_p$ in its integral closure $k(u)[\xi]^*_p$. By [15; Lemma, p. 269], $\mathbb{C} \cdot k(u)[\xi]_p = \mathbb{C}_p$. Therefore $\partial E/\partial z_{r+1} \in \mathbb{C}_p$. By Proposition 2, we have that $k(u)[\xi]_p$ is integrally closed if and only if $(\partial E/\partial z_{r+1}) \cdot k(u)[\xi]_p$ is unmixed.

2. Irreducibility of generic hyperplane section through a normal point. Let $V/k$ be a variety of dimension $r \geq 2$. Let $P \in V$ be a rational point. We are studying the generic hyperplane section
of $V$ through $P$. Without loss of generality, we may assume once for all in the sequel that $V$ passes through $(0)$ the origin of the affine space and that $P = (0)$. We shall denote the prime ideal of $V/k$ by $p$ in the sequel. Let $u_1, \ldots, u_n$ be $n$ indeterminates over $k$, and let $H_u$ be the generic hyperplane through $(0)$ defined by $u_1X_1 + \cdots + u_nX_n = 0$. We shall use $H_u$ in two senses whenever it is proper: (1) $H_u$ means the linear polynomial $u_1X_1 + \cdots + u_nX_n$ in $k(u)[X]$ ($= k(u_1, \ldots, u_n)$ $[X_1, \ldots, X_n]$), (2) $H_u$ stands for the hyperplane defined by $u_1X_1 + \cdots + u_nX_n = 0$. Let $k(u) = k(u_1, \ldots, u_n)$, $V$ is a variety over $k(u)$ and $V \cap H_u$ is defined over $k(u)$. Let $(\xi_1, \ldots, \xi_n)$ be a generic point of $W_u$ over $k(u)$. Since $(0) \in V$, $(\xi_1, \ldots, \xi_n)$ must contain at least one of the $p_1, i \leq s$, say $p_i$. Let us denote $p_i$ by $p_i$ and let $W_u$ be the variety over $k(u)$ of $p_i$ of dimension $r - 1$ as it is well known that any component of $V \cap H_u$, where $H$ is a hypersurface, is of dimension $r - 1$. Let $(\xi)$ be a generic point of $W_u$ over $k(u)$. Since $\text{tr. deg}_{k(\xi)} k(u; \xi) + \text{tr. deg}_{k(u)} k(u; \xi) = \text{tr. deg}_{k(u)} k(u; \xi) = \text{tr. deg}_{k(u)} k(u; \xi) = \text{tr. deg}_{k(u)} k(u; \xi) = n + r - 1$ and $\text{tr. deg}_{k(\xi)} k(u; \xi) = n - 1$, we have $\text{tr. deg}_{k(\xi)} k(u; \xi) = r$. But $(\xi) \in V$, therefore $\text{tr. deg}_{k(\xi)} k(u; \xi) = r$. We thus have

**Lemma 2.** If $\dim V \geq 2$, a generic point of $W_u$ over $k(u)$ is also a generic point of $V$ over $k$.

**Lemma 3.** If $\xi_j \neq 0$, then $u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_n$ are algebraically independent over $k(\xi)$.

**Proof.** Say

$$i = 1, \text{tr. deg}_{k(u_1, u_2, \ldots, u_n)} k(u_1, \ldots, u_n; \xi) + \text{tr. deg}_{k(u_2, \ldots, u_n)} k(u_2, \ldots, u_n) = n + r - 1.$$ 

Therefore $\text{tr. deg}_{k(u_1, u_2, \ldots, u_n)} k(u_1, \ldots, u_n; \xi) = r$.

Since

$$u_2\xi_2 + \cdots + u_n\xi_n \in k(u_2, \ldots, u_n; \xi_1, \ldots, \xi_n),$$

we have $k(u_1, \ldots, u_n; \xi) = k(u_2, \ldots, u_n; \xi)$. Now

$$\text{tr. deg}_{k(\xi)} k(u_2, \ldots, u_n; \xi) + r = r + n - 1.$$ 

Therefore $\text{tr. deg}_{k(\xi)} k(u_2, \ldots, u_n; \xi) = n - 1$, i.e., $u_2, \ldots, u_n$ are algebraically independent over $k(\xi)$.

