ON THE HYPERPLANE SECTION THROUGH A RATIONAL POINT OF AN ALGEBRAIC VARIETY

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Let $V/k$ be an irreducible affine algebraic variety of dimension $\geq 3$ defined over an infinite field $k$ with $\wp$ as its prime ideal in $k[X_1, \ldots, X_n]$. Let $P$ be a rational normal point on $V/k$. It is proved that (1) for a generic hyperplane $H_{u}$ through $P$, $(\wp, H_{u})$ is a prime ideal and $(\wp, H_{u})$ is quasi-absolutely (absolutely irreducible) if $\wp$ is quasi-absolutely (absolutely irreducible). (2) It is not true in general that $V \cap H_{u}$ is normal at $P$; however, $V \cap H_{u}$ is normal at $P$ if the local ring of $V/k$ at $P$ is also Cohen-Macaulay (Theorem 8).

It is well known [11] that if $V/k$ is a normal variety of dimension $\geq 2$, then for almost all hyperplanes $H$ the section $V \cap H$ is again a normal variety. This research is motivated by this result to study the following problem: If $V/k$ is normal at a rational point $P$ on $V$, will hyperplane sections of $V$ through $P$ be normal at $P$? Section 1 localizes some of the results of [11]. Section 2 describes the ideal decomposition of the generic hyperplane section through a given rational point of an irreducible variety, and Section 3 gives a negative answer to the problem of normality. As a consequence the converse of [3; Lemma 4, p. 360] is invalid in general.

1. Generalities. In the following and the subsequent sections, a variety $V/k$ shall mean an irreducible algebraic variety in the affine space $A^n$ defined over a field $k$ of arbitrary characteristic.

Recall the following definitions.

**Definition 1.** Let $V/k$ be a variety with $(\zeta) = (\xi_1, \ldots, \xi_n)$ as a generic point over $k$, and let $P$ be a point on $V$. Let $k[\zeta]_{\wp} = \left\{ \frac{f(\zeta)}{g(\zeta)} \mid f, g \in k[\zeta] \text{ and } g(P) \neq 0 \right\}$ be the local ring of $V$ at $P$ in the function field $k(\zeta)$ of $V$ over $k$. We say that $P$ is $k$-normal on $V$ if $k[\zeta]_{\wp}$ is integrally closed in $k(\zeta)$, that $P$ is $k$-simple on $V$ if $k[\zeta]_{\wp}$ is a regular local ring, and that $P$ is singular on $V$ if $P$ is not $k$-simple on $V$.

**Definition 2.** Let $V/k$ be a variety of dimension $r$, and let $P$ be a point on $V$. We say that $V/k$ is locally free of $s$-dimensional
singularities at \(P\) if every \(s\)-dimensional subvariety of \(V\) containing \(P\) is \(k\)-simple on \(V\).

**Definition 3.** Let \(R\) be a finite integral domain \(k[\xi_1, \ldots, \xi_s]\) over a field \(k\) or a localization thereof relative to a prime ideal of \(k[\xi_1, \ldots, \xi_s]\). Let \(\mathfrak{p}\) be a prime ideal of \(R\) we define

\[
ht \mathfrak{p} = \max (\text{length of chains of prime ideals contained in } \mathfrak{p}),
\]

\[
deepth \mathfrak{p} = \max (\text{length of chains of prime ideals containing } \mathfrak{p}),
\]

\[
dim \mathfrak{p} = \text{transcendence degree of the quotient field of } R/\mathfrak{p} \text{ over } k,
\]

\[
dim R = \text{transcendence degree of the quotient field of } R \text{ over } k.
\]

It is well known that \(ht \mathfrak{p} + \deepth \mathfrak{p} = \dim R\) and \(\dim \mathfrak{p} = \deepth \mathfrak{p}\).

The following criterion for local normality is parallel to [11; Th. 3, p. 363] and is well known [8; (12.9), p. 41].

**Proposition 1.** Let \(V/k\) be a variety of dimension \(r\) defined over a field \(k\), and let \(P\) be a point of dimension \(s\) on \(V\). \(P\) is \(k\)-normal on \(V\) if and only if

1. \(V/k\) is locally free of \((r-1)\)-dimensional singularities at \(P\),
2. every nonzero principal ideal \((a) k[\xi]_{\mathfrak{p}}\) is unmixed of dimension \(r-s-1\).

**Proposition 2.** Let \(V/k, (\xi), \) and \(P\) be the same as those in Proposition 1, let \(k[\xi]_{\mathfrak{p}}^*\) be the integral closure of \(k[\xi]_{\mathfrak{p}}\), and let \(\mathcal{C}_\mathfrak{p}\) be the conductor of \(k[\xi]_{\mathfrak{p}}\). If \(V\) is locally free of \((r-1)\)-dimensional singularities at \(P\) and if \(\mathcal{C}_\mathfrak{p} \neq (1)\), then every nonzero element of \(\mathcal{C}_\mathfrak{p}\) generates a mixed principal ideal.

**Proof.** Let \(\alpha \in k[\xi]_{\mathfrak{p}}^*\) not in \(k[\xi]_{\mathfrak{p}}\), and let \(c \in \mathcal{C}_\mathfrak{p}\), whence \(c\alpha \in k[\xi]_{\mathfrak{p}}\), say \(c\alpha = b, b \in k[\xi]_{\mathfrak{p}}\). Then \((c) \cdot k[\xi]_{\mathfrak{p}}\) must be mixed. Indeed, if \((c) k[\xi]_{\mathfrak{p}}\) were unmixed, and let \(\mathfrak{p}_1, \ldots, \mathfrak{p}_t\) be the associated prime ideals of \((c) k[\xi]_{\mathfrak{p}}\), then \(\dim \mathfrak{p}_i = r-s-1\), for \(i = 1, 2, \ldots, t\). \(\alpha\) is integral over \(k[\xi]_{\mathfrak{p}}\), hence integral over \((k[\xi]_{\mathfrak{p}})^{\mathfrak{p}_i}\) for \(i = 1, 2, \ldots, t\). By hypothesis \((k[\xi]_{\mathfrak{p}})^{\mathfrak{p}_i}\) is a regular local ring of dimension 1, for \(i = 1, 2, \ldots, t\), therefore \((k[\xi]_{\mathfrak{p}})^{\mathfrak{p}_i}\) is integrally closed for \(i = 1, 2, \ldots, t\). Hence \(\alpha \in \bigcap_{i=1}^t (k[\xi]_{\mathfrak{p}})^{\mathfrak{p}_i}\) and \(b \in (\bigcap_{i=1}^t (c) (k[\xi]_{\mathfrak{p}})^{\mathfrak{p}_i}) \cap k[\xi]_{\mathfrak{p}} = \bigcap_{i=1}^t q_i\), where \(q_1 \cap \cdots \cap q_t\) is a primary decomposition of \((c) k[\xi]_{\mathfrak{p}}\). Thus \(b \in (c) b[\xi]_{\mathfrak{p}}\), i.e., \(\alpha \in k[\xi]_{\mathfrak{p}}\), a contradiction.

