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# THE KAKUTANI THEOREM FOR TENSOR PRODUCTS OF $W^*$-ALGEBRAS

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**In a recent paper Bures proved a result concerning the classification of tensor products of a family of semi-finite $W^*$-algebras and showed that it constituted a non-commutative extension of the main part of Kakutani's theorem on infinite product measures. In this paper these results are extended, first by removing the semi-finiteness restriction, and secondly by completing the analogy with Kakutani's Theorem.**

In particular, it is shown in [1] that if $(\mathscr{A}_i)_{i \in I}$ is a family of semi-finite $W^*$-algebras, then the incomplete tensor products determined respectively by the families of normal states $(\mu_i)$ and $(\nu_i)$ are essentially the same (i.e., product isomorphic) if and only if $\Sigma_{i \in I}[d(\mu_i, \nu_i)]^2 < \infty$, where $d$ is a certain metric defined on the normal states (see Definition 1.1 below). In fact $d$ is a generalization of the metric defined by Kakutani on sets of measures and when each $\mathscr{A}_i$ is abelian the above result yields the first part of the theorem proved in [4].

By removing the semi-finiteness condition from Bures' product formula ([1], Th. 2.5), which relates the distance $d$ between product states to the distances between their components, we are able to obtain the same result for an arbitrary family of $W^*$-algebras. This then completes the classification of tensor products up to product isomorphism as given in ([2], p. 15). Moreover we prove the product formula for the case of infinite product states which gives the extension of the second part of Kakutani's Theorem.

**1. Preliminaries.** If $\mathscr{A}$ is a $W^*$-algebra we let $\Sigma_{\mathscr{A}}$ denote the set of all normal states on $\mathscr{A}$. (We always consider a state $\mu$ to be normalized so that $\mu(1) = 1$). If $\mu \in \Sigma_{\mathscr{A}}$ and $T \in \mathscr{A}$ is such that $\mu(TT^*) = 1$, we define $\mu_T \in \Sigma_{\mathscr{A}}$ by $\mu_T(A) = \mu(TAT^*)$ for all $A \in \mathscr{A}$. For $\mu \in \Sigma_{\mathscr{A}}$ we let $S(\mu)$ denote the support of $\mu$.

Suppose that $\mathscr{A}$ and $\mathscr{B}$ are $W^*$-algebras and that $\mu \in \Sigma_{\mathscr{A}}$, $\nu \in \Sigma_{\mathscr{B}}$. Then $\mu \otimes \nu$ denotes the unique element of $\Sigma_{\mathscr{A} \otimes \mathscr{B}}$ (where $\mathscr{A} \otimes \mathscr{B}$ is the $W^*$-tensor product) such that

$$(\mu \otimes \nu)(A \otimes B) = [\mu(A)][\nu(B)] \text{ for all } A \in \mathscr{A}, B \in \mathscr{B} .$$

A *homomorphism* between two $W^*$-algebras will always mean $a$ *-preserving identity preserving, algebraic homomorphism.

By a *representation* $\phi$ of a $W^*$-algebra $\mathscr{A}$ on a Hilbert space $H$

we always mean a one-to-one homomorphism from $\mathscr{A}$ into $\mathscr{L}(H)$ such that $\phi(\mathscr{A})$ is a von Neumann algebra on $H$. For $\mu \in \Sigma_{\mathscr{A}}$, we say that the vector $x \in H$ *induces* $\mu$ relative to $\phi$ if $\mu(A) = (\phi(A)x \mid x)$ for all $A \in \mathscr{A}$.

DEFINITION 1.1. Let $\mathscr{A}$ be a $W^*$-algebra and let $\mu$ and $\nu \in \Sigma_{\mathscr{A}}$. We define:

$$Q(\mu, \nu) = \{[\phi, x, y]\colon \phi \text{ is a representation of } \mathscr{A} \text{ on } H, \text{ and } x, y \in H \text{ induce } \mu, \nu \text{ respectively relative to } \phi.\}$$

$$d(\mu, \nu) = \inf\{\|x - y\|\colon [\phi, x, y] \in Q(\mu, \nu).\}$$

$$\rho(\mu, \nu) = \sup\{|(x \mid y)|\colon [\phi, x, y] \in Q(\mu, \nu).\}\ .$$

The quantities $d$ and $\rho$ were introduced in [1] where it is shown that $d$ is a metric on $\Sigma_{\mathscr{A}}$ and that $d$ and $\rho$ are related by the formula

$$[d(\mu, \nu)]^2 = 2[1 - \rho(\mu, \nu)]\ . \tag{1.1}$$

The number $d(\mu, \nu)$ can vary from 0 to $\sqrt{2}$ and is equal to 0 if and only if $\mu = \nu$. We consider the other extreme.

LEMMA 1.2. *$d(\mu, \nu) = \sqrt{2}$ if and only if $S(\mu)S(\nu) = 0$.*

*Proof.* Suppose that $S(\mu)S(\nu) = 0$. Choose any $[\phi, x, y] \in Q(\mu, \nu)$. A direct calculation shows that $\phi(S(\mu))x = x$ and $\phi(S(\nu))y = y$, so that $(x \mid y) = 0$. It follows that $\rho(\mu, \nu) = 0$, and from (1.1) $d(\mu, \nu) = \sqrt{2}$.