**Proposition 4.** Let $(\xi), p_u$ and $W_u$ be as above. Then $(p, H_u)$:
Let \( F(u_1, \ldots, u_n; X) \in \mathfrak{p}_u \) be a polynomial, we may assume \( F(u_1, \ldots, u_n; X) \in k(u_1, \ldots, u_n)[X] \). If \( \xi_i \neq 0 \), \( F(-u_i \xi_i + \cdots + u_n \xi_n, u_2, \ldots, u_n; \xi) = 0 \). Hence there exists a nonnegative integer \( \sigma \) such that \( X_i \).

\[
F\left(-\frac{u_2X_2 + \cdots + u_nX_n}{X_1}, \ldots, \frac{u_nX_n}{X_1}; X\right) \in k(u_2, \ldots, u_n)[X]
\]

vanishes at \((\xi)\). By Lemma 3, the prime ideal determined by \((\xi)\) in \( k(u_2, \ldots, u_n)[X] \) is \( pk(u_2, \ldots, u_n)[X] \). Thus

\[
X_i \cdot F\left(-\frac{u_2X_2 + \cdots + u_nX_n}{X_1}, \ldots, \frac{u_nX_n}{X_1}; X\right) \in \mathfrak{p} \cdot k(u_1, \ldots, u_n)[X]
\]

for sufficiently large \( \sigma \). But

\[
X_i \cdot F(u_1, \ldots, u_n; X) = 0
\]

mod \((u_1, X_1 + \cdots + u_nX_n) \cdot k(u)[X]\) for sufficiently large \( \sigma \). We have

\[
X_i \cdot F(u_1, \ldots, u_n; X) \in (\mathfrak{p}, H_\sigma) \cdot k(u)[X]
\]

for sufficiently large \( \sigma \). Therefore, there exists \( m_i(u; X), n_i(u; X) \in k(u)[X] \) such that \( h(u; X)g(u; X) = \sum_{i=1}^{r} m_i(u; X) \cdot F_i(X) + n(u; X)H_\sigma \), where \( (F_1, \ldots, F_r) \cdot k[X] = \mathfrak{p} \).

**THEOREM 1.** If \( V/k \) is of dimension \( r \geq 2 \), then \((\mathfrak{p}, H_\sigma) \cdot k(u)[X] = \mathfrak{p}_u \).
is either a prime ideal \( p_u \) or an intersection of the prime ideal \( p_u \) with a primary ideal of which \( (X_1, \ldots, X_n) \cdot k(u)[X] \) is its radical.

**Proof.** Let \( \mathfrak{B} = (p, H_u) \) and let \( \mathfrak{B} = q_1 \cap \cdots \cap q_t \) be the irredundant primary representation of \( \mathfrak{B} \) with \( p_1, \ldots, p_t \) as the associated prime ideals. By the corollary, there exists only one isolated prime component, say \( q_i \), and denote \( p_i \) by \( p_u \). Let \( m = (X_1, \ldots, X_n) \cdot k(u)[X] \). Since \( \mathfrak{B} : m^\lambda = p_u \) for sufficiently large \( \rho \), we have \( (q_i : m^\lambda) = p_u \). There are two possibilities (I) no \( p_i \) contains \( m^\lambda \) for any nonnegative integer \( \lambda \), or (II) some of \( p_i \) contains a power of \( m \). (I) leads to \( \mathfrak{B} = p_u \). In case of (II), say \( p_2 \) contains \( m^\lambda \) for some \( \lambda \) then \( m = p_2 \). We may assume that there is no other \( p_i \) to contain \( m^\lambda \) for any \( 0 \leq \lambda \in \mathbb{Z} \). Thus for \( \lambda = 1, 3, 4, \ldots, r \), \( q_i : m^\lambda = q_i \) for any \( 0 \leq \lambda \in \mathbb{Z} \). Since \( q_2 : m^\lambda = k(u)[X] \) for large \( \rho \), hence \( \mathfrak{B} : m^\lambda = (q_1 : m^\lambda) \cap (q_2 : m^\lambda) \cap \cdots \cap (q_t : m^\lambda) = q_1 \cap q_2 \cap q_4 \cap \cdots \cap q_t \) and thus \( p_u \cap q_2 = (p, H_u) \).

**Corollary 1.** If \( V \) is normal over \( k \), then \( (p, H_u) = p_u \).

**Proof.** Passing to the coordinate ring of \( V, k(u)[\gamma] \), we have that \( (u, \gamma_t + \cdots + u_s \gamma_s) \cdot k(u)[\gamma] \) is unmixed. Letting \( \overline{p}_s = p_u / p, \overline{q}_2 = q_2 / p \) we have \( (\sum u_i, \gamma_i) = \overline{p}_s \cap \overline{q}_2 \) or \( (\sum u_i, \gamma_i) = \overline{p}_s \), by Theorem 1. The unmixedness implies that \( (\sum u_i, \gamma_i) = \overline{p}_s \), i.e., \( (p, H_u) = p_u \).

**Corollary 2.** If \( V \) is \( k \)-normal at \( (0) \), then \( (p, H_u) = p_u \) i.e., \( (p, H_u) \) is a prime ideal.

