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A root system Λ is a partially ordered set having the property that no two incomparable elements λ and μ have a common lower bound. $\Pi(\Lambda, \mathbf{R}_{\lambda})$ will denote the direct product of copies of \mathbf{R} , the set of real numbers, one for each $\lambda \in \Lambda$. $V(\Lambda, \mathbf{R}_{\lambda})$ is the following subgroup: $v \in V = V(\Lambda, \mathbf{R}_{\lambda})$ if the support of v has no infinite ascending sequences. We put a lattice order on v by setting $v \ge 0$ if v = 0 or else every maximal component of v is positive in \mathbf{R} .

This paper has two main results: we first show that the cone of any finite dimensional vector lattice G can be obtained as the union of an increasing sequence $P_1, P_2 \cdots$ of archimedean vector lattice cones on G such that $(G, P_1) \cong (G, P_2) \cong \cdots$, as vector lattices. Next, generalizing this, we show that for any root system Λ the cone of the ℓ -group $V = V(\Lambda, \mathbf{R}_{\lambda})$ can be obtained as the union of a family of archimedean vector ℓ -cones $\{P_7: \gamma \in \Gamma\}$ on V, where $(V, P_7) \cong (V, P_{\delta})$, as vector lattices, for all $\gamma, \delta \in \Gamma$.

It is proved in [1], Theorem 2.2, that $V(\Lambda, \mathbf{R}_i)$ is indeed an \checkmark group when Λ is a root system. In an \checkmark -group K, $x \in K$ is a strong order unit if $x \geq 0$, and for each $0 < a \in K$ there is an $n = 1, 2, \cdots$ such that $nx \geq a$. The symbol \boxplus will denote the cardinal sum of \checkmark groups; that is, if $K_i(i \in I)$ are \checkmark -groups then $K = \boxplus \{K_i: i \in I\}$ means that K is the direct sum of the K_i , as groups, and $0 \leq x \in K$ if and only if $0 \leq x_i \in K_i$, for each $i \in I$. Finally, if r is a real number, $\langle r \rangle$ will denote the smallest integer exceeding r.

Throughout the paper the pair (G, P) will denote an abelian \not -group; that is, G is an abelian group, and P is the cone for a lattice-group order on G. An \not -group (G, P) is said to be archimedean if for any pair $a, b \in P$ there is a positive integer n such that $na \not\leq b$; P is then called an archimedean \not -cone. We restrict our considerations to abelian groups since archimedean \not -groups are necessarily abelian (see [2]).

Let (G, Q) be an \checkmark -group; we say that Q can be approximated by the archimedean \checkmark -cone P if there is a family $\{P_{\gamma}: \gamma \in \Gamma\}$ of archimedean \checkmark -cones on G, such that (i) $(G, P_{\gamma}) \cong (G, P_{\delta})$, for all $\gamma, \delta \in \Gamma$, (ii) $Q = \bigcup \{P_{\gamma}: \gamma \in \Gamma\}$ and (iii) $P = P_{\gamma}$, for some $\gamma \in \Gamma$. The \checkmark -group (G, Q) is then called a *limit A-group*. If the approximating family is directed by set inclusion (resp. a chain under set inclusion) we call (G, Q) a directed (resp. linear) limit A-group. If $\Gamma = \{1, 2, \dots\}$ and $P_n \subseteq P_{n+1}$ for all $n = 1, 2, \dots$, we call (G, Q) a sequential limit A-group.

(G, Q) is a vector lattice if G is a real vector space, and in addition to being an \checkmark -cone, P is closed under scalar multiplication by positive real numbers. The vector lattice (G, Q) can be approximated by the archimedean vector lattice cone P if there is a family $\{P_{\tau}: \gamma \in \Gamma\}$ of archimedean vector \checkmark -cones on G, such that (i) $(G, P_{\tau}) \cong (G, P_{\delta})$, as vector lattices, for all $\gamma, \delta \in \Gamma$, (ii) $Q = \bigcup \{P_{\tau}: \gamma \in \Gamma\}$ and (iii) $P = P_{\tau}$, for some $\gamma \in \Gamma$. In this case we call (G, Q) a limit A-space. By a directed (resp. linear, resp. sequential) limit A-space (G, Q) we mean one where the approximating vector \checkmark -cones form a directed set (resp. a chain, resp. an increasing sequence.)

It will be useful to denote a limit A-group (G, Q) by (G, Q, P), where $P \cong P_{\gamma}$ for all $\gamma \in \Gamma$; this way we can keep track of what approximation is being used.

Let (G, Q, P) be a limit A-group (resp. limit A-space); we call it a strong limit A-group (resp. strong limit A-space) if Q is essential over each P_{γ} . (Let (G, P) be an \checkmark -group, Q be an extension of the cone P. Q is an essential extention of P if every \checkmark -ideal of (G, Q) is an \checkmark -ideal of (G, P). For further discussion on essential extensions see [3]). Suppose the family $\{P_{\gamma}: \gamma \in \Gamma\}$ has a smallest member (which is once again denoted by P); it follows from a remark in [3] concerning essential extensions, that (G, Q, P) is a strong limit Agroup if and only if Q is essential over P.