Let \(V/k\) be a variety of dimension \(r\) defined over a field \(k\) with \((\xi)\) as a generic point, and let \(P\) be a point on \(V\). Let \(u\) be an indeterminate over \(k(\xi)\), it is well known that \(V\) is a variety over \(k(u)\) with \((\xi)\) as a generic point of \(V\) over the pure transcendental extension field \(k(u)\). Let \(k(u)[\xi]_{\mathfrak{p}} = \{ f(u; \xi)/g(u; \xi) \mid f, g \in k(u)[\xi] \text{ and } g(u; \mathfrak{p}) \neq 0\} \)
be the local ring of \( V \) at \( P \) over \( k(\omega) \). We have, by [10, (d), p. 64], the following lemma.

**Lemma 1.** \( k[\xi]_p \) is integrally closed if and only if \( k(\omega)[\xi]_p \) is integrally closed.

Recall the definition of the ground form of an unmixed \( r \)-dimensional ideal \( \mathfrak{a}' \), [11; p. 373], as following: Let \( \mathfrak{a} \) be an unmixed \( r \)-dimensional ideal in the polynomial ring \( k[X_1, \cdots, X_n] \), we form \( r + 1 \) linear forms in the \( X_i \)'s with indeterminates coefficients \( u_{i,j} \); \( z_i = u_{i,1} x_1 + \cdots + u_{i,n} x_n \), \( i = 1, 2, \cdots, r + 1 \), and consider the ideal \( \mathfrak{a} \cdot k(\omega)[X] \cap k(\omega)[z_1, \cdots, z_{r+1}] \), where \( k(\omega)[X] = k(\omega_1, \cdots, \omega_{r+1})[X_1, \cdots, X_n] \), which is a principal ideal \( (E(z_1, \cdots, z_{r+1}; \omega)) \) in \( k(\omega)[X] \). If \( E \) is normalized so as to be a polynomial in the \( u_{i,j} \) and primitive in them, so that \( E \) is defined to within a factor in \( k \), then \( E \) is the elementary divisor form or the ground form of \( \mathfrak{a} \). The polynomial \( E \) is integral in any \( z_i \) over the other \( z_i \)'s and is a polynomial in \( z_1, \cdots, z_{r+1} \) of least degree in \( z_{r+1} \), which is in \( \mathfrak{a} \cdot k(\omega)[X] \). If \( \mathfrak{a} \) is prime, then its ground form is irreducible, the converse is not true in general; but \( \mathfrak{a} \) is primary if and only if its ground form is a power of an irreducible polynomial [9; Th. 9, p. 252]. \( \mathfrak{a} \) is prime and absolutely irreducible if and only if \( (E) \) is prime and absolutely irreducible [9; Th. 15, p. 259]. If \( \mathfrak{a} \) is prime and quasi-absolutely irreducible, then \( (E) \) is prime and quasi-irreducible [11, p. 373].

**Proposition 3.** Let \( V/k \) be an \( r \)-dimensional variety defined over a field \( k \) with \( \mathfrak{p} \) as its prime ideal in \( k[X] (=k[X_1, \cdots, X_n]) \). Let \( P \) be a point on \( V \) and let \( E \) be the ground form of \( \mathfrak{p} \). Then \( V \) is \( k \)-normal at \( P \) if and only if \( (\mathfrak{p}, \partial E/\partial z_{r+1}) \cdot k(\omega)[X]_\mathfrak{p} \) is unmixed.

**Proof.** By Lemma 1, \( V \) is \( k \)-normal at \( P \) if and only if \( V \) is \( k(\omega) \)-normal at \( P \). By [13; Lemma 2, p. 132] \( V/k(\omega) \) is free of \((r - 1)\)-dimensional singularities at \( P \). Let \( (\xi) \) be a generic point of \( V/k(\omega) \), and pass to \( k(\omega)[\xi] \), we assert that \( k(\omega)[\xi]_\mathfrak{p} \) is integrally closed if and only if \( (\partial E/\partial z_{r+1}) \cdot k(\omega)[\xi]_\mathfrak{p} \) is unmixed, where the bar denotes residue. By the proof of [11; Th. 5, p. 365], we have \( \partial E/\partial z_{r+1} \in \mathcal{C} \), the conductor of \( k(\omega)[\xi] \) in its integral closure \( k(\omega)[\xi] \). Let \( \mathcal{C}_\mathfrak{p} \) be the conductor of \( k(\omega)[\xi]_\mathfrak{p} \) in its integral closure \( k(\omega)[\xi]_\mathfrak{p} \). By [15; Lemma, p. 269], \( \mathcal{C} \cdot k(\omega)[\xi]_\mathfrak{p} = \mathcal{C}_\mathfrak{p} \). Therefore \( \partial E/\partial z_{r+1} \in \mathcal{C}_\mathfrak{p} \). By Proposition 2, we have that \( k(\omega)[\xi]_\mathfrak{p} \) is integrally closed if and only if \( (\partial E/\partial z_{r+1}) \cdot k(\omega)[\xi]_\mathfrak{p} \) is unmixed.

2. Irreducibility of generic hyperplane section through a normal point. Let \( V/k \) be a variety of dimension \( r \geq 2 \). Let \( P \in V \) be a rational point. We are studying the generic hyperplane section
of $V$ through $P$. Without loss of generality, we may assume once for all in the sequel that $V$ passes through $(0)$ the origin of the affine space and that $P = (0)$. We shall denote the prime ideal of $V/k$ by $p$ in the sequel. Let $u_1, \ldots, u_n$ be $n$ indeterminates over $k$, and let $H_u$ be the generic hyperplane through $(0)$ defined by $u_1X_1 + \cdots + u_nX_n = 0$. We shall use $H_u$ in two senses whenever it is proper: (1) $H_u$ means the linear polynomial $u_1X_1 + \cdots + u_nX_n$ in $k(u)[X]$ ($= k(u_1, \ldots, u_n)$ [$X_1, \ldots, X_n$]), (2) $H_u$ stands for the hyperplane defined by $u_1X_1 + \cdots + u_nX_n = 0$. Let $k(u) = k(u_1, \ldots, u_n)$, $V$ is a variety over $k(u)$ and $V \cap H_u$ is defined over $k(u)$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$, $s \leq t$, be the isolated prime ideals. Since $(0) \in V$, $(\mathfrak{p}_1, \ldots, \mathfrak{p}_t) \subset (X_1, \ldots, X_n) \cdot k(u)[X]$. Hence $(X_1, \ldots, X_n) \cdot k(u)[X]$ must contain at least one of the $\mathfrak{p}_i$, $i \leq s$, say $\mathfrak{p}_t$. Let us denote $\mathfrak{p}_i$ by $p_i$ and let $W_u$ be the variety over $k(u)$ of $\mathfrak{p}_u \cdot W_u$ is of dimension $r - 1$ as it is well known that any component of $V \cap H$, where $H$ is a hypersurface, is of dimension $r - 1$. Let $(\xi)$ be a generic point of $W_u$ over $k(u)$. Since $\text{tr. deg}_{k(\xi)} k(u; \xi) + \text{tr. deg}_{k} k(\xi) = \text{tr. deg}_{k} k(u; \xi) + \text{tr. deg}_{k(u)} k(u; \xi) = n + r - 1$ and $\text{tr. deg}_{k(\xi)} k(u; \xi) \leq n - 1$, we have $\text{tr. deg}_{k(\xi)} k(u; \xi) \geq r$. But $(\xi) \in V$, therefore $\text{tr. deg}_{k} k(\xi) = r$. We thus have