**Proof.** By Theorem 1, \( (p, H_u) = p_u \) or \( (p, H_u) = p_u \cap q_2 \). Passing to the local ring \( k(u)[\gamma]_{(0)} \) of \( V \) at \( (0) \), we have \( (\sum u_i, \gamma_i) k(u)[\gamma]_{(0)} = \overline{p}_s \) or \( \overline{p}_s \cap \overline{q}_2 \) where \( \overline{p}_s = p_u / p, \overline{q}_2 = q_2 / p \overline{p}_s \) and \( \overline{q}_2 \), are extensions of \( \overline{p}_s \) and \( \overline{q}_2 \) in \( k(u)[\gamma]_{(0)} \) respectively. Since \( k(u)[\gamma]_{(0)} \) is integrally closed, the unmixedness of \( (\sum u_i, \gamma_i) \cdot k(u)[\gamma]_{(0)} \) implies that \( (\sum u_i, \gamma_i) k(u)[\gamma] = \overline{p}_s \) and \( (p, H_u) = p_u \).

Recall that \( V/k \) is a quasi-approximately irreducible variety if \( k \) is quasi-algebraically closed in the field \( k(\xi_1, \ldots, \xi_s) \) of rational functions on \( V/k \); a prime ideal \( \mathfrak{A} \) in \( k[X_1, \ldots, X_n] \) is quasi-approximately irreducible if \( k[X_1, \ldots, X_n][\mathfrak{A}] \) is primary, where \( \overline{k} \) is the algebraic closure of \( k \). By [11; Th. 10, p. 371], \( p \) is quasi-approximately irreducible if and only if \( V/k \) is quasi-approximately irreducible. \( V/k \) is approximately irreducible if \( k \) is approximately closed in \( k(\xi) \) and \( k(\xi) \) is separable over \( k \). A prime ideal \( \mathfrak{A} \) in \( k[X_1, \ldots, X_n] \) is approximately irreducible if \( k[X_1, \ldots, X_n][\mathfrak{A}] \) is a prime ideal. It is well known that the prime ideal \( p \) of \( V/k \) is approximately irreducible if and only if \( V/k \) is.

**Theorem 2.** If \( V/k \) is quasi-approximately irreducible of dimension
Proof. Let \((\eta)\) be a generic point of \(V \cap H_u\) over 
\[ k(u) = k(u_1, \ldots, u_n). \]
By Lemma 2, \((\eta)\) is a generic point of \(V\) over \(k\). Let \(\eta_1, \eta_2,\) and \(\eta_n\) be algebraically independent over \(k\). By Lemma 3, \((\eta)\) is a generic point of \(V\) over \(k(u_2, \ldots, u_n)\). By [11; Lemma 5, p. 368], \(k(u_2, \ldots, u_n)\) is quasi-algebraically closed in \(k(u_1, \ldots, u_n)(\eta)\). Let \(\Sigma = k(u_2, \ldots, u_{n-1})\) \((\eta), u_n\) is algebraically independent over \(\Sigma\). Viewing \(k(u_2, \ldots, u_{n-1})\) as the field \(k\) and \(u_n\) as the \(u\) in [11; corollary, p. 369], we have \(\Sigma(u_n) = k(u_2, \ldots, u_{n-1})(\eta) = k(u)(\xi)\). Let \(\xi_1\) and \(\xi_2\) in [11; corollary, p. 369] be replaced by \(-(u_2\xi_2 + \cdots + u_{n-1}\xi_{n-1})/\xi_1\) and \(-\eta_{n-1}/\eta_1\), respectively, one sees that \(-(u_2\xi_2 + \cdots + u_{n-1}\xi_{n-1})/\xi_1\) and \(\eta_{n}/\eta_1\) are algebraically independent over \(k(u_2, \ldots, u_{n-1})\). Hence by the same corollary we have that 
\[ k(u_2, \ldots, u_{n-1})(u_n)(-(u_2\xi_2 + \cdots + u_{n-1}\xi_{n-1})/\xi_1 - u_n\xi_n/\xi_1) = k(u_2, \ldots, u_{n-1})(u_n)(u) = k(u) \]
quasi-algebraically closed in \(\Sigma(u_n) = k(u)(\eta)\).

**Lemma 4.** Let \(K\) be a regular finitely generated extension of an infinite field \(k\) with \(\text{tr. deg} K \geq 3\). Let \(x, y, z\) be three elements of \(K\) algebraically independent over \(k\), and \(z/x \in K^p/k\), where \(p\) is the characteristic of \(k\). Then for all but a finite number of constants \(c \in k\), \(K\) is a regular extension of \(k(y + cz/x)\). Moreover, let \(\tau\) be an indeterminate \(K(\tau)\) is regular over \(k(\tau)(y + \tau z/x)\).