PROPOSITION 1. The cardinal sum of (strong) sequential limit Agroups is a (strong) sequential limit A-group. The same statement holds for (strong) sequential limit A-spaces.

Proof. Let $(G, Q) = \boxplus (G_i, Q_i)$, $i \in I$. Suppose each Q_i is the limit of the sequence $\{P_{n,i}: n = 1, 2, \dots\}$ of archimedean \checkmark -cones on G_i , and $(G_i, P_{1,i}) \cong (G_i, P_{2,i}) \cong \cdots$, for all $i \in I$. Fix n, and let P_n be the \checkmark -cone of the cardinal sum of the $(G_i, P_{n,i})$. Since each $P_{n,i}$ is archimedean, so is P_n ; clearly $P_n \subseteq P_{n+1}$, for each $n = 1, 2, \cdots$, and $P_n \subseteq Q$.

So let $y \in Q$ and i_1, i_2, \dots, i_k be the nonzero components of y. Then each y_{i_m} is in Q_{i_m} , for $m = 1, 2, \dots, k$, and there exists an n(m) such that $y_{i_m} \in P_{n(m),i_m}$. Let $n = \max \{n(m): m = 1, 2, \dots, k\}$; then each $y_{i_m} \in P_{n,i_m}$, which implies that $y \in P_n$. This show that $Q = \bigcup_{n=1}^{\infty} P_n$; it is obvious that $(G, P_1) \cong (G, P_2) \cong \cdots$. It follows therefore that (G, Q, P_1) is a sequential limit A-group.

Now suppose Q_i is essential over each $P_{n,i}$, $i \in I$. (This is equi-

valent to saying that each \checkmark -ideal of (G_i, Q_i) is an \checkmark -ideal of $(G_iP_{n,i})$.) Let K be an \checkmark -ideal of (G, Q); then $K = \boxplus \{K_i: i \in I\}$, where $K_i = K \cap G_i$. Each K_i is an \checkmark -ideal of (G_i, Q_i) , and hence an \checkmark -ideal of $(G_i, P_{n,i})$. Thus K is an \checkmark -ideal of (G, P_n) , proving that Q is essential over P_n , that is, (G, Q, P_i) is a strong sequential limit A-group.

The above proposition can be generalized, in a sense:

PROPOSITION 2. The cardinal sum of (strong) directed limit Agroups is a (strong) directed limit A-group. The same statement holds for cardinal products.

Proof. Let $(G, Q) = \bigoplus (G_i, Q_i)$, $i \in I$. Suppose $(G_i, Q_i) = (G_i, Q_i, P_i)$ is a directed limit A-group, and $\{P_{\gamma_i}: \gamma_i \in \Gamma^{(i)}\}$ is the approximating family. Let $\Gamma = \pi\{\Gamma^{(i)}: i \in I\}$ and consider the family $\{P_{\gamma}: \gamma \in \Gamma\}$ of \checkmark -cones defined by: $x \in P_{\gamma}$ if for each $i \in I$ $x_i \in P_{\gamma_i}(\gamma_i \in \Gamma^{(i)})$. Each P_{γ} is clearly an archimedean \checkmark -cone for G, and $(G, P_{\gamma}) \cong (G, P_{\delta})$, for $\gamma \neq \delta$. The P_{γ} obviously form a directed system, and finally, if $y \in Q$ then $y_i = 0$ or $y_i \in Q_i$; in either case $y_i \in P_{\delta_i}$, for some $\delta_i \in \Gamma^{(i)}$, and therefore $y \in P_{\delta}$, where $\delta = (\cdots, \delta_i, \cdots) \in \Gamma$. Thus Q is the join of the P_{γ} and we're done.

Notice that the above proof works for the cardinal product of directed limit A-groups. If each (G_i, Q_i, P_i) is a strong limit A-group then one uses the technique of the proof of Proposition 1 to show that (G, Q, P) is also a strong limit A-group. We should also point out once more, that a similar version of this theorem holds for directed limit A-spaces.

It is not known whether the cardinal sum (resp. product) of linear limit A-groups is again a linear limit A-group. By Proposition 2 it is certainly a directed limit A-group.

THEOREM 3. Let (G, Q, P_1) be a strong sequential limit A-space having a strong order unit. Let $K = \mathbf{R} \bigoplus G$ and $Q' = \{r + g: r > 0,$ or else r = 0 and $g \in Q\}$. Then $(K, Q', \mathbf{R}^+ \bigoplus P_1)$ is a strong sequential limit A-space.

Proof. Let $u \in G$ be a strong order unit relative to Q; without loss of generality we can assume $u \in P_n$ for each $n = 1, 2, \dots$. Let v be any positive real number and define

$$v^{(n)} = \left(\frac{1}{n}\right)v + \left(\frac{1-n}{n}\right)u$$
, for $n = 1, 2, \cdots$.