**Lemma 2.** If dim $V \geq 2$, a generic point of $W_u$ over $k(u)$ is also a generic point of $V$ over $k$.

**Lemma 3.** If $\xi_j \neq 0$, then $u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_n$ are algebraically independent over $k(\xi)$.

**Proof.** Say

$$i = 1, \text{tr. deg}_{k(u_2, \ldots, u_n)} k(u_1, \ldots, u_n; \xi)$$

$$+ \text{tr. deg}_{k} k(u_2, \ldots, u_n) = n + r - 1.$$ 

Therefore $\text{tr. deg}_{k(u_2, \ldots, u_n)} k(u_1, \ldots, u_n; \xi) = r$.

Since

$$\frac{u_2\xi_2 + \cdots + u_n\xi_n}{\xi_1} \in k(u_2, \ldots, u_n; \xi_1, \ldots, \xi_n),$$

we have $k(u_1, \ldots, u_n; \xi) = k(u_2, \ldots, u_n; \xi)$. Now

$$\text{tr. deg}_{k(\xi)} k(u_2, \ldots, u_n; \xi) + r = r + n - 1.$$ 

Therefore $\text{tr. deg}_{k(\xi)} k(u_2, \ldots, u_n; \xi) = n - 1$, i.e., $u_2, \ldots, u_n$ are algebraically independent over $k(\xi)$.

**Proposition 4.** Let $(\xi)$, $p_u$ and $W_u$ be as above. Then $(p, H_u)$:
(X_1, \cdots, X_n)^\rho = p_u$ for sufficiently large integers $\rho$, where $(X_1, \cdots, X_n) = (X, \cdots, X_n) \cdot k(u)[X]$.

Proof. Let $F(u_1, \cdots, u_n; X) \in p_u$ be a polynomial, we may assume $F(u_1, \cdots, u_n; X) \in k[u_1, \cdots, u_n][X]$. If $\xi \neq 0$, $F(u_1, \cdots, u_n; \xi) = 0$ implies that $F(-u_2^2 + \cdots + u_n^2 + \xi, u_2, \cdots, u_n; \xi) = 0$. Hence there exists a nonnegative integer $\sigma$ such that $X_i^\sigma$.

$$F(-u_2X_2 + \cdots + u_nX_n, u_2, \cdots, u_n; X) \in k(u_2, \cdots, u_n)[X]$$

vanishes at $(\xi)$. By Lemma 3, the prime ideal determined by $(\xi)$ in $k(u_2, \cdots, u_n)[X]$ is $p \cdot k(u_1, \cdots, u_n)[X]$. Thus

$$X_i^\sigma F(-u_2X_2 + \cdots + u_nX_n, u_2, \cdots, u_n; X) \in p \cdot k(u_1, \cdots, u_n)[X]$$

for sufficiently large $\sigma$. But

$$X_i^\sigma F(-u_2X_2 + \cdots + u_nX_n, u_2, \cdots, u_n; X) = 0 \mod u_1^\sigma, \cdots, u_n^\sigma$$

for sufficiently large $\sigma$. We have $X_i^\sigma F(u_1, \cdots, u_n; X) \in (p, H_u) \cdot k(u)[X]$ for sufficiently large $\sigma$. The above discussion is symmetric with respect to those $\xi_i \neq 0$. Therefore for any $\xi_i \neq 0$, we have $X_i^\sigma F(u_1, \cdots, u_n; X) \in (p, H_u)$ for sufficiently large integer $\sigma_i$ and for all $F \in p_u$. For any $j$ such that $\xi_j = 0$, $X_j \in p$. Thus $X_i^\sigma F(u_1, \cdots, u_n; X) \in (p, H_u)$ for any positive integer $\sigma_j$ and for all $F \in p_u$. Thus $(p, H_u) \cdot (X_1, \cdots, X_n)^\sigma \supset p_u$ for sufficiently large integer $\rho$. We now show the other inclusion. Let $g(u_1, \cdots, u_n; X)$ be an element in $(p, H_u) : (X_1, \cdots, X_n)^\sigma$. Then for any $h(u_1, \cdots, u_n; X) \in (X_1, \cdots, X_n)^\sigma$, $h(u_1; X) \cdot g(u_1; X) \in (p, H_u)$. Therefore, there exists $m_i(u; X), n(u; X) \in k(u)[X]$ such that $h(u; X) \cdot g(u; X) = \sum_{i=1}^r m_i(u; X) \cdot F_i(X) + n(u; X)H_u$, where $(F_1, \cdots, F_r) \cdot k[X] = p$. Thus $h(u; \xi) \cdot g(u; \xi) = 0$. If $g(u; \xi) \neq 0$, then $h(u; X) = 0$ at $(\xi)$ for all $h(u; X) \in (X_1, \cdots, X_n)^\sigma$, which implies that $(\xi) = 0$, a contradiction. Thus $g(u; X) = 0$ at $(\xi)$ and therefore $p \supset (p, H_u) : (X_1, \cdots, X_n)^\sigma$.

COROLLARY. $(p, H_u)$ has only one isolated component.

Proof. Suppose $p_2$ is another isolated component, by Proposition 4, we have $(p, H_u) : (X_1, \cdots, X_n)^\rho = p_u$, for sufficiently large integer $\rho'$. Hence we have $p_2 = (p, H_u) = (X_1, \cdots, X_n)^\rho = p_u$.