**Proof.** [5; Lemma 3].

**Theorem 3.** If \(V/k\) is an absolutely irreducible variety of dimension \(r \geq 3\) defined over an infinite field \(k\), then \(V \cap H_u/k(u)\) is an absolutely irreducible variety.

**Proof.** \(V \cap H_u/k(u)\) is irreducible. Let \((\xi)\) be a generic point of \(V \cap H_u\) over \(k(u)\). By Lemma 3, \((\xi)\) is a generic point of \(V\) over \(k\), hence \(\text{tr. deg} k(\xi) \geq 3\) and \(k(\xi)\) is a regular extension over \(k\) by [12; Proposition 1, p. 69]. Let \(\xi_1, \xi_2,\) and \(\xi_n\) be three elements in a separable transcendental basis of \(k(\xi)\) over \(K\). Let \(K = k(u_2, \ldots, u_{n-1})(\xi), u_n\) is algebraically independent over \(K\). Viewing \(k(u_2, \ldots, u_{n-1})(\xi)\) as the field \(k\) and \(u_n\) as the \(\tau\) in Lemma 4, we have \(K(u_n) = k(u)(\xi)\). Let 
\[ y = -(u_2\xi_2 + \cdots + u_{n-1}\xi_{n-1}), z = \xi_n \text{ and } x = \xi_1, \]
then \(x, y\) and \(z\) are
algebraically over $k(u_2, \ldots, u_n)$. By [6, Proposition 1, p. 185] and [6; corollary to Proposition 2, p. 186], $\frac{z}{x} = -\frac{\xi_n}{\xi_1} \in K^\prime k(u_2, \ldots, u_n)$, we have that $K(u_n)$ is a regular extension over

$$k(u_2, \ldots, u_n)(u_n)\left(\frac{y - u_{n-1}}{k}\right) = k(u) \, .$$

Therefore $k(u)(\xi)$ is a regular extension over $k(u)$, hence $V \cap H_n/k(u)$ is an absolutely irreducible variety.

Let $\{F_1, \ldots, F_s\}$ be a set of generators of $p$ in $k[x]$. Let $P$ be a point on $V$. According to [14], $P$ is $k$-simple on $V$ if and only if the mixed Jacobian of $\{F_1, \ldots, F_s\}$ is of rank $n - r$ at $P$. When $k(P)$ is separable over $k$, $P$ is $\Lambda$-simple on $V$ if and only if the classical Jacobian of $\{F_1, \ldots, F_s\}$ is of rank $n - r$ at $P$.

Following Theorem 1, we denote $p_a$ as the sole isolated component of $(p, H_n)$ and $W_a/k(u)$ as its variety in the sequel.

**THEOREM 4.** Let $V/k$ be of dimension $r \geq 2$. Then $P \in W_a$ is $k(u)$-simple if and only if $P$ is $k$-simple on $V$.

**Proof.** Let $P \in W_a$ be $k$-simple on $V$. By Theorem 1, $(p, H_n) = p_a \cap \mathfrak{A}$, where $\mathfrak{A}$ is the embedded component with $(X_1, \ldots, X_n)$ as radical. Let $(\eta)$ be a generic point of $V$ over $k(u)$, and let $(\xi)$ be a generic point of $W_a$ over $k(u)$. Let $k(u)[\eta]_p$ and $k(u)[\xi]_p$ be the local rings of $V$ and $W_a$ at $P$ respectively. $k(u)[\eta]_p$ is regular and

$$k(u)[\xi]_p \cong k(u)[\eta]_p/\overline{p}_a \cdot k(u)[\eta]_p \, ,$$

where $\overline{p}_a$ is the residue of $p_a$ modulo $p$. If $P \neq (0)$, let $\mathfrak{A}$ be the residue of $\mathfrak{A}$ modulo $p$ and let $m_p$ be the maximal ideal of $k(u)[\eta]_p$, then $\mathfrak{A}k(u)[\eta] \subset m_p$. For otherwise $(\eta_1, \ldots, \eta_n)_p \subset m_p$ for some integer $\rho > 0$, as $(X_1, \ldots, X_n)_p \subset \mathfrak{A}$. Thus $P = (0)$, a contradiction. Therefore, when $P \neq (0)$, $(\Sigma u_i \eta_i) \cdot k(u)[\eta]_p = \overline{p}_a \cdot k(u)[\eta]_p$, and $k(u)[\xi]_p \cong k(u)[\eta]_p/(\Sigma u_i \eta_i)k(u)[\eta]_p$. By [16; Th. 26, p. 303], to show that $k(u)[\xi]_p$ is regular it is sufficient to show that $\Sigma u_i \eta_i \in m_p^\rho$. But this is the case, for if $\Sigma u_i \eta_i \in m_p^\rho$, taking partial derivatives with respect to $u_i$ for $i = 1, 2, \ldots, n$, we have $\eta_i \in m_p$ for $i = 1, 2, \ldots, n$, i.e., $P = (0)$ a contradiction. Therefore $k(u)[\xi]_p$ is regular. If $P = (0)$, then $0$ is $k$-normal on $V$. By Corollary 2 to Theorem 1, $(p, H_n) = p_a$. In viewing [14, Th. 7, p. 28], we let $F_1, \ldots, F_s$ be a basis of $p$, and let $F_i$'s and $X_i$'s be so arranged that $(\det(\partial F_i/\partial X_j))_{(0)} \neq 0$, where $i, j = 1, 2, \ldots, n - r$, and the subscript $(0)$ means that we replace $(X)$ by $(0)$ after the determinant of the Jacobian is formed, as the rank of