Let $V^{(n)} = \{rv^{(n)}: r \in \mathbf{R}\}; V^{(n)}$ is a one-dimensional space, and clearly $V^{(n)} \cap G = 0$, so $K = V^{(n)} \oplus G$. Now let $P'_n = \{rv^{(n)} + g: 0 \leq r \text{ and } g \in P_n\}$; then (K, P'_n) is the cardinal sum of $V^{(n)}$, ordered as the reals, and (G, P_n) . Since each P_n is archimedean it follows that each P'_n is also. Notice that $V^{(1)} = \mathbf{R}$ and $P'_1 = \mathbf{R} \boxplus P_1$. If H is an \checkmark -ideal of (K, Q') then either H = K or H = G, or else H is a proper \checkmark -ideal of (G, Q); in any case H is an \checkmark -ideal of (K, P'_1) , since Q is essential over P_1 . Notice also that $(K, P'_n) \cong (K, P'_{n+1})$, for all n.

We must show (1) $P'_n \subseteq P'_{n+1} \subseteq Q'$ and (2) $Q' = \bigcup_{n=1}^{\infty} P'_n$.

(1) We show first that $P'_1 \subseteq P'_k \subseteq Q'$, for all $k = 1, 2, \cdots$. The first inequality will follow if we can prove that $v \in P'_k$, the second, if $v^{(k)} \in Q'$, because we know that $P_1 \subseteq P_k \subseteq Q$. That $v^{(k)}$ is in Q' is clear since (1/n)v > 0. One can easily show that

$$v = k v^{(k)} + (k-1) u$$
 ,

proving that $v \in P'_k$.

But now observe that for each $n = 1, 2, \cdots$ we have

$$v^{(n)} - v^{(n+1)} = rac{1}{n(n+1)} (v+u) \in P'_1 \subseteq P'_{n+1}$$
 ,

so $v^{(n)}$ is the sum of two elements in P'_{n+1} , and hence $v^{(n)} \in P'_{n+1}$. That is enough to show that $P'_n \subseteq P'_{n+1}$.

(2) Let $y \in Q'$; we have the following expressions for $y: y = sv + y_0 = s^{(n)}v^{(n)} + y^{(n)}$, with $s, s^{(n)} \in \mathbf{R}$ and $y_0, y^{(n)} \in G$. This forces certain relations:

(1)
$$s^{(n)} = ns \ge 0$$
 (since $y \in Q'$),

and

(2)
$$\left(\frac{(1-n)}{n}\right)s^{(n)}u + y^{(n)} = y_0.$$

Thus each $s^{(n)} \ge 0$; moreover, the above equations give

(2')
$$y^{(n)} = (n-1)su + y_0$$
.

Writing y_0 as the difference of its positive and negative parts relative [to Q, we obtain

$$(2'') y^{(n)} = (n -)su + y_0^+ - y_0^- .$$

Observe that since u is a strong order unit of (G, Q), then so is su. Therefore if n is large enough, $(n-1)su > y_0^-$ (rel. Q). But since the P_n form a chain we can certainly find an n_0 such that $y_0^+, y_0^- \in P_{n_0}$ and $(n_0 - 1)su > y_0^-$ (rel. P_{n_0}). Thus $y_0^{(n)} \in P_{n_0}$; together with the fact that $s_0^{(n)} \ge 0$ this implies that $y \in P_{n_0}$. This proves the theorem. COROLLARY 3.1. Every finite dimensional vector lattice is a strong sequential limit A-space.

Proof. Note at the outset that every finite dimensional vector lattice has a strong order unit. For if (V, Q) is a *t*-dimensional vector lattice, we may regard (V, Q) as $V(\Lambda, \mathbf{R}_{\lambda})$, where Λ is a root system of *t* elements, and for each $\lambda \in \Lambda$, $\mathbf{R}_{\lambda} = \mathbf{R}$. ([1], Theorem 5.11) Then $x = (1, 1, \dots, 1)$ is a strong order unit.

We proceed by induction on t:

Case I. Λ has a largest element λ_0 . Let $\Lambda' = \Lambda \setminus \{\lambda_0\}$; then (V, Q) is a direct lexicographic extension of $V(\Lambda', \mathbf{R}_{\lambda})$ by **R**. But $V(\Lambda', \mathbf{R}_{\lambda})$ has dimension t - 1, so it is a strong sequential limit A-space. By Theorem 3 (V, Q) is also a strong sequential limit A-space.

Case II. Λ has no largest element. Then Λ can be written as the union of two nonempty, disjoint subsets Λ_1 and Λ_2 having the property that λ is incomparable to μ , for all $\lambda \in \Lambda_1$ and $\mu \in \Lambda_2$. It follows that $(V, Q) = V(\Lambda_1, \mathbf{R}_2) \boxplus V(\Lambda_2, \mathbf{R}_2)$, and both these summands have dimension less than t; thus they both are strong sequential limit Λ -spaces, and by Proposition 1 so is (V, Q).

