BEHAVIOR OF GREEN LINES AT THE KURAMOCHI BOUNDARY OF A RIEMANN SURFACE

Mitsuru Nakai and Leo Sario
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BOUNDARY OF A Riemann surface

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We shall establish necessary and sufficient conditions, in
terms of Green lines, for a point of the Kuramochi boundary
Γ^k of a hyperbolic Riemann surface R to be of positive har-
monic measure.

Explicitly, let ℬ be the bundle of all Green lines l issuing
from a fixed point of R. It forms a measure space with the
Green measure. We call a subset ℱ of ℬ a distinguished
bundle if it has positive measure and there exists a point p
in Γ^k such that almost every l in ℱ terminates at p. The
point p will be referred to as the end of ℱ.

Our main result is that a point p of Γ^k has positive
measure if and only if there exists a distinguished bundle ℱ
whose end is p.

We shall also give an intrinsic characterization of the
latter property, without reference to points of Γ^k: A bundle
ℱ is distinguished if and only if it has positive measure and
for every HD-function u there exists a real number c_u such
that u has the limit c_u along almost every l in ℱ.

1. Green lines

1. Let R be a hyperbolic Riemann surface, the hyperbolicity
characterized by the existence of Green's functions. Fix a point z_0 ∈ R
and denote by g(z) = g(z, z_0) the Green's function on R with singularity
z_0. Consider the differential equations

\begin{align}
   (1) \quad \frac{dr(z)}{r(z)} &= -dg(z), \quad r(z_0) = 0, \\
   (2) \quad d\theta(z) &= -\ast dg(z).
\end{align}

Equation (1) has the unique solution r(z) = e^{-g(z)} on R with 0 ≤ r(z) < 1.
In any simply connected subregion of R - z_0 where dg(z) ≠ 0, equation
(2) also has a solution \( \theta(z) \), unique up to an additive constant. The
global solution \( \theta(z) \), however, is a multivalued harmonic function.

Set \( G_\rho = \{ z \in R | r(z) < \rho \} \), \( C_\rho = \partial G_\rho (0 < \rho < 1) \). For a sufficiently
small \( \rho \), the analytic function \( w = \varphi(z) = r(z)e^{i\theta(z)} \) is single-valued and
gives a univalent conformal mapping of \( G_\rho \) onto the disk \( |w| < \rho \).
Denote by \( \rho_0 \) the supremum of all \( \rho \) with this property.

2. An open arc α is called a Green arc if \( dg(z) \neq 0 \) for all \( z \in \alpha \),
and a branch of \( \theta \) is constant on \( \alpha \). The set of Green arcs is partially ordered by inclusion. A maximal Green arc in this partially ordered set is called a Green line.

A Green line \( l \) is said to issue from \( z_0 \) if \( z_0 \in \overline{l} \). We denote by \( \mathcal{B} \) the set of Green lines issuing from \( z_0 \) and use the suggestive term bundle for a subset \( \mathcal{A} \) of \( \mathcal{B} \), with the case \( \mathcal{A} = \mathcal{B} \) not excluded.

For a fixed \( \rho \in (0, \rho_0) \) and a given \( p \in C_\rho \) let \( l(p) \) be the Green line in \( \mathcal{B} \) passing through \( p \). Making use of the function \( w = \varphi(z) = r(z)e^{i\theta(z)} \) we see that the mapping \( p \rightarrow l(p) \) is bijective; let \( p(l) \) be the inverse mapping. We call a bundle \( \mathcal{A} \subset \mathcal{B} \) measurable if \( p(\mathcal{A}) \) is measurable in \( C_\rho \), and define the Green measure of \( \mathcal{A} \) by

\[
(3) \quad m(\mathcal{A}) = \frac{1}{2\pi} \int_{p(\mathcal{A})} d\theta(z) = -\frac{1}{2\pi} \int_{p(\mathcal{A})} *d\varphi(z) .
\]

The space \((\mathcal{B}, m)\) is a probability space, i.e., a measure space of total measure unity. The definition is independent of the choice of \( \rho \in (0, \rho_0) \).