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1 If $P \neq 0$, and if $P$ is $k$-simple on $V$, then $P$ remains simple on $W_a/k(u)$ follows also from [13; the theorem of Bertini, p. 138].
Consider
\[ J(F_1, \ldots, F_s; X_1, \ldots, X_n)(0) = n - r. \]

where \( \eta - r + 1 \leq j < \eta \). If \( \Delta_j = 0 \) for some \( j \) then \( u_1, \ldots, u_{\eta-r}, u_j \)
are algebraically dependent over \( k \). This is a contradiction, hence \( (0) \) is \( k \)-simple on \( W_u \). Conversely, assume that \( P \in W_u \) is \( k(u) \)-simple on \( W_u \). If \( P \neq (0) \), we have \( k(u)[\xi,p] \cong k(u)[\eta,p] \cdot k(u)[\eta,p] \) from the above. If \( P = (0) \), then \( P \) is \( k(u) \)-normal on \( W_u \). By Theorem 6 in the following \( V/k \) is normal at \( (0) \), therefore \( (p, H_u) = p,u \) and \( k(u)[\xi,p] \cong k(u)[\eta,p] \cdot k(u)[\eta,p] \). Therefore \( k(u)[\xi,p] \cong k(u)[\eta,p] \cdot k(u)[\eta,p] \) if \( P \) is \( k(u) \)-simple on \( W_u \). Since \( h((\Sigma u_i \eta_i) \cdot k(u)[\eta,j]) = 1 \), it follows from [8; (9; 11), p. 28] that \( k(u)[\eta,p] \) is a regular local ring. Hence \( P \)
is \( k \)-simple on \( V \).

By an argument similar to the proof of Lemma 2, we have the following.

**COROLLARY.** If \( V/k \) is of dimension \( r \geq 3 \) and if \( V/k \) is locally free of \( (r-1) \)-dimensional singularities, then \( V \cap H_u/k(u) \) is locally free of \( (r-2) \)-dimensional singularities.

Note. If \( r = 2 \), the corollary is clearly false as one sees by taking \( V \) to be a cone with vertex at \( (0) \).

**THEOREM 5.** If \( V/k \) is a complete intersection of dimension \( \geq 3 \) and if \( V \) is \( k \)-normal at \( (0) \), then the generic hyperplane section \( V \cap H_u \) is also \( k(u) \)-normal at \( (0) \).

Proof. \( V/k(u) \) is \( k(u) \)-normal at \( (0) \), by Lemma 1. By corollary to Theorem 1, \( (p, H_u) = p,u \) is prime. For any polynomial \( F \neq 0 \) in \( k(u)[X] \), by [7; Th. p. 49] or [16; Th. 26, p. 203], \( (p,u,F) = (p, H_u,F) \) is unmixed. Hence, passing to the quotient modulo \( p,u \), we have that every nonzero principal ideal in the coordinate ring \( k(u)[\xi] \) of \( V \cap H_u \)
is unmixed. It follows that every nonzero principal ideal in the local ring of \( V \cap H_u \) at \( (0) \), \( k(u)[\xi,0] \), is also unmixed. Since \( V/k \) is \( k \)-normal at \( (0) \), therefore \( V/k \) is locally free of \( (r-1) \)-dimensional singularities at \( (0) \). By the above corollary, \( V \cap H_u \) is locally free of \( (r-2) \)-dimensional singularities at \( (0) \). It follows from Proposition 1 that \( V \cap H_u \) is \( k(u) \)-normal at \( (0) \).
Theorem 6. If $V \cap H_u$ is $k(u)$-normal at $(0)$, then $V/k$ is normal at $(0)$.