Let Λ be a root system, $\Pi = \Pi(\Lambda, \mathbf{R}_{\lambda})$, $V = V(\Lambda, \mathbf{R}_{\lambda})$ and $P = V \cap \Pi^+$, where $\Pi^+ = \{x: x_{\lambda} \ge 0$, for all $\lambda \in \Lambda\}$. The following discussion will establish that V is a limit Λ -space. (Of course we consider V as a vector lattice relative to the cone $V^+ = \{v: \text{ all the maximal nonzero components of } v$ are positive}.) Notice that (V, P) is an \checkmark -subgroup of Π . For each $x \in P$ let s(x) denote the support of x, m(x) the set of maximal nonzero components of x. Choose a family $\{n_{\lambda}: \lambda \in m(x)\}$ of positive integers, and define a map $\theta_{x,\{n_{\lambda}\}}$ on Π by:

$$(y\theta_{x}, {}_{\{n_{\lambda}\}})_{\lambda} = \begin{cases} y_{\lambda} & \text{if } \lambda \in s(x) \text{ or } \lambda \in m(x); \\ y_{\lambda} - n_{\lambda(x)}^{\langle x_{\lambda} \rangle} y_{\lambda(x)} & \text{if } \lambda \in s(x) \setminus m(x) \text{ and } \lambda \text{ has no successor in } s(x); \\ y_{\lambda} - n_{\lambda(x)}^{\langle x_{\lambda} \rangle} y_{\lambda-1} & \text{if } \lambda \in s(x) \setminus m(x) \text{ and } \lambda - 1 \text{ is the seccessor of } \lambda \text{ in } s(x). \end{cases}$$

(Note: $\lambda(x)$ is the maximal component of x that exceeds λ .) This map has an inverse $\theta_{x, \{n_{\lambda}\}}^{-1}$:

$$(a\theta_{x,\{n_{\lambda}\}}^{-1})_{\lambda} = \begin{cases} y_{\lambda} & \text{if } \lambda \notin s(x) \text{ or } \lambda \in m(x); \\ n_{\lambda(x)}^{\langle x_{\lambda} \rangle} y_{\lambda(x)} + y_{\lambda} & \text{if } \lambda \in s(x) \setminus m(x) \text{ and } \lambda \text{ has no successor in } s(x); \\ n_{\lambda(x)}^{\langle x_{\lambda} \rangle^{2} + \dots + \sqrt{x} \lambda_{k} \rangle} y_{\lambda_{1}} + \dots + n_{\lambda(x)}^{\langle x_{\lambda} \rangle} y_{\lambda_{k-1}} + y_{\lambda_{k} = \lambda} \\ & \text{if } \lambda \in s(x) \setminus m(x) \text{ and } \lambda_{i-1} \text{ is the successor of } \lambda_{i}; \text{ also } \lambda_{1} = \lambda(x): \\ n_{\lambda(x)}^{\langle x_{\lambda_{1}} \rangle + \dots + \langle x_{\lambda_{k}} \rangle} y_{\lambda(x)} + n_{\lambda(x)}^{\langle x_{\lambda_{2}} \rangle + \dots + \langle x_{\lambda_{k}} \rangle} y_{\lambda_{1}} + \dots + y_{\lambda_{k} = \lambda} \\ & \text{if } \lambda \in s(x) \setminus m(x) \text{ and } \lambda_{i-1} \text{ is the successor of } \lambda_{i}; \lambda_{1} \text{ has no successor.} \end{cases}$$

Clearly then $\theta_{x,\{n_{\lambda}\}}$ is a vector space isomorphism of Π onto itself. Let $P_{x,\{n_{\lambda}\}} = P\theta_{x,\{n_{\lambda}\}}$; we claim first that, restricted to V, each $\theta_{x,\{n_{\lambda}\}}$ is an isomorphism of V onto itself. This is due to the fact that for all $y \in \Pi$

$$s(y) \subseteq s(x) \cup s(y\theta_{x,\{n_2\}})$$
 and $s(y\theta_{x,\{n_2\}}) \subseteq s(y) \cup s(x)$.

A quick look at the definition of $\theta_{x,\{n_{\lambda}\}}^{-1}$ readily shows that $P\theta_{x,\{n_{\lambda}\}} \subseteq P$, that is: $P \subseteq P_{x,\{n_{\lambda}\}}$. Thus $P_{x,\{n_{\lambda}\}}$ is an archimedean vector lattice order on V, and $(V, P) \cong (V, P_{x,\{n_{\lambda}\}})$, for all $x \in P$ and $\{n_{\lambda}: \lambda \in m(x)\}$.