THEOREM 1. If $V/k$ is of dimension $r \geq 2$, then $(p, H_u) \cdot k(u)[X]$
is either a prime ideal \( p_u \) or an intersection of the prime ideal \( p_u \) with a primary ideal of which \( (X_1, \ldots, X_n) \cdot k(u)[X] \) is its radical.

**Proof.** Let \( \mathfrak{B} = (p, H_u) \) and let \( \mathfrak{B} = q_1 \cap \cdots \cap q_t \) be the irredundant primary representation of \( \mathfrak{B} \) with \( p_1, \ldots, p_t \) as the associated prime ideals. By the corollary, there exists only one isolated prime component, say \( q_i \), and denote \( p_t \) by \( p_u \). Let \( m = (X_1, \ldots, X_n) \cdot k(u)[X] \). Since \( S_3: m \cdot p = \mathfrak{m} \) for sufficiently large \( \rho \), we have \( (q_i: m) = \mathfrak{m} \). There are two possibilities (I) no \( p_i \) contains \( m^\lambda \) for any nonnegative integer \( \lambda \), or (II) some of \( p_i \) contains a power of \( m \). (I) leads to \( S_3 = p \cdot \mathfrak{m} \). In case of (II), say \( p_2 \) contains \( m^\lambda \) for some \( \lambda \) then \( m = p_2 \). We may assume that there is no other \( p_j \) to contain \( m^\lambda \) for any \( 0 \leq \lambda \in \mathbb{Z} \). Thus for \( i = 1, 3, 4, \ldots, r \), \( q_i: m^\lambda = q_i \cdot m^\lambda \cdot \cdots \cdot (q_r: m^\lambda) = q_i \cap q_3 \cap q_t \cap \cdots \cap q_i \), and thus \( p_u \cap q_i = (p, H_u) \).

**COROLLARY 1.** If \( V \) is normal over \( k \), then \( (p, H_u) = p_u \).

**Proof.** Passing to the coordinate ring of \( V, k(u)[\eta] \), we have that \( (u_1\eta_1 + \cdots + u_n\eta_n) \cdot k(u)[\eta] \) is unmixed. Letting \( \tilde{p}_u = p_u/p, \tilde{q}_3 = q_3/p \) we have \( (\sum u_i\eta_i) = \tilde{p}_u \cap \tilde{q}_3 \) or \( (\sum u_i\eta_i) = \tilde{p}_u \), by Theorem 1. The unmixedness implies that \( (\sum u_i\eta_i) = \tilde{p}_u, \) i.e., \( (p, H_u) = p_u \).

**COROLLARY 2.** If \( V \) is \( k \)-normal at \( (0) \), then \( (p, H_u) = p_u \) i.e., \( (p, H_u) \) is a prime ideal.

**Proof.** By Theorem 1, \( (p, H_u) = p_u \) or \( (p, H_u) = p_u \cap q_3 \). Passing to the local ring \( k(u)[\eta]_{(0)} \), we have \( (\sum u_i\eta_i) \cdot k(u)[\eta]_{(0)} = \tilde{p}_u \) or \( \tilde{p}_u \cap \tilde{q}_3 \) where \( \tilde{p}_u = p_u/p, \tilde{q}_3 = q_3/p \tilde{p}_u \) and \( \tilde{q}_3 \), are extensions of \( \tilde{p}_u \) and \( \tilde{q}_3 \) in \( k(u)[\eta]_{(0)} \) respectively. Since \( k(u)[\eta]_{(0)} \) is integrally closed, the unmixedness of \( (\sum u_i\eta_i) \cdot k(u)[\eta]_{(0)} \) implies that \( (\sum u_i\eta_i) \cdot k(u)[\eta] = \tilde{p}_u \) and \( (p, H_u) = p_u \).

Recall that \( V/k \) is a quasi-absolutely irreducible variety if \( k \) is quasi-algebraically closed in the field \( k(\xi_1, \ldots, \xi_n) \) of rational functions on \( V/k \); a prime ideal \( \mathfrak{a} \) in \( k[X_1, \ldots, X_n] \) is quasi-absolutely irreducible if \( k[X_1, \ldots, X_n]\mathfrak{a} \) is primary, where \( k \) is the algebraic closure of \( k \). By [11; Th. 10, p. 371], \( p \) is quasi-absolutely irreducible if and only if \( V/k \) is quasi-absolutely irreducible. \( V/k \) is absolutely irreducible if \( k \) is algebraically closed in \( k(\xi) \) and \( k(\xi) \) is separable over \( k \). A prime ideal \( \mathfrak{a} \) in \( k[X_1, \ldots, X_n] \) is absolutely irreducible if \( k[X_1, \ldots, X_n]\mathfrak{a} \) is a prime ideal. It is well known that the prime ideal \( p \) of \( V/k \) is absolutely irreducible if and only if \( V/k \) is.

**Theorem 2.** If \( V/k \) is quasi-absolutely irreducible of dimension...
if $k$ is infinite, then $V \cap H_u/k(u)$ is quasi-absolutely irreducible.

**Proof.** Let $(γ)$ be a generic point of $V \cap H_u$ over

$$k(u) = k(u_1, \ldots, u_n).$$

By Lemma 2, $(γ)$ is a generic point of $V$ over $k$. Let $γ_1, γ_3,$ and $γ_n$ be algebraically independent over $k$. By Lemma 3, $(γ)$ is a generic point of $V$ over $k(u_2, \ldots, u_n)$. By [11; Lemma 5, p. 368], $k(u_2, \ldots, u_n)$ is quasi-algebraically closed in $k(u_2, \ldots, u_n)(γ)$. Let $Σ = k(u_2, \ldots, u_{n-1})(γ)$, $u_n$ is algebraically independent over $Σ$. Viewing $k(u_2, \ldots, u_{n-1})$ as the field $k$ and $u_n$ as the $u$ in [11; corollary, p. 369], we have $Σ(u_n) = k(u_2, \ldots, u_{n-1})(u_n)(γ) = k(u)(ξ)$. Let $ξ_1$ and $ξ_2$ in [11; corollary, p. 369] be replaced by $-(u_2γ_2 + \cdots + u_{n-1}γ_{n-1})/γ_1$ and $-γ_n/γ_1$ respectively, one sees that $-(u_2γ_2 + \cdots + u_{n-1}γ_{n-1})/γ_1$ and $γ_n/γ_1$ are algebraically independent over $k(u_2, \ldots, u_{n-1})$. Hence by the same corollary we have that

$$k(u_2, \ldots, u_{n-1})(u_n)(- (u_2γ_2 + \cdots + u_{n-1}γ_{n-1})/γ_1 - u_nγ_n/γ_1) = k(u_2, \ldots, u_{n-1})(u_n)(u) = k(u)$$

quasi-algebraically closed in $Σ(u_n) = k(u)(γ)$.