Proof. This theorem is really a consequence of [3; Lemma 4, p. 360] ([8; (36.9), p. 134]). Indeed, let $(\gamma)$ be a generic point of $V$ over $k(u)$. Passing to $k(u)[\gamma]$, by Theorem 1, we have $(u, \gamma_1, \ldots + u_n \gamma_n) \cdot k(u)[\gamma] = \bar{p}_u \cap \bar{q}$, where $\bar{p}_u$ and $\bar{q}$ are residues of $p_u$ and $q$ modulo $p$ respectively. It is clear that (1) $(u, \gamma_1, \ldots + u_n \gamma_n) \cdot k(u)[\gamma]_{(0)} = \bar{p}_u \cdot k(u)[\gamma]_{(0)}$, (2) $(u, \gamma_1, \ldots + u_n \gamma_n) \cdot k(u)[\gamma]_{(0)} = \bar{p}_u = \bar{p}_u \cdot k(u)[\gamma]_{(0)}$, and (3) let $(\xi)$ be a generic point of $V \cap H_u$ over $(0)$, then

$$\frac{k(u)[\gamma]_{(0)}}{\bar{p}_u, k(u)[\gamma]_{(0)}} \cong k(u)[\xi]_{(0)},$$

which is integrally closed as $V \cap H_u$ is $k(u)$-normal at $(0)$. Moreover, let $k(u)[\gamma]_{(0)}$ be the integral closure of $k(u)[\gamma]_{(0)}$ in $k(u)[\gamma]$, and let $p'$ be a minimal prime divisor of $(u, \gamma_1, \ldots + u_n \gamma_n) \cdot k(u)[\gamma]_{(0)}$. It follows from [2; Th. 2, p. 253] and [2; Th. 3; p. 254] that $ht(p' \cap k(u)[\gamma]_{(0)}) = htp = 1$. Therefore $p' \cap k(u)[\gamma]_{(0)} = p_u$, i.e., every minimal prime divisor of $(u, \gamma_1, \ldots + u_n \gamma_n) \cdot k(u)[\gamma]_{(0)}$ lies over $p_u$. The above verify the conditions of [3; Lemma 4, p. 360], therefore $k(u)[\gamma]_{(0)}$ is integrally closed.

3. The local normal problem. Throughout this section let $V/k$ be a variety of dimension $r \geq 3$, passing through $(0)$ with $(\xi)$ as a generic point over $k$ and let $H_u: u_1 X_1 + \ldots + u_n X_n = 0$ be a generic hyperplane through $(0)$. If $V/k$ is normal at $(0)$, is it true that $H_u \cap V$ is $k(u)$-normal at $(0)$? If $V/k$ is a complete intersection then by Theorem 5, the answer to the question is yes. However we shall prove the answer to the question is negative in general.

Definition 4. (a) Let $R$ be a Noetherian ring. Subset $\{a_1, \ldots, a_q\}$ of $R$ is a prime sequence if for each $i = 1, 2, \ldots, q$, $a_i$ is not a zero divisor in the ring $R/(a_1, \ldots, a_i)$. If $R/(a_1, \ldots, a_i)$ is a local ring, the number of elements of a maximal prime sequence in $R$ is called the homological co-dimension of $R$, and is denoted by $\text{cod}_R(A)$. If $\text{cod}_R(A) = \dim A$, we say that $A$ is a Cohen-Macaulay ring.

For a general commutative ring $R$ and a multiplicative system $S$ which does not contain 0, it is well known [15, p. 219] that $(\mathfrak{a}; \mathfrak{b})^e \subset \mathfrak{a}^e$ and $(\mathfrak{x}; \mathfrak{y})^e \subset \mathfrak{x}^e; \mathfrak{y}^e$, where $(*)^e = (* \cdot R_S)$, $(*)^e = f^{-1}(*), f$ is the canonical homomorphism of $R$ into $R_S$ and where $\mathfrak{a}, \mathfrak{b}$ are two ideals in $R$, and $\mathfrak{x}, \mathfrak{y}$ are two ideals in $R_s$. 
PROPOSITION 5. Let $\mathfrak{A}, \mathfrak{B}, X$ and $Y$ be the same as above. Then 
(a) $(\mathfrak{A}; \mathfrak{B})^r = \mathfrak{A}; \mathfrak{B}$; if $\mathfrak{A} \supset \text{Ker } f$ and $\mathfrak{B}$ is finitely generated, also (b) $(X; Y)^r = X; \mathfrak{B}$ if $Y$ is finitely generated.