3. Fix an \( l \in \mathcal{B} \). The number \( a(l) = \sup_{z \in l} r(z) \) is in \((0, 1]\). If \( a(l) < 1 \), then \( l \) terminates at a point of \( R \) at which \( \varphi = 0 \). Such an \( l \) is called singular. If \( a(l) = 1 \), then \( l \) tends to the ideal boundary of \( R \) and is called regular. The bundle \( \mathcal{B}_r \) of regular Green lines “almost” comprises \( \mathcal{B} \), that is, \( m(\mathcal{B}_r) = 1 \). This is a result of Brelot-Choquet [1] (cf. [7], [8]).

2. Compactifications.

4. Let \( R^e \) be a compactification of \( R \), i.e., a compact Hausdorff space containing \( R \) as its open dense subspace. For a bounded continuous function \( \varphi \) on the ideal boundary \( \Gamma^e = R^e - R \) of \( R \), denote by \( U_\varphi^{R^e} \) the class of superharmonic functions \( s \) on \( R \) such that

\[
\lim \inf_{z \in R, z \to p} s(z) \geq \varphi(p)
\]

for every \( p \in \Gamma^e \). The function

\[
H_\varphi^{R^e}(z) = \inf_{s \in U_\varphi^{R^e}} s(z)
\]

is harmonic on \( R \). We assume that \( R^e \) is a resolutive compactification (cf. Constantinescu-Cornea [2]), that is, \( \varphi \rightarrow H_\varphi^{R^e}(z) \) is a continuous linear functional. Then for \( z_0 \in R \) there exists a measure \( \mu^e \), called the harmonic measure on \( \Gamma^e \), and a function \( P^e(z, p) \) on \( R \times \Gamma^e \) with properties \( P^e(z, p) = 1 \),

\[
(4) \quad H_\varphi^{R^e}(z) = \int_{\Gamma^e} P^e(z, p)\varphi(p)d\mu^e(p) .
\]
This representation extends to bounded Borel measurable functions \( \varphi \) on \( \Gamma^c \).

Let \( \widetilde{HD}(R) \) be the class of harmonic functions \( u \geq 0 \) on \( R \) such that there exists a decreasing sequence \( \{u_n\} \subset \widetilde{HD}(R) \) with \( u = \lim_n u_n \) on \( R \). A function \( u \in \widetilde{HD}(R) \) is said to be \( \widetilde{HD} \)-minimal if for every \( v \in \widetilde{HD}(R) \) with \( v \leq u \) on \( R \) there exists a constant \( c_v \) such that \( v = c_v u \) on \( R \). We shall call the compactification \( R^e \) \( \widetilde{HD} \)-compatible if the following condition is satisfied: \( u \in \widetilde{HD}(R) \) is \( \widetilde{HD} \)-minimal if and only if there exists a point \( p_0 \in \Gamma^c \) with \( \mu^e(p_0) > 0 \) and a number \( k > 0 \) such that

\[
(5) \quad u(z) = k \int_{p_0} P^e(z, p) d\mu^e(p).
\]

5. The Royden compactification \( R^* \) of \( R \), with the Royden boundary \( \Gamma' = R^* - R \), is a typical example of an \( \widetilde{HD} \)-compatible compactification (see [6], [8]). We let \( \mu \) and \( P \) stand for \( \mu^e \) and \( P^e \) corresponding to \( R^* \).

A compactification \( R^e \) is said to lie below \( R^* \) if there exists a continuous mapping \( \pi = \pi^e \) of \( R^* \) onto \( R^e \) such that \( \pi|_R \) is the identity and \( \pi^{-1}(R) = R \). Clearly \( \pi \) is unique and we have

\[
(6) \quad \int_{R^e} P^e(z, p) \varphi(p) d\mu^e(p) = \int_{R} P(z, p^*) \varphi(\pi(p^*)) d\mu(p^*)
\]

for every bounded Borel function \( \varphi \) on \( \Gamma^c \).