Now if $y \in V^+$ then consider $x = |y|_p$; of course s(x) = s(y) and m(x) = m(y). We proceed by induction on the maximal chains of s(x). Let μ be a fixed maximal component of x; of course $(y\theta_x^{-1}, (n_\lambda))_\lambda = y_\lambda$ for all $\lambda \ge \mu$ and every choice of integers $\{n_\lambda: \lambda \in m(x)\}$. So assume $\lambda < \mu$ and $\lambda \in s(x)$; if λ has no successor in s(x), let n_μ be the smallest positive integer ≥ 2 such that $n_\mu x_\mu \ge 2$. If $y_\lambda > 0$ then $n_\mu^{\langle x_\lambda \rangle} y_\mu + y_\lambda \ge 1$, since $x_\mu = y_\mu$. If $y_\lambda < 0$ then $y_\lambda = -x_\lambda$; now if $x_\lambda > 1$ we get $n_\mu^{\langle x_\lambda \rangle -1} \ge x_\lambda$, for all $n_\mu \ge 2$. This implies that $n_\mu^{\langle x_\lambda \rangle} y_\mu \ge 2x_\lambda \ge x_\lambda + 1$. If $0 > y_\lambda \ge -1$ then $n_\mu^{\langle x_\lambda \rangle} y_\mu = n_\mu y_\mu \ge 2 = 1 + 1 \ge x_\lambda + 1$. Hence in any of the above cases $n_\mu^{\langle x_\lambda \rangle} y_\mu + y_\mu \ge 1$, for large enough n_μ . Notice that n_μ is independent of λ .

If λ does have a successor in s(x) there are two cases for $(y\theta_x^{-1}, (n_2))_{\lambda}$.

Case I. $(y\theta_x^{-1}, (n_\lambda))_{\lambda} = n_{\mu}^{\langle x_{\lambda_k} \rangle} y_{\lambda_1} + \cdots + n_{\mu}^{\langle x_{\lambda_k} \rangle} y_{\lambda_{k-1}} + y_{\lambda_k}$, where $\lambda_k = \lambda, \lambda_{i-1}$ is the successor of λ_i in s(x) and $\lambda_1 = \mu$. Thus

$$(y heta_{x}^{-1}, \{n_{2}\})_{\lambda} = n_{\mu}^{\langle x \lambda k \rangle} [n_{\mu}^{\langle x \lambda_{2} \rangle + \dots + \langle x \lambda_{k-1} \rangle} y_{\mu} + \cdots + y_{\lambda_{k-1}}] + y_{\lambda_{k}}$$

and by induction the sum in the square brackets is ≥ 1 ; so

$$(y heta_x^{-1},_{\{n_j\}})_\lambda \geqq n_\mu^{\langle x_{\lambda k}
angle} + y_{\lambda_k} \geqq 1$$
 .

(The last inequality holds since for any real number $r, n^{\langle |r| \rangle} \ge r + 1$, for all $n \ge 2$.)

Case II.

$$(y heta_x^{-1},y_\lambda)_\lambda=n_\mu^{\langle x_{\lambda_1}
angle+\langle x_{\lambda_2}
angle+\cdots+\langle x_{\lambda_k}
angle}y_\mu+n_\mu^{\langle x_{\lambda_2}
angle+\cdots+\langle x_{\lambda_k}
angle}y_{\lambda_1}+\cdots+n_\mu^{\langle x_{\lambda_k}
angle}y_{\lambda_{k-1}}+y_{\lambda_k}$$
 ,

where $\lambda_k = \lambda$, λ_{i-1} is the successor of λ_i in s(x) and λ_1 has no successor in s(x). Again

$$(y heta_x^{x},_{\{n_\lambda\}})_\lambda = n_\mu^{\langle x_{\lambda k}
angle} [n_\mu^{\langle x_{\lambda_1}
angle+\cdots+\langle x_{\lambda k-1}
angle} y_\mu + \cdots + y_{\lambda_{k-1}}] + y_{\lambda_k} \; ,$$

and again by induction the bracketed sum is ≥ 1 ; so

$$(y heta_x^{-1},_{\{n_j\}})_\lambda \geqq n^{\langle x_{\lambda k}
angle} + y_{\lambda_k} \geqq 1 \; .$$

Out of all of this we get that if $\lambda < \mu$ and $\lambda \in s(x)$ then there is an $n_{\mu}(\text{independent of }\lambda)$ such that $(y\theta_{x}^{-1}, (n_{\lambda}))_{\lambda} \geq 1$. This works for every $\mu \in m(x) = m(y)$, and so we can find integers $\{n_{\lambda}: \lambda \in m(x)\}$ such that $y\theta_{x}^{-1}, \{n_{\lambda}\} \in P$. (Remark: if $\lambda < \mu$ in the above arguments, but $x_{\lambda} = y_{\lambda} = 0$, then there is no problem; any θ^{-1} will fix this component.) Putting it differently: we've discovered an x in P and integers $\{n_{\lambda}: \lambda \in m(x)\}$ such that $y \in P_{x, \{n_{\lambda}\}}$; hence

$$V^+ \subseteq \bigcup \{P_{x,\{n_j\}}: x \in P, \{n_{\lambda}: \lambda \in m(x)\}\}$$
.