Proof. Let $\mathfrak{B} = (b_1, \cdots, b_i)R$, we have $\mathfrak{B}^r = (f(b_1), \cdots, f(b_i)) \cdot R^r$. Let $x \in \mathfrak{A}; \mathfrak{B}^r$. Then $x\mathfrak{B}^r \subset \mathfrak{A}^r$ and $xf(b_i) = f(a_i)/f(b_i)$ for some $a_i \in \mathfrak{A}$ and $b_i \in S$. Therefore $f(\pi, s)xf(b_i) \in f(\mathfrak{A})$. For each $b \in f(\mathfrak{B})$, $b = \sum_i f(r_i) f(b_i)$ for some $r_i \in R$. Now $f(\pi, s) x b = \sum j f(\pi, s) x f(r_i) f(b_i) \in f(\mathfrak{A})$, which implies that $f(\pi, s) x \in f(\mathfrak{A}) \cdot f(\mathfrak{B})$. Hence $x \in (f(\mathfrak{A}) \cdot f(\mathfrak{B}))^r$. Since $\mathfrak{A} \supset \text{Ker } f$, by [15; (15), p. 148], $f(\mathfrak{A}) \cdot f(\mathfrak{B}) = f(\mathfrak{A}; \mathfrak{B})$. Therefore $x \in (\mathfrak{A}; \mathfrak{B})^r$ and $\mathfrak{A}^r; \mathfrak{B}^r = (\mathfrak{A}; \mathfrak{B})^r$. The proof of (b) is similar.

LEMMA 5. $k(u)[\xi]_0$ is Cohen-Macaulay if and only if $k[\xi]_0$ is Cohen-Macaulay, where $k[\xi]$ is the coordinate ring of $V/k$, and $u$ is an indeterminate over $k(\xi)$.

Proof. If $k[\xi]_0$ is Cohen-Macaulay, then there exist $\zeta, \cdots, \zeta_r$ such that $(\zeta, \cdots, \zeta_r)$ forms a maximal prime sequence, where $r = \text{dim } V$. Thus $(\zeta, \cdots, \zeta_r) k[\xi]_0: (\zeta_{i+1}) \cdot k[\xi]_0 = (\zeta, \cdots, \zeta_r) \cdot k[\xi]_0$ for $i = 1, 2, \cdots, r$. By [15; (1), p. 227], [15; (15), (21), p. 148] Proposition 5 and [16; (3), p. 221] one has $(\zeta, \cdots, \zeta_r) k(u)[\xi]_0: (\zeta_{i+1}) k(u)[\xi]_0 = (\zeta, \cdots, \zeta_r) k(u)[\xi]_0$ for $i = 1, 2, \cdots, r$. Therefore $(\zeta, \cdots, \zeta_r)$ remains as a maximal prime sequence of $k(u)[\xi]_0$. Thus $k(u)[\xi]_0$ is Cohen-Macaulay.

Conversely, let $k(u)[\xi]_0$ be Cohen-Macaulay, let $(\zeta(u; \xi), \cdots, \zeta(u; \xi))$ be a maximal prime sequence of $k(u)[\xi]_0$. Then, for $i = 1, 2, \cdots, r$, we have $(\zeta(u; \xi), \cdots, \zeta(u; \xi)) k(u)[\xi]_0: (\zeta_{i+1}(u; \xi)) k(u)[\xi]_0 = (\zeta(u; \xi), \cdots, \zeta(u; \xi)) k(u)[\xi]_0$. By [15; (21), p. 148], going back to the polynomial ring $k(u)[x]$, we have $(\zeta(u; x), \cdots, \zeta(u; x), p) k(u)[x]_0: (\zeta_{i+1}(u; x), p) k(u)[x]_0 = (\zeta(u; x), \cdots, \zeta(u; x), p) k(u)[x]_0$. In viewing [4; Satz 3, p. 59], one sees that

$$(\zeta(u; x), \cdots, \zeta(u; x), p) k(u)[x]_0$$

almost always for $i = 1, 2, \cdots, r$, where the bar means specialization of $u$ to elements in $k$. Passing to the local ring of $V/k(u)$ at $(0)$, by [15; (15), p. 148], we have $\overline{(\zeta(u; \xi), \cdots, \zeta(u; \xi)) k(u)[\xi]_0} = (\zeta(u; \xi), \cdots, \zeta(u; \xi)) k(u)[\xi]_0$ almost always for $i = 1, 2, \cdots, r$. Let $a \in k$ be such that the above holds and $\zeta_i(a; \xi) \neq 0$, for $i = 1, 2, \cdots, r$, then $(\zeta_i(a; \xi), \cdots, \zeta_i(a; \xi)) k[\xi]_0: (\zeta_{i+1}(a; \xi)) k[\xi]_0 = (\zeta_i(a; \xi), \cdots, \zeta_i(a; \xi)) k[\xi]_0$ for $i = 1, 2, \cdots, r$. Therefore $(\zeta_i(a, \xi), \cdots, \zeta_r(a, \xi))$ forms a system of prime sequence of $k[\xi]_0$. Hence $k[\xi]_0$ is Cohen-Macaulay.