6. We are interested in the behavior of \( l \in \mathcal{B} \) in \( R^e \). We set

\[
(7) \quad e^c(l) = \overline{l^c} - l \cup \{z_0\},
\]

with \( \overline{l^c} \) the closure of \( l \) in \( R^c \), and call \( e^c(l) \) the end part of \( l \) in \( R^c \). It is a compact set in \( \Gamma^c \). If

\[
\mathcal{B}^c = \{l \in \mathcal{B} | e^c(l) \text{ is a single point}\}
\]

is of measure \( m(\mathcal{B}^c) = 1 \), then we call \( R^e \) Green-compatible.

We shall make use of a result of Maeda [4]: A metrizable compactification \( R^e \) which lies below \( R^* \) is Green-compatible.

7. A compactification \( R^e \) of \( R \) is said to be of type \( G \) if \( R^e \) is metrizable, \( \widetilde{HD} \)-compatible, and lies below \( R^* \). Note that \( R^e \) is then Green-compatible. An important example:

**Proposition.** The Kuramochi compactification \( R^k \) of \( R \) is of type \( G \).

In fact, metrizability and \( \widetilde{HD} \)-compatibility of \( R^k \) are immediate
consequences of related results of Constantinescu-Cornea [2, pp. 171 and 169]. That \( R^k \) lies below \( R^* \) follows from the definition of the Kuramochi compactification given in [2, p. 167].

\( R^k \) is actually the only significant compactification of type \( G \) known thus far. For a general discussion of its properties we also refer to [5].

3. Distinguished bundles.

8. Let \( R^e \) be a compactification of \( R \) of type \( G \). We call a bundle \( \mathcal{A} \subset \mathcal{B} \) \( R^e \)-distinguished if \( m(\mathcal{A}) > 0 \) and there exists a point \( p \in \Gamma^e \) such that \( e^e(l) = p \) for almost every \( l \in \mathcal{A} \). The point \( p \) will be referred to as the end of \( \mathcal{A} \). In the case \( R^e = R^* \) we simply say that \( \mathcal{A} \) is distinguished.

We shall characterize points \( p \in \Gamma^e \) of positive measure in terms of \( R^e \)-distinguished bundles:

**Theorem.** Let \( R^e \) be a compactification of type \( G \) of a hyperbolic Riemann surface \( R \). A point \( p \in \Gamma^e = R^e - R \) has positive harmonic measure if and only if there exists an \( R^e \)-distinguished bundle \( \mathcal{A} \) with end \( p \).

The proof will be given in 9–13.

9. Let \( \Gamma^* = R^* - R \) be the Royden boundary of \( R \). For \( l \in \mathcal{B} \), denote by \( e(l) \) the set \( \bar{l} - l \cup \{z_0\} \) in \( \Gamma \), with \( \bar{l} \) the closure of \( l \) in \( R^* \). Given a subset \( S \subset \Gamma^* \) we write

\[
\check{S} = \{l \in \mathcal{B} | e(l) \cap S \neq \emptyset\}, \quad \check{S} = \{l \in \mathcal{B} | e(l) \subset S\}.
\]

We shall employ the following auxiliary result ([7], [8]): For every \( F_e \)-set \( K \) (resp. \( G_e \)-set \( U \)) in \( \Gamma^* \)

\[
(9) \quad \check{m}(\bar{K}) \leq \mu(K), \quad \check{m}(\bar{U}) \geq \mu(U),
\]

where \( \check{m} \) and \( \check{m} \) are the outer and inner measures induced by \( m \).