**Lemmma 4.** Let $K$ be a regular finitely generated extension of an infinite field $k$ with $\text{tr. deg}_k K \geq 3$. Let $x, y, z$ be three elements of $K$ algebraically independent over $k$, and $z/x \in K^p k$, where $p$ is the characteristic of $k$. Then for all but a finite number of constants $c \in k$, $K$ is a regular extension of $k(y + cz/x)$. Moreover, let $τ$ be an indeterminate $K(τ)$ is regular over $k(τ)(y + τz/x)$.

**Proof.** [5; Lemma 3].

**Theorem 3.** If $V/k$ is an absolutely irreducible variety of dimension $r \geq 3$ defined over an infinite field $k$, then $V \cap H_u/k(u)$ is an absolutely irreducible variety.

**Proof.** $V \cap H_u/k(u)$ is irreducible. Let $(ξ)$ be a generic point of $V \cap H_u$ over $k(u)$. By Lemma 3, $(ξ)$ is a generic point of $V$ over $k$, hence $\text{tr. deg}_k k(ξ) \geq 3$ and $k(ξ)$ is a regular extension over $k$ by [12; Proposition 1, p. 69]. Let $ξ_1, ξ_2$ and $ξ_n$ be three elements in a separable transcendental basis of $k(ξ)$ over $k$. Let $K = k(u_2, \ldots, u_{n-1})(ξ)$, $u_n$ is algebraically independent over $K$. Viewing $k(u_2, \ldots, u_{n-1})$ as the field $k$ and $u_n$ as the $τ$ in Lemma 4, we have $K(u_n) = k(u)(ξ)$. Let

$$y = -(u_2ξ_2 + \cdots + u_{n-1}ξ_{n-1}), \quad z = ξ_n \quad \text{and} \quad x = ξ_1,$

then $x, y$ and $z$ are
algebraically over \( k(u_2, \cdots, u_{n-1}) \). By [6, Proposition 1, p. 185] and [6; corollary to Proposition 2, p. 186], \( z/x = -\xi_n/\xi_1 \in K^p k(u_2, \cdots, u_{n-1}) \), we have that \( K(u_n) \) is a regular extension over 
\[ k(u_2, \cdots, u_{n-1})(u_n) \left( \frac{y - u_n z}{k} \right) = k(u) . \]

Therefore \( k(u)(\xi) \) is a regular extension over \( k(u) \), hence \( V \cap H_u/k(u) \) is an absolutely irreducible variety.

Let \( \{F_1, \cdots, F_s\} \) be a set of generators of \( \mathfrak{p} \) in \( k[x] \). Let \( P \) be a point on \( V \). According to [14], \( P \) is \( k \)-simple on \( V \) if and only if the mixed Jacobian of \( \{F_1, \cdots, F_s\} \) is of rank \( n - r \) at \( P \). When \( k(P) \) is separable over \( k \), \( P \) is \( k \)-simple on \( V \) if and only if the classical Jacobian of \( \{F_1, \cdots, F_s\} \) is of rank \( n - r \) at \( P \).

Following Theorem 1, we denote \( p_u \) as the sole isolated component of \( (\mathfrak{p}, H_u) \) and \( W_u/k(u) \) as its variety in the sequel.

**Theorem 4.** Let \( V/k \) be of dimension \( r \geq 2 \). Then \( P \in W_u \) is \( k(u) \)-simple if and only if \( P \) is \( k \)-simple on \( V \).

**Proof.** Let \( P \in W_u \) be \( k \)-simple on \( V \). By Theorem 1, \((\mathfrak{p}, H_u) = p_u \cap \mathfrak{A} \), where \( \mathfrak{A} \) is the embedded component with \( (X_1, \cdots, X_n) \) as radical. Let \( (\eta) \) be a generic point of \( V \) over \( k(u) \), and let \( (\xi) \) be a generic point of \( W_u \) over \( k(u) \). Let \( k(u)[\eta]_p \) and \( k(u)[\xi]_p \) be the local rings of \( V \) and \( W_u \) at \( P \) respectively. \( k(u)[\eta]_p \) is regular and
\[
k(u)[\xi]_p \cong \frac{k(u)[\eta]_p}{\mathfrak{p}_u \cdot k(u)[\eta]_p} ,
\]
where \( \mathfrak{p}_u \) is the residue of \( p_u \) modulo \( \mathfrak{p} \). If \( P \neq (0) \), let \( \mathfrak{A} \) be the residue of \( \mathfrak{A} \) modulo \( \mathfrak{p} \) and let \( \mathfrak{m}_p \) be the maximal ideal of \( k(u)[\eta]_p \), then \( \mathfrak{A}k(u)[\eta]_p \not\subset \mathfrak{m}_p \). For otherwise \( (\eta_1, \cdots, \eta_n)^\rho \subset \mathfrak{m}_p \) for some integer \( \rho > 0 \), as \( (X_1, \cdots, X_n)^\rho \subset \mathfrak{A} \). Thus \( P = (0) \), a contradiction. Therefore, when \( P \neq (0) \), \((\sum u_i \eta_i) \cdot k(u)[\eta]_p = \mathfrak{p}_u \cdot k(u)[\eta]_p \), and \( k(u)[\xi]_p \cong \frac{k(u)[\eta]_p}{(\sum u_i \eta_i)k(u)[\eta]_p} \). By [16; Th. 26, p. 303], to show that \( k(u)[\xi]_p \) is regular it is sufficient to show that \( \sum u_i \eta_i \in \mathfrak{m}_p^2 \). But this is the case, for if \( \sum u_i \eta_i \in \mathfrak{m}_p^2 \), taking partial derivatives with respect to \( u_i \) for \( i = 1, 2, \cdots, n \), we have \( \eta_i \in \mathfrak{m}_p \) for \( i = 1, 2, \cdots, n \), i.e., \( P = (0) \) a contradiction. Therefore \( k(u)[\xi]_p \) is regular. If \( P = (0) \), then \( k \)-normal on \( V \). By Corollary 2 to Theorem 1, \((\mathfrak{p}, H_u) = p_u \). In viewing [14, Th. 7, p. 28], we let \( F_1, \cdots, F_s \) be a basis of \( \mathfrak{p} \), and let \( F_i \)'s and \( X_i \)'s be so arranged that \( (\det (\partial F_i/\partial X_j))_{(0)} \neq 0 \), where \( i, j = 1, 2, \cdots, n - r \), and the subscript \( (0) \) means that we replace \( (X) \) by \( (0) \) after the determinant of the Jacobian is formed, as the rank of