To show the reverse containment we show a little bit more. The maps $\theta_{x,\{n_{\lambda}\}}$ all take V^{+} into itself. For if $a \in V^{+}$ and $\mu \in m(a)$ then $(a\theta_{x,\{n_{\lambda}\}})_{\mu} = a_{\mu}$. And if $\lambda > \mu$ then $(a\theta_{x,\{n_{\lambda}\}})_{\lambda} = a_{\lambda} = 0$; thus $m(a) \subseteq m(a\theta_{x,\{n_{\lambda}\}})$. One shows in a similar fashion that $m(a\theta_{x,\{n_{\lambda}\}}) \subseteq m(a)$, and hence equality holds. This clearly shows that $V^{+}\theta_{x,\{n_{\lambda}\}} = V^{+}$ and therefore $P_{x,\{n_{\lambda}\}} \subseteq V^{+}$, for all $x \in P$ and $\{n_{\lambda}: \lambda \in m(x)\}$.

In addition V^+ is essential over P, in view of Proposition 2.5 in [3]. We've thus proved the following theorem:

THEOREM 4. If Λ is any root system, then $V = V(\Lambda, \mathbf{R}_{\lambda})$ is a strong limit A-space.

Again let Λ be a root system, and $F = F(\Lambda, \mathbf{R}_{\lambda}) = \{v \in V: s(v) \text{ is contained in the union of finitely many maximal chains;} F is then an <math>\lambda$ -subgroup of V. In the above construction we can throw out quite a few of the $P_{x,\{n_{\lambda}\}}$; in this case we take for each $x \in Q = P \cap F$ and $n = 1, 2, \dots$, mappings $\partial_{x,\{n_{\lambda}\}}$ where each $n_{\lambda} = n$. We abbreviate the notation to $\theta_{x,n}$ and $P_{x,n}$ respectively. (We mention in passing

that (F, Q) is an \checkmark -subgroup of (V, P).) For each $a \in Q$ and each positive integer n, we denote by $Q_{a,n}$ the cone $P_{a,n} \cap F = (P \cap F)\theta_{a,n} = Q\theta_{a,n}$. Notice that since $s(b) \subseteq s(a) \cup s(b\theta_{a,n})$ and $s(b\theta_{a,n}) \subseteq s(a) \cup s(b)$ it follows that $F\theta_{a,n} = F$. This means that $Q_{a,n}$ is an \checkmark -cone for F and $(F, Q) \cong (F, Q_{a,n})$.

If $y \in F^+ = F \cap V^+$ then $x = |y|_F \in F$; pick n_0 to be the smallest integer ≥ 2 such that $n_0 x_{\mu_j} \geq 2$, for all $j = 1, \dots, k$, with $m(x) = m(y) = \{\mu_1, \dots, \mu_k\}$. With this notation, we can follow the technique of the proof of Theorem 4 and show that $y \in Q_{x,n_0}$. We get therefore that $F^+ = \bigcup \{Q_{x,n}: x \in Q, n = 1, 2, \dots\}$, and we've proved the following:

THEOREM 5. If Λ is a root system, then $F = F(\Lambda, \mathbf{R}_{\lambda})$ is a strong limit A-space.

REMARK. Once again in view of 2.5 in [3] we can conclude that F^+ is essential over Q.

Now let Λ be a root system having finitely many maximal chains and no infinite ascending sequences; note that in this case $V = \Pi$. Let $m(\Lambda)$ denote the set of maximal components of Λ . For each $x \in P$ define $\Psi_{x,n}$ on Π by

$$(y\Psi_{x,n})_{\lambda} = \begin{cases} y_{\lambda} & \text{if } \lambda \in m(\Lambda); \\ y_{\lambda} - n^{\langle x_{\lambda} \rangle} y_{\lambda^{*}} & \text{if } \lambda \in m(\Lambda) \text{ and } \lambda \text{ has no successor in } \\ \lambda; \\ y_{\lambda} - n^{\langle x_{\lambda} \rangle} y_{\lambda-1} & \text{if } \lambda \in m(\Lambda) \text{ and } \lambda-1 \text{ is its successor } \\ \text{in } \Lambda. \end{cases}$$

(Note: λ^* denotes the maximal entry of Λ exceeding λ .) As before $\Psi_{x,n}$ is a vector space isomorphism on V, and $Q_{x,n} = P\Psi_{x,n} \supseteq P$, for all $x \in P$ and $n = 1, 2, \cdots$. Once again $(V, P) \cong (V, Q_{x,n})$; and if $y \in V^+$ and $x = |y|_P$ we pick n_0 to be the smallest integer ≥ 2 such that $n_0 x_{\mu_j} \ge 2$, for all maximal components $\mu_1, \mu_2, \cdots, \mu_k$ of x. Then as in the proof of Theorem 4, with the various cases, one shows that for all $\lambda < \mu_j$ $(j = 1, \cdots, k)$ we get $(y \Psi_{x,n_0})_j \ge 1$. (We have to assume here that $x_{\mu_j} \ge 1$, for each j, but this can be done without loss of generality.) Therefore $V^+ = \bigcup \{Q_{x,n}: x \in P, n = 1, 2, \cdots\}$.

But in this case we can say more: the system $\{Q_{x,n}: x \in P, n = 1, 2, \dots\}$ is directed. To prove this we show that if $m \leq n$ are positive integers then $Q_{x,m} \subseteq Q_{x,n}$; and if $0 \leq x \leq y$ (rel. p) then $Q_{x,n} \subseteq Q_{y,n}$. First suppose $m \leq n$; let $a \in P$ and consider $a \Psi_{x,m} \Psi_{x,n}^{-1}$: given $\lambda \in A$

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there are four cases to consider.