Let \( p^* \) be on the Royden harmonic boundary \( \Lambda \) of \( R \). The set

\[
\Lambda_{p^*} = \{q^* \in \Gamma^* | \check{u}(q^*) = u(p^*) \text{ for all } u \in \text{HBD}(R)\}
\]

is called a block at \( p^* \). It is known ([7], [8]) that it has a measurable \( \check{\Lambda}_{p^*}, \)

\[
(10) \quad \check{m}(\check{\Lambda}_{p^*}) = \mu(p^*),
\]

and that

\[
(11) \quad u(p^*) = \lim_{z \in \bar{l}, l(z) \rightarrow 1} u(z)
\]

for every \( u \in \text{HD}(R) \) and almost every \( l \in \check{\Lambda}_{p^*} \).
10. Suppose $\mathcal{A}$ is an $R^e$-distinguished bundle with end $p \in \Gamma^e$. We are to prove that $\mu^e(p) > 0$. Take the projection $\pi = \pi^e$ of $R^e$ onto $R^e$ (see 5). The set $K = \pi^{-1}(p)$ is compact and clearly $\mathcal{A} \subset \tilde{K}$. By (9),

$$0 < m(\mathcal{A}) \leq \bar{m}(\tilde{K}) \leq \mu(K).$$

From (6) it follows that $\mu(K) = \mu(\pi^{-1}(p)) = \pi^e(p)$. Therefore

$$0 < m(\mathcal{A}) \leq \mu^e(p).$$

11. Conversely suppose that $p \in \Gamma^e$ and $\mu^e(p) > 0$. Since $R^e$ is $\tilde{H}D$-compatible, the function $u(z) = \int_{\mathcal{A}} P(z, q)d\mu^e(q)$ is $\tilde{H}D$-minimal on $R$. By (6) we see that

$$u(z) = \int_{\pi^{-1}(p)} P(z, q^*)d\mu(q^*).$$

Since $R^e$ is also $\tilde{H}D$-compatible and the integral representation (12) of the $\tilde{H}D$-function $u$ is unique up to a boundary function vanishing $\mu$-almost everywhere on $\Gamma$ ([6], [8]), we conclude that there exists a point $p^* \in \pi^{-1}(p)$ with $\mu(p^*) = \mu(\pi^{-1}(p)) > 0$. Observe that

$$m(\tilde{A}_{p^*}) = \mu(p^*) > 0.$$ 

In view of the Green-compatibility of $R^e$, there exists a measurable subset $\mathcal{A} \subset \tilde{A}_{p^*}$ with $m(\tilde{A}_{p^*}) = m(\mathcal{A})$ and such that $e^e(l)$ is a single point in $\Gamma^e$ for each $l \in \mathcal{A}$.

To conclude that $\mathcal{A}$ is an $R^e$-distinguished bundle with end $p$, we must show that $\mathcal{A}' = \{l \in \mathcal{A} | e^e(l) \neq p\}$ is of $m$-measure zero. For this purpose take a sequence $\{U_n\}$ of open sets in $\Gamma^e$ with

$$U_{n+1} \subset \tilde{U}_{n+1} \subset U_n, \quad \bigcap_{1}^{\infty} U_n = \{p\}.$$ 

Let $\mathcal{A}'_n = \{l \in \mathcal{A}' | e^e(l) \in U_n\}$. Since $\mathcal{A}' = \bigcup_{n=1}^{\infty} \mathcal{A}'_n$, it suffices to show that $m(\mathcal{A}'_n) = 0$ for every $n$.

12. First we assume that $R \in O_{HD}$. For an arbitrarily fixed $n$ there exists a $u_n \in HBD(R)$ such that

$$0 \leq u_n | \Delta \leq 1, \quad u_n | \pi^{-1}(U_{n+1}) \cap \Delta = 1, \quad u_n | (\Delta - \pi^{-1}(U_n)) = 0.$$ 

In view of (11), there exists a measurable subset $\mathcal{A}'' \subset \mathcal{A}'$ with $m(\mathcal{A}' - \mathcal{A}'') = 0$ and

$$1 = u_n(p^*) = \lim_{z \in \Gamma, z \rightarrow p^*} u_n(z)$$

for every $l \in \mathcal{A}''$. The set $E_n = \{q^* \in \Gamma | u_n(q^*) < \frac{1}{2}\}$ is open in $\Gamma$. By
(15), \( e(l) \cap E_n = \emptyset \) for every \( l \in \mathcal{U}_n \). Because of the definition of \( \mathcal{U}_n \), it is also clear that \( e(l) \cap \pi^{-1}(U_n) = \emptyset \) for every \( l \in \mathcal{U}_n \). Since the set \( K_n = \Gamma - \pi^{-1}(U_n) \cup E_n \) is compact and \( \pi^{-1}(U_n) \cup E_n \supset \Delta \), we have \( K_n \subset \Gamma - \Delta \) and a fortiori \( \mu(K_n) = 0 \).