\[ 1 \text{ If } P \neq 0, \text{ and if } P \text{ is } k \text{-simple on } V, \text{ then } P \text{ remains simple on } W_u/k(u) \text{ follows also from } [13; \text{ the theorem of Bertini, p. 138}]. \]
Consider
\[ A_j = \det \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_{n-r}} & \frac{\partial F_j}{\partial X_j} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial F_{n-r}}{\partial X_1} & \cdots & \frac{\partial F_{n-r}}{\partial X_{n-r}} & \frac{\partial F_{n-r}}{\partial X_j} \end{bmatrix}_{0} \]

where \( \eta - r + 1 \leq j < \eta \). If \( A_j = 0 \) for some \( j \) then \( u_1, \ldots, u_{n-r}, u_j \) are algebraically dependent over \( k \). This is a contradiction, hence \( (0) \) is \( k \)-simple on \( W_u \). Conversely, assume that \( P \in W_u \) is \( k(\xi) \)-simple on \( W_u \). If \( P \neq (0) \), we have \( k(\xi)[\xi]_{P} \cong k(\xi)[\gamma]_{P}/(\Sigma u_i \gamma_i) \cdot k(\xi)[\gamma]_{P} \) from the above. If \( P = (0) \), then \( P \) is \( k(\xi) \)-normal on \( W_u \). By Theorem 6 in the following \( V/k \) is normal at \( (0) \), therefore \( (p, H_u) = p_u \) and \( k(\xi)[\xi]_{(0)} \cong k(\xi)[\xi]_{(0)}/(\Sigma u_i \gamma_i) \cdot k(\xi)[\gamma]_{(0)}. \)

Therefore \( k(\xi)[\xi]_{P} \cong k(\xi)[\gamma]_{P}/(\Sigma u_i \gamma_i) \cdot k(\xi)[\gamma]_{P} \) if \( P \in k(\xi) \)-simple on \( W_u \). Since \( h^t((\Sigma u_i \gamma_i) \cdot k(\xi)[\gamma]_{P}) = 1 \), it follows that \( k(\xi)[\gamma]_{P} \) is a regular local ring. Hence \( P \) is \( k(\xi) \)-simple on \( V \).

By an argument similar to the proof of Lemma 2, we have the following.

**Corollary.** If \( V/k \) is of dimension \( r \geq 3 \) and if \( V/k \) is locally free of \((r - 1)\)-dimensional singularities, then \( V \cap H_u/k(\xi) \) is locally free of \((r - 2)\)-dimensional singularities.

**Note.** If \( r = 2 \), the corollary is clearly false as one sees by taking \( V \) to be a cone with vertex at \( (0) \).

**Theorem 5.** If \( V/k \) is a complete intersection of dimension \( \geq 3 \) and if \( V \) is \( k \)-normal at \( (0) \), then the generic hyperplane section \( V \cap H_u \) is also \( k(\xi) \)-normal at \( (0) \).

**Proof.** \( V/k(\xi) \) is \( k(\xi) \)-normal at \( (0) \), by Lemma 1. By corollary to Theorem 1, \((p, H_u) = p_u \) is prime. For any polynomial \( F \neq 0 \) in \( k(\xi)[X] \), by [7; Th. p. 49] or [16; Th. 26, p. 203], \((p_u, F) = (p, H_u, F) \) is unmixed. Hence, passing to the quotient modulo \( p_u \), we have that every nonzero principal ideal in the coordinate ring \( k(\xi)[\xi] \) of \( V \cap H_u \) is unmixed. It follows that every nonzero principal ideal in the local ring of \( V \cap H_u \) at \( (0) \), \( k(\xi)[\xi]_{(0)} \), is also unmixed. Since \( V/k \) is \( k \)-normal at \( (0) \), therefore \( V/k \) is locally free of \((r - 1)\)-dimensional singularities at \( (0) \). By the above corollary, \( V \cap H_u \) is locally free of \((r - 2)\)-dimensional singularities at \( (0) \). It follows from Proposition 1 that \( V \cap H_u \) is \( k(\xi) \)-normal at \( (0) \).
THEOREM 6. If $V \cap H_u$ is $k(u)$-normal at $(0)$, then $V/k$ is normal at $(0)$.

Proof. This theorem is really a consequence of [3; Lemma 4, p. 360] ([8; (36.9), p. 134]). Indeed, let $(\eta)$ be a generic point of $V$ over $k(u)$. Passing to $k(u)[\eta]$, by Theorem 1, we have

$$k(u)[\eta]_0 = \bar{p}_u \cap \bar{q},$$

where $\bar{p}_u$ and $\bar{q}$ are residues of $p_u$ and $q$ modulo $\bar{v}$ respectively. It is clear that

$$(1) \ (u, \eta_1 + \cdots + u_n\eta_n) \subset (u_1, \eta_1 + \cdots + u_n\eta_n) \cap k(u)[\eta]_0 = \bar{p}_u \cap k(u)[\eta]_0 \cap \bar{q};$$

$$(2) \ (u, \eta_1 + \cdots + u_n\eta_n) \cap (u_1, \eta_1 + \cdots + u_n\eta_n) = p_u \cap k(u)[\eta]_0 \bar{p}_u;$$

and

$$(3) \ (u_1, \eta_1 + \cdots + u_n\eta_n) \cap (u_1, \eta_1 + \cdots + u_n\eta_n) = p_u \cap k(u)[\eta]_0 \bar{p}_u,$$

which is integrally closed as $V \cap H_u$ is $k(u)$-normal at $(0)$. Moreover, let $k(u)[\eta]_0^*$ be the integral closure of $k(u)[\eta]_0$ in $k(u)(\eta)$, and let $p'$ be a minimal prime divisor of $(u, \eta_1 + \cdots + u_n\eta_n) \cdot k(u)[\eta]_0^*$. It follows from [2; Th. 2, p. 253] and [2; Th. 3; p. 254] that $ht(p' \cap k(u)[\eta]_0^*) = ht(p) = 1$. Therefore $p' \cap k(u)[\eta]_0 = \bar{p}_u$, i.e., every minimal prime divisor of $(u, \eta_1 + \cdots + u_n\eta_n) \cdot k(u)[\eta]_0^*$ lies over $p_u$. The above verify the conditions of [3; Lemma 4, p. 360], therefore $k(u)[\eta]_0$ is integrally closed.

3. The local normal problem. Throughout this section let $V/k$ be a variety of dimension $r \geq 3$, passing through $(0)$ with $(\xi)$ as a generic point over $k$ and let $H_u: u_1X_1 + \cdots + u_nX_n = 0$ be a generic hyperplane through $(0)$. If $V/k$ is normal at $(0)$, is it true that $H_u \cap V$ is $k(u)$-normal at $(0)$? If $V/k$ is a complete intersection then by Theorem 5, the answer to the question is yes. However we shall prove the answer to the question is negative in general.