THEOREM 7. Let $V/k$ and $H_u$ be the same as the above. It is not
true in general that if $V/k$ is $k$-normal at $(0)$, then $V \cap H_u/k(u)$ is $k(u)$-normal at $(0)$.

Proof. Suppose that if $V/k$ is $k$-normal at $(0)$, then $V \cap H_u/k(u)$ is $k(u)$-normal at $(0)$. Let $(\xi)$ be a generic point of $V$ over $k$ and let $(\eta)$ be that of $V \cap H_u$ over $k(u)$. Applying the supposition to $V \cap H_u/k(u)$, we get $(V \cap H_u) \cap H_u(\eta)$-normal at $(0)$, where

$$H_u(\eta): u_1X_1 + \cdots + u_nX_n = 0$$

is a generic hyperplane through $(0)$ on $V \cap H_u/k(u)$ and $u(2) = \{u_1, \ldots, u_n\}$ are algebraically independent over $k(\xi, \eta)$. Repeating the supposition and Corollary 2 to Theorem 1 in this way until dimension $r$ of $V$ is cut down to 2, we have then

$$V \cap H_u \cap H_u(\eta) \cap \cdots \cap H_u(\eta_{r-2}) \cap k(u, u(2), \ldots, u(\gamma - 2))$$

at $(0)$, where $u(i) = \{u_{i1}, \ldots, u_{im}\}$, and {$u_{i1}, \ldots, u_{im}$} are indeterminates over $k(u, u(2), \ldots, u(i - 1)(\xi, \eta, \eta_{i1}, \ldots, \eta_{im})$ being a generic point of $V \cap H_u \cap H_u(\eta) \cap \cdots \cap H_u(\eta_{i-1})$ over $k(u, u(2), \ldots, u(i))$. Let $U = \{u, u(2), \ldots, u(\gamma - 2)\}$, then $k(U) = k(u, u(2), \ldots, u(\gamma - 2))$. Consider $V/k(U), (\xi)$ is a generic point of $V$ over $k(U)$. Correspondingly in the coordinate ring $k(U)[\xi]$ of $V$ over $k(U)$, we have then $r - 2$ quantities $\xi = u_1\xi_1 + \cdots + u_m\xi_m, i = 1, 2, \ldots, r - 2$, such that $(\xi, \ldots, \xi)$ is a prime ideal in $k(U)[\xi]$. Thus {$\xi, \ldots, \xi_{r-2}$} is a prime sequence in the local ring $k(U)[\xi]$ over $k(U)$ we have then $r - 2$ quantities $\xi = u_1\xi_1 + \cdots + u_m\xi_m, i = 1, 2, \ldots, r - 2$, such that $(\xi, \ldots, \xi)$ is a prime ideal in $k(U)[\xi]$. Therefore $\dim k(U)[\xi] = \text{codim} k(U)[\xi]$ and hence $k(U)[\xi]$ is a Cohen-Macaulay ring. It follows from Lemma 5 that $k[\xi]_{(0)}$ is a Cohen-Macaulay ring. So under the supposition, we conclude that $k[\xi]_{(0)}$ is integrally closed implies that $k[\xi]_{(0)}$ is Cohen-Macaulay. But on the other hand, [1; Proposition, p. 655] and [1; Th. 5, p. 653] yield an example of a local ring of an algebraic variety at a rational point which is a factorial local ring (hence normal), but not a Cohen-Macaulay local ring. Hence the above supposition yields a contradiction.

**Theorem 8.** If $V/k$ is normal at $(0)$, and the local ring $k[\xi]_{(0)}$ is a Cohen-Macaulay ring, then $V \cap H_u/k(u)$ is normal at $(0)$.

**Proof.** By the corollary to Theorem 4, $(\nu, H_u)$ is free of $(\gamma - 2)$.
dimensional singularities. By Lemma 5, $k(u)[\xi](0)$ is Cohen-Macaulay. For any nonzero $a(u; \xi)k(u)[\xi](0)$ not in the prime ideal 

$$(u, \xi, u\xi_1 + \cdots + u_n\xi_n) \in k(u)[\xi](0)$$

forms a prime sequence of $k(u)[\xi](0)$, therefore by [16; Lemma 5, p. 401], $(a(u, \xi), u\xi_1 + \cdots + u_n\xi_n)k(u)[\xi](0)$, is unmixed. Hence every nonzero principal ideal of $k(u)[\xi](0)/(u, \xi, u\xi_1 + \cdots + u_n\xi_n)k(u)[\xi](0)$, is unmixed. It follows from Proposition 1 that $V \cap H_u$ is $k(u)$-normal at $(0)$.

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