On the other hand, \( e(l) \subset K_n \) for every \( l \in \mathcal{U}_n \). Therefore \( \mathcal{U}_n \subset K_n \subset \mathcal{K}_n \). In view of (9), we obtain

\[
m(\mathcal{U}_n) \leq \bar{m}(\mathcal{K}_n) \leq \mu(K_n) = 0
\]

and conclude that \( m(\mathcal{U}_n) = m(\mathcal{U}_n') = 0 \).

13. If \( R \in O_{HD} \), then \( \Delta \) consists of a single point and consequently \( \Delta = \{ p^* \} \). The set \( F_n = \Gamma - \pi^{-1}(U_n) \) is compact in \( \Gamma - \Delta \) and hence \( \mu(F_n) = 0 \). By the definition of \( \mathcal{U}_n \) we have \( \mathcal{U}_n \subset \mathcal{F}_n \subset \bar{F}_n \). Therefore

\[
m(\mathcal{U}_n) \leq m(\mathcal{F}_n) \leq \mu(F_n) = 0.
\]

The proof of Theorem 8 is herewith complete.


14. We next give necessary and sufficient conditions for a bundle to be distinguished, without referring to its end:

**Theorem.** Let \( R^* \) be a compactification of type \( G \) of a hyperbolic Riemann surface \( R \). A bundle \( \mathcal{U} \subset \mathcal{B} \) is \( R^* \)-distinguished if and only if \( m(\mathcal{U}) > 0 \) and for each \( u \in HD(R) \) there exists a number \( c_u \) such that

\[
\lim_{z \in l, r(z) \to 1} u(z) = c_u
\]

for almost every \( l \in \mathcal{U} \).

The proof will be given in 15-18.

15. First suppose \( \mathcal{U} \) is \( R^* \)-distinguished with end \( p \in \Gamma^* \). Then by 10 and 11, there exists a point \( p^* \in K = \pi^{-1}(p) \) such that

\[
0 < \mu^*(p) = \mu(K) = \mu(p^*).
\]

Fix a \( u \in HD(R) \). By the Godefroid theorem [3] (see also [7], [8]),

\[
u(l) = \lim_{z \in l, r(z) \to 1} u(z)
\]

exists for almost every \( l \in \mathcal{B} \). On omiting from \( \mathcal{U} \) a set of measure zero we may assume that \( u(l) \) in (17) exists for every \( l \in \mathcal{U} \). We may also suppose that \( e^*(l) = p \) and a fortiori \( e(l) \subset K \) for every \( l \in \mathcal{U} \).

Since \( \mu(p^*) > 0 \), \( |u(p^*)| < \infty \) (cf. [6], [8]). Let

\[
\mathcal{U}' = \{ l \in \mathcal{U} | u(l) - u(p^*) \neq 0 \}
\]
and
\[ K_n = \{ q^* \in K \mid |u(q^*) - u(p^*)| \geq 1/n \}. \]
Clearly \( K_n \) is a compact set. For \( l \in \mathcal{U}' \) and \( q^* \in e(l) \), we have \( u(l) = u(q^*) \) by (17) and the continuity of \( u \) on \( R^* \). Therefore \( |u(q^*) - u(p^*)| \geq 1/n \) for some \( n \) and a fortiori \( e(l) \subset K_n \). It follows that
\[ \mathcal{U}' \subset \bigcup_{n=1}^{\infty} \tilde{K}_n \subset \bigcup_{n=1}^{\infty} K_n, \]
which by (9) gives
\[ m(\mathcal{U}') \leq \bar{m} \left( \bigcup_{n=1}^{\infty} \tilde{K}_n \right) \leq \sum_{n=1}^{\infty} \bar{m}(\tilde{K}_n) \leq \sum_{n=1}^{\infty} \mu(K_n). \]
From \( K_n \subset K - p^* \) and \( \mu(K) = \mu(p^*) \), we obtain \( \mu(K_n) = 0 \). Consequently \( m(\mathcal{U}') = 0 \) and, since
\[ \lim_{z \to l^*} u(z) = u(l) = u(p^*) \]
for every \( l \in \mathcal{U} - \mathcal{U}' \), we have (16) for almost every \( l \in \mathcal{U} \).