DEFINITION 4. (a) Let $R$ be a Noetherian ring. Subset $\{a_i, \cdots, a_q\}$ of $R$ is a prime sequence if for each $i = 1, 2, \cdots, q, a_i$ is not a zero divisor in the ring $R/(a_1, \cdots, a_{i-1}) \cdot R$.

(b) Let $R$ be a local ring, the number of elements of a maximal prime sequence in $R$ is called the homological co-dimension of $R$, and is denoted by $\text{cod} \ h(A)$. If $\text{cod} \ h(A) = \dim A$, we say that $A$ is a Cohen-Macaulay ring.

For a general commutative ring $R$ and a multiplicative system $S$ which does not contain 0, it is well known [15, p. 219] that $(\mathfrak{A}; \mathfrak{B})^e \subset \mathfrak{A}^e; \mathfrak{B}^e$ and $(\mathfrak{C}; \mathfrak{D})^e \subset \mathfrak{C}^e; \mathfrak{D}^e$, where $(*)^e = (**); R_s, (**); f^{-1}(*)$, $f$ is the canonical homomorphism of $R$ into $R_s$ and where $\mathfrak{A}, \mathfrak{B}$ are two ideals in $R$, and $\mathfrak{C}, \mathfrak{D}$ are two ideals in $R_s$. 
PROPOSITION 5. Let \( \mathcal{A}, \mathcal{B}, \mathfrak{x} \) and \( \mathfrak{y} \) be the same as above. Then (a) \((\mathfrak{x}; \mathfrak{y})^c = \mathfrak{x}^c; \mathfrak{y}^c\); if \( \mathfrak{A} \supset \text{Ker } f \) and \( \mathcal{B} \) is finitely generated, also (b) \((\mathfrak{x}; \mathfrak{y})^c = \mathfrak{x}^c; \mathfrak{y}^c\) if \( \mathfrak{y} \) is finitely generated.

Proof. Let \( \mathcal{B} = (b_1, \ldots, b_r)R \), we have \( \mathcal{B}^c = (f(b_1), \ldots, f(b_r)) \cdot R^c \). Let \( x \in \mathfrak{x}^c; \mathcal{B}^c \). Then \( x^c \mathcal{B}^c \subseteq \mathfrak{x}^c \) and \( xf(b_i) = f(a_i)/f(s_i) \) for some \( a_i \in \mathfrak{A} \) and \( s_i \in S \). Therefore \( f(\pi, s_i)xf(b_i) \in f(\mathfrak{A}) \). For each \( b \in f(\mathcal{B}) \), \( b = \sum f(r_j)b_j \) for some \( r_j \in R \). Now \( f(\pi, s_i)xb = \sum_j f(\pi, s_i)xf(r_j)b_j \in f(\mathfrak{A}) \), which implies that \( f(\pi, s_i)x \in f(\mathfrak{A}) : f(\mathcal{B}) \). Hence \( x \in (f(\mathfrak{A}) : f(\mathcal{B})) \cdot R^c \). Since \( \mathfrak{A} \supset \text{Ker } f \), by \([15; (1), p. 148]\), \( f(\mathfrak{A}) : f(\mathcal{B}) = f(\mathfrak{A}; \mathcal{B}) \). Therefore \( x \in (\mathfrak{A}; \mathcal{B})^c \) and \( \mathfrak{x}^c; \mathcal{B}^c = (\mathfrak{x}^c; \mathcal{B})^c \). The proof of (b) is similar.

LEMMA 5. \( k(u)[\xi]_0 \) is Cohen-Macaulay if and only if \( k[\xi]_0 \) is Cohen-Macaulay, where \( k[\xi] \) is the coordinate ring of \( \mathcal{V}/k \), and \( u \) is an indeterminate over \( k(\xi) \).

Proof. If \( k[\xi]_0 \) is Cohen-Macaulay, then there exist \( \zeta_i, \ldots, \zeta_r \) such that \( \{\zeta_i, \ldots, \zeta_r\} \) forms a maximal prime sequence, where \( r = \text{dim } \mathcal{V} \). Thus \( \{\zeta_i, \ldots, \zeta_r\}k[\xi]_0 : (\zeta_{i+1}) \cdot k[\xi]_0 = (\zeta_i, \ldots, \zeta_r) \cdot k[\xi]_0 \) for \( i = 1, 2, \ldots, r \). By \([15; (1), p. 227]\), \([15; (15), (21), p. 148]\) Proposition 5 and \([16; (3), p. 221]\) one has \( (\zeta_i, \ldots, \zeta_r)k(u)[\xi]_0 : (\zeta_{i+1})k(u)[\xi]_0 = (\zeta_i, \ldots, \zeta_r)k(u)[\xi]_0 \) for \( i = 1, 2, \ldots, r \). Therefore \( \{\zeta_i, \ldots, \zeta_r\} \) remains as a maximal prime sequence of \( k(u)[\xi]_0 \). Thus \( k(u)[\xi]_0 \) is Cohen-Macaulay.

Conversely, let \( k(u)[\xi]_0 \) be Cohen-Macaulay, let \( \{\zeta_i(u; \xi), \ldots, \zeta_r(u; \xi)\} \) be a maximal prime sequence of \( k(u)[\xi]_0 \). Then, for \( i = 1, 2, \ldots, r \), \( \zeta_i(u; \xi), \ldots, \zeta_r(u; \xi) \cdot k(u)[\xi]_0 : (\zeta_{i+1}(u; \xi))k(u)[\xi]_0 = (\zeta_i(u; \xi), \ldots, \zeta_r(u; \xi))k(u)[\xi]_0 \). By \([15; (21), p. 148]\), going back to the polynomial ring \( k(u)[x] \), we have \( (\zeta_i(u; x), \ldots, \zeta_r(u; x), \overline{p})k(u)[x]_0 : (\zeta_{i+1}(u; x), \overline{p})k(u)[x]_0 = (\zeta_i(u; x), \ldots, \zeta_r(u; x), \overline{p})k(u)[x]_0 \). In viewing \([4; \text{Satz } 3, p. 59]\), one sees that

\[
(\zeta_i(u; x), \ldots, \zeta_r(u; x), \overline{p})k(u)[x]_0 = (\zeta_i(u; x), \ldots, \zeta_r(u; x), \overline{p})k(u)[x]_0
\]

almost always for \( i = 1, 2, \ldots, r \), where the bar means specialization of \( u \) to elements in \( k \). Passing to the local ring of \( \mathcal{V}/k(u) \) at \( 0 \), by \([15; (15), p. 148]\), we have \( \zeta_i(u; \xi), \ldots, \zeta_r(u; \xi)k(u)[\xi]_0 : (\zeta_{i+1}(u; \xi))k(u)[\xi]_0 = (\zeta_i(u; \xi), \ldots, \zeta_r(u; \xi))k(u)[\xi]_0 \) almost always for \( i = 1, 2, \ldots, r \). Let \( a \in k(u) \) be such that the above holds and \( \zeta_i(a; \xi) \neq 0 \), for \( i = 1, 2, \ldots, r \), then \( (\zeta_i(a; \xi), \ldots, \zeta_r(a; \xi))k(\xi)_0 : (\zeta_{i+1}(a; \xi))k(\xi)_0 = (\zeta_i(a; \xi), \ldots, \zeta_r(a; \xi)) \cdot k(\xi)_0 \) for \( i = 1, 2, \ldots, r \). Therefore \( \{\zeta_i(a; \xi), \ldots, \zeta_r(a; \xi)\} \) forms a system of prime sequence of \( k(\xi)_0 \). Hence \( k[\xi]_0 \) is Cohen-Macaulay.