(1) $\lambda \in m(\Lambda)$; then $(a \Psi_{x,m} \Psi_{x,n}^{-1})_{\lambda} = a_{\lambda} \geq 0.$

(2) $\lambda \notin m(\Lambda)$ and λ has no successor in Λ ; then

$$egin{array}{lll} (a arPsi_{x,m} arPsi_{x,n}^{-1})_{\lambda} &= n^{\langle x_{\lambda}
angle} (a arPsi_{x,m})_{\lambda^{st}} + \left(a arPsi_{\chi,m}
ight)_{\lambda} \ &= n^{\langle x_{\lambda}
angle} a_{\lambda^{st}} + a_{\lambda} - m^{\langle x_{\lambda}
angle} a_{\lambda^{st}} \ &= a_{\lambda} + (n^{\langle x_{\lambda}
angle} - m^{\langle x_{\lambda}
angle}) a_{\lambda^{st}} \geq 0 \;. \end{array}$$

(3) $\lambda \notin m(\Lambda)$ and λ_{i-1} is the successor of λ_i , where $\lambda_k = \lambda$ and $\lambda_1 \in m(\Lambda)$. Then

$$\begin{array}{l} (a \varPsi_{x,m} \varPsi_{x,n} \varPsi_{x,n})_{\lambda} \\ = n^{\langle x_{\lambda 2} \rangle + \dots + \langle x_{\lambda k} \rangle} (a \varPsi_{x,m})_{\lambda_{1}} + \dots + n^{\langle x_{\lambda k} \rangle} (a \varPsi_{x,m})_{\lambda_{k-1}} + (a \varPsi_{x,m})_{\lambda_{k}} \\ = n^{\langle x_{\lambda 2} \rangle + \dots + \langle x_{\lambda k} \rangle} a_{\lambda_{1}} + \dots + n^{\langle x_{\lambda k} \rangle} (a_{\lambda_{k-1}} - m^{\langle x_{\lambda k-1} \rangle} a_{\lambda_{k-2}}) + a_{\lambda_{k}} - m^{\langle x_{\lambda k} \rangle} a_{\lambda_{k-1}} \\ = n^{\langle x_{\lambda 3} \rangle + \dots + \langle x_{\lambda k} \rangle} (n^{\langle x_{\lambda 2} \rangle} - m^{\langle x_{\lambda 2} \rangle}) a_{\lambda_{1}} + \dots + (n^{\langle x_{\lambda k} \rangle} - m^{\langle x_{\lambda k} \rangle}) a_{\lambda_{k-1}} + a_{\lambda_{k}} \ge 0 . \end{array}$$

(4) $\lambda \notin m(\Lambda)$ and λ_{i-1} is the successor of $\lambda_i, \lambda_k = \lambda$ and λ_1 has no successor. As in (3) one shows that $(\alpha \Psi_{x,m} \Psi_{x,n}^{-1})_{\lambda} \ge 0$. This proves that $P\Psi_{x,m}\Psi_{x,n}^{-1} \subseteq P$, or $Q_{x,m} \subseteq Q_{x,n}$.

Next, suppose $0 \leq x \leq y$ (rel. p) and n is a positive integer. Consider $(a \Psi_{x,n} \Psi_{y,n}^{-1})_{\lambda}$ with $a \in P$; once again there are four cases.

(1) $\lambda \in m(\Lambda)$; then $(a \Psi_{x,n} \Psi_{y,n}^{-1})_{\lambda} = a_{\lambda} \ge 0$.

(2) $\lambda \notin m(\Lambda)$ and λ has no successor in Λ ; then one can check that $(a \Psi_{x,n} \Psi_{y,n}^{-1})_{\lambda} = a_{\lambda} + (n^{\langle y_{\lambda} \rangle} - n^{\langle x_{\lambda} \rangle})a_{\lambda^*} \ge 0$, since $\langle y_{\lambda} \rangle \ge \langle x_{\lambda} \rangle$.

(3) $\lambda \notin m(\Lambda)$ and λ_{i-1} is the successor of $\lambda_i, \lambda_k = \lambda$ and λ_1 is a maximal component of Λ . One easily verifies that

$$(a \varPsi_{x,n} \varPsi_{y,n})_{\lambda} = n^{\langle y_{\lambda_3}
angle + \cdots + \langle y_{\lambda_k}
angle} (n^{\langle y_{\lambda_2}
angle} - n^{\langle x_{\lambda_2}
angle}) a_{\lambda_1} + \cdots + (n^{\langle y_{\lambda_k}
angle} - n^{\langle x_{\lambda_k}
angle}) a_{\lambda_{k-1}} + a_{\lambda_k} \ge 0 \; .$$

(4) $\lambda \notin m(\Lambda)$ and λ_{i-1} is the successor of λ_i , where $\lambda_k = \lambda$ but λ_1 has no successor in Λ . One checks as in the other cases that $(a \Psi_{x,n} \Psi_{y,n}^{-1}) \geq 0$. Thus $P \Psi_{y,n} \Psi_{x,n}^{-1} \subseteq P$, that is $Q_{x,n} \subseteq Q_{y,n}$.