16. Conversely suppose that, for a bundle \( \mathcal{U} \subset \mathcal{B} \) with \( m(\mathcal{U}) > 0 \), (16) is satisfied. We may assume that \( e^*(l) \) is a single point in \( \Gamma^c \) for every \( l \in \mathcal{U} \).

First consider the case \( R \in O_{HD} \). The harmonic boundary \( \Delta \) consists of a single point \( p^* \) and \( \mu(p^*) > 0 \). Let \( p = \pi(p^*) \). Take a sequence \( \{U_n\}_n \) of open sets in \( \Gamma^c \) such that \( \bigcup_{n=1}^{\infty} U_n = \{p\} \). For the bundles \( \mathcal{U}_n = \{ l \in \mathcal{U} \mid e^*(l) \notin U_n \} \), \( n = 1, 2, \ldots \), and
\[ \mathcal{U}' = \{ l \in \mathcal{U} \mid e^*(l) \neq p \} \]
we have \( \mathcal{U}' = \bigcup_{l}^\infty \mathcal{U}_n \). Set \( K_n = \Gamma - \pi^{-1}(U_n) \subset \Gamma - \Delta \). Every \( l \in \mathcal{U}_n \) has \( e(l) \subset K_n \) and we obtain \( \mathcal{U}_n \subset \tilde{K}_n \subset \tilde{K}_n \). Hence
\[ m(\mathcal{U}_n) \leq \bar{m}(\tilde{K}_n) \leq \mu(K_n) = 0 \]
and therefore \( m(\mathcal{U}') = 0 \), i.e., \( e^*(l) = p \) for almost every \( l \in \mathcal{U} \). This proves that \( \mathcal{U} \) is \( R^c \)-distinguished.

17. Next suppose \( R \in O_{HD} \). The family
\[ T(\mathcal{U}) = \{ u \in HBD(R) \mid 0 \leq u \leq 1 \text{ on } R, u(l) = 1 \text{ for almost every } l \in \mathcal{U} \} \]
is a Perron family and
\[ s(z) = \inf \{ u(z) \mid u \in T(\mathcal{U}) \} \]
is an \( \hat{H}D \)-minimal function on \( R \) (see [7], [8]). We can therefore choose a decreasing sequence \( \{h_n\} \subset T(\mathcal{U}) \) such that
(19) \[ s(z) = \lim_{n \to \infty} h_n(z) \]
on R. Let \( \mathcal{U}_0 \) be a measurable subset of \( \mathcal{U} \) with \( m(\mathcal{U}) = m(\mathcal{U}_0) \) such that \( h_n(l) \) exists and equals unity for every \( n = 1, 2, \ldots \), and for every \( l \in \mathcal{U}_0 \). We set \[ \bar{s}(l) = \limsup_{z \in \mathcal{U}, r(z) \to 1} s(z) \]
and observe that
\[ s(z_0) = \int_{\mathcal{U}} s(re^{it})dm(l) \leq \int_{\mathcal{U}} h_n(re^{it})dm(l) = h_n(z_0) \]
for every \( r \in (0, 1) \) (see [7], [8]). By Fatou’s lemma
\[ s(z_0) \leq \int_{\mathcal{U}} \bar{s}(l)dm(l) \leq \int_{\mathcal{U}} h_n(l)dm(l) = h_n(z_0) . \]
Let \( h(l) = \lim_{n} h_n(l) \). Since \( h_n(l) \geq \bar{s}(l) \) and
\[ 0 \leq \int_{\mathcal{U}} (h(l) - \bar{s}(l))dm(l) \leq \lim_{n \to \infty} (h_n(z_0) - s(z_0)) = 0 , \]
we conclude that \( \bar{s}(l) = h(l) \) almost everywhere on \( \mathcal{U} \). In view of \( h(l) = 1 \) for every \( l \in \mathcal{U}_0 \) we may suppose that