THEOREM 7. Let \( \mathcal{V}/k \) and \( H_u \) be the same as the above. It is not
true in general that if \( V/k \) is \( k \)-normal at \((0)\), then \( V \cap H_u/k(u) \) is \( k(u) \)-normal at \((0)\).

**Proof.** Suppose that if \( V/k \) is \( k \)-normal at \((0)\), then \( V \cap H_u/k(u) \) is \( k(u) \)-normal at \((0)\). Let \((\xi)\) be a generic point of \( V \) over \( k \) and let \((\eta)\) be that of \( V \cap H_u \) over \( k(u) \). Applying the supposition to \( V \cap H_u/k(u) \), we get \( (V \cap H_u) \cap H_u(2,k(u),u(2)) \)-normal at \((0)\), where

\[
H_u(2): u_{21}X_1 + \cdots + u_{2n}X_n = 0
\]

is a generic hyperplane through \((0)\) on

\[
V \cap H_u/k(u) \quad \text{and} \quad u(2) = \{u_{21}, \ldots, u_{2n}\}
\]

are algebraically independent over \( k(u)(\xi, \eta) \). Repeating the supposition and Corollary 2 to Theorem 1 in this way until dimension \( r \) of \( V \) is cut down to 2, we have then

\[
V \cap H_u \cap H_u(2) \cap \cdots \cap H_u(r-2)k(u, u(2), \ldots, u(\gamma - 2))-
\]

normal at \((0)\), where \( u(i) = \{u_{i1}, \ldots, u_{in}\} \), and \( \{u_{i1}, \ldots, u_{in}\} \) are indeterminates over \( k(u, u(2), \ldots, u(i-1)(\xi, \eta, \eta_2, \eta_{i-1}) \) with \( \eta_j = (\eta_{j1}, \ldots, \eta_{jn}) \) being a generic point of \( V \cap H_u \cap H_u(2) \cap \cdots \cap H_u(j) \) over \( k(u, u(2), \ldots, u(j)) \).

Let \( U = \{u, u(2), \ldots, u(\gamma - 2)\} \), then \( k(U) = k(u, u(2), \ldots, u(\gamma - 2)) \). Consider \( V/k(U), (\xi) \) is a generic point of \( V \) over \( k(U) \). Correspondingly in the coordinate ring \( k(U)[\xi] \) of \( V \) over \( k(U) \) we have then \( r - 2 \) quantities \( \zeta_i = u_{i1}\xi_1 + \cdots + u_{in}\xi_n, i = 1, 2, \ldots, r - 2 \), such that \( \zeta_i, \ldots, \zeta_r \) is a prime ideal in \( k(U)[\xi](o) \) and \( \zeta_i = (\zeta_i, \ldots, \zeta_r)k(U)[\xi](o) \). Thus \( \zeta_i, \ldots, \zeta_{r-2} \) is a prime sequence in the local ring \( k(U)[\xi](o) \). Let \( R \) be \( k(U)[\xi](o)/(\zeta_i, \ldots, \zeta_{r-2})k(U)[\xi](o) \), then \( R \) is integrally closed of dimension 2. By [16; (3), p. 397], \( R \) is Cohen-Macaulay. Let \( a, b \in k(U)[\xi](o) \) be such that their residues modulo \( (\zeta_i, \ldots, \zeta_{r-2})k(U)[\xi](o) \) form a maximal prime sequence of \( R \), then \( (\zeta_i, \ldots, \zeta_{r-2}, a, b) \) is a prime sequence of \( k(U)[\xi](o) \). Therefore \( \dim k(U)[\xi](o) = \text{codim} k(U)[\xi](o) \) and hence \( k(U)[\xi](o) \) is a Cohen-Macaulay ring. It follows from Lemma 5 that \( k[\xi](o) \) is a Cohen-Macaulay ring. So under the supposition, we conclude that \( k[\xi](o) \) is integrally closed implies that \( k[\xi](o) \) is Cohen-Macaulay. But on the other hand, [1; Proposition, p. 655] and [1; Th. 5, p. 653] yield an example of a local ring of an algebraic variety at a rational point which is a factorial local ring (hence normal), but not a Cohen-Macaulay local ring. Hence the above supposition yields a contradiction.

**Theorem 8.** If \( V/k \) is normal at \((0)\), and the local ring \( k[\xi](o) \) is a Cohen-Macaulay ring, then \( V \cap H_u/k(u) \) is normal at \((0)\).

**Proof.** By the corollary to Theorem 4, \( (p, H_u) \) is free of \((\gamma - 2)\)-
dimensional singularities. By Lemma 5, \( k(u)[\xi]_{(0)} \) is Cohen-Macaulay. For any nonzero \( a(u; \xi) \) in \( k(u)[\xi]_{(0)} \) not in the prime ideal
\[
(a(u, \xi), u_1\xi_1 + \cdots + u_n\xi_n)
\]
forms a prime sequence of \( k(u)[\xi]_{(0)} \), therefore by [16; Lemma 5, p. 401], \( (a(u, \xi), u_1\xi_1 + \cdots + u_n\xi_n) \cdot k(u)[\xi]_{(0)} \), is unmixed. Hence every nonzero principal ideal of \( k(u)[\xi]_{(0)}/(u_1\xi_1 + \cdots + u_n\xi_n) \cdot k(u)[\xi]_{(0)} \), is unmixed. It follows from Proposition 1 that \( V \cap H_u \) is \( k(u) \)-normal at \( (0) \).

I would like to take this opportunity to express my thanks to Professor A. Seidenberg for suggesting the problem, his valuable advice and continuous encouragement.

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Received December 16, 1969. Part of the results of this paper over a field \( k \) of characteristic 0 forms part of my doctoral thesis written in 1966 under Professor A. Seidenberg in the University of California at Berkeley and was partially supported by NSF under GP3990.
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