So if $Q_{a,m}$ and $Q_{b,n}$ are given, with $a, b \in P$, then we may assume $m \leq n$ and so $Q_{a,m} \cup Q_{b,n} \subseteq Q_{av_Pb,n}$; this proves that the system of the $Q_{x,n}$ is directed. Hence:

THEOREM 6. If Λ is a root system having finitely many roots and no infinite ascending sequences, then $V = V(\Lambda, \mathbf{R}_{\lambda}) = \Pi(\Lambda, \mathbf{R}_{\lambda})$ and V is a strong directed limit A-space.

As an easy corollary of Theorem 4 we prove the following:

PROPOSITION 7. Let Λ be a root system, and D be an \prime -subgroup of $V = V(\Lambda, \mathbf{R}_{\lambda})$ having the property that

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(a) D is an ∠-subgroup of (V, P); P = {x ∈ V: x_λ ≥ 0, all λ ∈ Λ}.
(b) And if a, b ∈ D, c ∈ V and s(c) ⊆ s(a) ∪ s(b), this implies that c ∈ D.

Then $(D, D \cap V^+)$ is a limit A-group.

Proof. Condition (a) guarantees, of course, that $(D, D \cap P)$ is an \checkmark -group. Condition (b) says that for each $x \in D \cap P$ and each family $\{n_{\lambda}: \lambda \in m(x)\}$ the isomorphism $\theta_{x,\{n_{\lambda}\}}$ takes D onto D. Thus

$$(D, D \cap P) \cong (D, D \cap P_{x, \{n\}})$$

and

$$D = \bigcup \{D \cap P_{x, \{n\}}\}.$$

This completes the proof.

In particular $\Sigma = \Sigma(\Lambda, \mathbf{R}_{\lambda}) = \{x \in V: s(x) \text{ is finite}\}$ satisfies (a) and (b) in Proposition 7, and so $(\Sigma, \Sigma \cap V^+, \Sigma \cap P)$ is a limit A-space.

In closing we point out that it is unknown whether the construction of Theorem 4 or 5 yields a directed system. Even if this should not be the case, some subsystem might be directed and still fill out V^+ . A case in point is $\Sigma = \Sigma (\Lambda, \mathbf{R}_{\lambda})$; one can show (the proof being long, but in the spirit of that of Theorems 4 and 5) that Σ is a directed limit A-space, by taking an appropriate subsystem of the $\{P_{x,\{n_{\lambda}\}}\}$.

Suppose we have an 1-group (G, Q); if we knew under what conditions G admitted an archimedean \checkmark -order P, of which Q was a very essential extension, we could perhaps make a construction on P along the lines of the construction of Theorem 4. It is doubtful that the construction of Theorem 4 applies to too many \checkmark -subgroups of V. The reason being that the archimedean \checkmark -cones $P_{x, \{n_{\lambda}\}}$ are of a very special type, namely they have a basis.

A question which has some interest on its own: what groups G admit archimedean lattice orders? They must of course be abelian and torsion free, and if G is divisible then G does certainly admit such a cone. There is no guarantee however, that an archimedean \not -cone on the divisible closure G^* of G will even induce an \not -cone on G.

In view of Corollary 3.1 one can ask of course: what \checkmark -groups are (strong) sequential (or linear) limit A-groups. Let us give one example to show that 3.1 does not give all the strong sequential limit Aspaces. This is also an example of a strong sequential limit A-space with infinite descending chains of \checkmark -ideals; one can give examples of strong sequential limit A-spaces which have infinite ascending chains of \checkmark -ideals. It is even possible to find strong sequential limit A-spaces with descending chains (or ascending chains) of arbitrary length.

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Let $G = \mathbb{R} \boxplus \mathbb{R} \boxplus \mathbb{R} \boxplus \mathbb{R} \boxplus \cdots = \{\text{all finitely nonzero real sequences}\}$. Let Q be the lexicographic total order by ordering from the left; let $P = G^+$. Let θ_n be a map defined by

$$x\theta_n = (x_1, x_2 - nx_1, \cdots, x_n - nx_{n-1}, x_{n+1}, x_{n+2}, \cdots)$$
.

In the notation of the proof of Theorem 5 $\theta_n \equiv \theta_{x_n,n}$, where $x_n = (1, 1, \dots, 1, 0, 0, \dots)$; (the last 1 is the *n*-th position.) We therefore know that θ_n is an isomorphism of G onto itself, and $P_n = P\theta_n \supseteq P$. It can be shown further that $P_n \subseteq P_{n+1}$, for each $n = 1, 2, \dots$, and finally $Q = \bigcup_{n=1}^{\infty} P_n$. Thus (G, Q, P) is a strong sequential limit A-space, for Q is very essential over P.

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