(20) \[ \bar{s}(l) = 1 \quad (l \in \mathcal{U}) . \]

18. The remainder of the proof is analogous to that in 11–12. In fact, since \( s \) is \( \widetilde{H}D \)-minimal, there exist points \( p \) and \( p^* \) in \( I^c \) and \( I^r \) respectively such that \( \mu^c(p) = \mu(p^*) > 0, p^* \in \pi^{-1}(p) \), and
\[ s(z) = \int_{p} P^c(z, q)d\mu^c(q) = \int_{p^*} P(z, q^*)d\mu(q^*) . \]
We wish to show that \( e^c(l) = p \) for almost every \( l \in \mathcal{U} \), that is, \( \mathcal{U} \) is \( R^c \)-distinguished with end \( p \). For this purpose set \( \mathcal{U}' = \{ l \in \mathcal{U} | e^c(l) \neq p \} \). To see that \( m(\mathcal{U}') = 0 \) take a sequence \( \{ U_n \} \) of open sets in \( I^c \) such that
\[ \overline{U}_{n+1} \subset U_n, \quad n \cup U_n = \{ p \} . \]
For \( \mathcal{U}' = \{ l \in \mathcal{U} | e^c(l) \in U_n \} \) we have \( \mathcal{U}' = \bigcup_{1}^{n=\infty} \mathcal{U}' \) and it suffices to show that \( m(\mathcal{U}_n') = 0 \) for every \( n = 1, 2, \ldots \). Take a function \( u_n \in HBD(R) \) with
\[ 0 \leq u_n |A| \leq 1, u_n |\pi^{-1}(U_{n+1}) \cap A| = 1, u_n |(A - \pi^{-1}(U_n))| = 0 . \]
We may suppose \( u_n(l) \) exists for every \( l \in \mathcal{U} \). Since \( 1 \geq u_n \geq s \) on \( R \), (20) implies that

(21) \[ u_n(l) = 1 \quad (l \in \mathcal{U}) . \]
Clearly \( e(l) \subset \Gamma - \pi^{-1}(U_n) \) for every \( l \in \mathcal{U}' \). Moreover, if we set \( E_n = \{ q^* \in \Gamma \mid u_n(q^*) < \frac{1}{2} \} \), then \( e(l) \subset \Gamma - E_n \cup \pi^{-1}(U_n) = K_n \) for every \( l \in \mathcal{U}' \). Since \( K_n \) is compact and contained in \( \Gamma - \Delta \),

\[
\mathcal{U}_n \subset \overline{K}_n \subset \overline{K}_n
\]

implies that

\[
m(\mathcal{U}_n) \leq \overline{m}(\overline{K}_n) = \mu(K_n) = 0.
\]

The proof of Theorem 14 is herewith complete.

5. Conclusion.

19. Recall that a bundle \( \mathcal{U} \subset \mathcal{V} \) is distinguished with end \( p \) on the Kuramochi boundary if \( m(\mathcal{U}) > 0 \) and almost every Green line in \( \mathcal{U} \) terminates at \( p \). Since the Kuramochi compactification is of type \( G \), Theorems 8 and 14 imply:

**Theorem.** A point \( p \) of the Kuramochi boundary of a hyperbolic Riemann surface \( R \) has positive measure if and only if there exists a distinguished bundle \( \mathcal{U} \) of Green lines with end \( p \).

A bundle \( \mathcal{U} \) of Green lines with \( m(\mathcal{U}) > 0 \) is distinguished if and only if, for every \( u \in \text{HD}(R) \), there exists a number \( c_u \) such that the "radial limit" \( \lim_{z \to \infty, r(z) \to 1} u(z) \) exists and equals \( c_u \) for almost every \( l \in \mathcal{U} \).

**References**


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