Conversely, suppose that $d(\mu, \nu) = \sqrt{2}$ so that $\rho(\mu, \nu) = 0$. Choose any $[\phi, x, y] \in Q(\mu, \nu)$. It is a well known fact that $\phi(S(\mu)) =$ the uniform closure of the set $\{(\phi(\mathscr{A}))'x\}$, and similary for $\phi(S(\nu))$ with $y$ replacing $x$. Therefore, to show that $S(\mu)S(\nu) = 0$ it is enough to show that

$$(A'x \mid B'y) = 0$$

for all $A', B' \in (\phi(\mathscr{A}))'$. Clearly it is sufficient to consider the case where $A'$ and $B'$ are unitaries. But then a direct calculation shows that

$$[\phi, A'x, B'y] \in Q(\mu, \nu), \text{ so that}$$
$$|(A'x \mid B'y)| \leqq \rho(\mu, \nu) = 0\ .$$

**2. The product formula for $\rho$.** In this section we prove in general the product formula for $\rho$ which was obtained in ([1], Th. 2.5) for semi-finite algebras. The key step is Lemma 2.1 which is similar in statement and proof to ([1], Lemma 1.6). However by dealing with only one element of the algebra we are able to avoid the use of a trace.

LEMMA 2.1. *Let $\mathscr{A}$ be a $W^*$-algebra. Suppose that $\mu \in \Sigma_{\mathscr{A}}$, and $T \in \mathscr{A}^+$ is such that $\mu(T^2) = 1$. Then,*

$$\rho(\mu, \mu_T) = \mu(T) .$$

*Proof.* Choose any $[\phi, x, y] \in Q(\mu, \mu_T)$. A direct calculation shows that $[\phi, x, \phi(T)x]$ also $\in Q(\mu, \mu_T)$. Therefore

$$\rho(\mu, \mu_T) \geqq |(x|\phi(T)x)| = \mu(T) . \tag{2.1}$$

On the other hand, since $y$ and $\phi(T)x$ induce the same state relative to $\phi$ it is a standard result that $y = U'\phi(T)x$ for some partial isometry $U'$ in $(\phi(\mathscr{A}))'$ (see [3], Chapt. 1, § 4, Lemma 3). Therefore

$$\begin{aligned} |(x|y)| &= |(x|\, U'\phi(T)x)| \\ &= |(\,U'^*\phi(T)^{1/2}x|\,\phi(T)^{1/2}x)| \\ &\leqq ||\phi(T)^{1/2}x||^2 \\ &= \mu(T) . \end{aligned}$$

Taking the supremum over all $[\phi, x, y] \in Q(\mu, \nu)$ we obtain that

$$\rho(\mu, \mu_T) \leqq \mu(T) ,$$

which together with (2.1) completes the proof.

We now consider two $W^*$-algebras $\mathscr{A}_1$ and $\mathscr{A}_2$. For $j = 1$ or 2 let $\mu_j$ and $\nu_j$ be elements of $\Sigma_{\mathscr{A}_j}$. We want to prove the following:

$$\rho(\mu_1 \otimes \mu_2, \nu_1 \otimes \nu_2) = [\rho(\mu_1, \nu_1)][\rho(\mu_2, \nu_2)] . \tag{2.2}$$

LEMMA 2.2. *Suppose that for $j = 1$ or 2, $\mu_j = (\nu_j)_{T_j}$ for some $T_j \in \mathscr{A}_j^+$. Then* (2.2) *holds.*

*Proof.* $\mu_1 \otimes \mu_2 = (\nu_1)_{T_1} \otimes (\nu_2)_{T_2}$ which is easily seen to be equal to $(\nu_1 \otimes \nu_2)_{T_1 \otimes T_2}$. The result now follows from a direct calculation, using Lemma 2.1.

LEMMA 2.3. *For any $0 \leqq \delta \leqq 1$, let $\nu'_j = (1 - \delta)\nu_j + \delta\mu_j$, $j = 1$ or 2. Then*

(a) $|\rho(\mu_1 \otimes \mu_2, \nu_1 \otimes \nu_2) - \rho(\mu_1 \otimes \mu_2, \nu'_1 \otimes \nu'_2)| \leqq k\delta^{1/2}$,

(b) $|\rho(\mu_i, \nu_j) - \rho(\mu_j, \nu'_j)| \leqq k\delta^{1/2}$ ,

*where $k$ is a constant independent of $\delta$.*

*Proof.* From ([1], Proposition 1.8 (A))

$$\begin{aligned} & d(\nu_1 \otimes \nu_2, \nu_1 \otimes \nu_2') \\ &= d(\nu_1 \otimes \nu_?, [(1-\delta)\nu_1 \otimes \nu_2 + \delta(\nu_1 \otimes \mu_2)]) \\ &\leqq 2\delta^{1/2} . \end{aligned}$$

Similarly, $d(\nu_1 \otimes \nu_2', \nu_1' \otimes \nu_2') \leqq 2\delta^{1/2}$ so by the triangle inequality

$$d(\nu_1 \otimes \nu_2, \nu_1' \otimes \nu_2') \leqq 4\delta^{1/2} ,$$

and (a) follows from ([1], Proposition 1.9 $(B)$). Part (b) follows in a similar manner.

THEOREM 2.4. *Formula* (2.2) *holds in general.*

*Proof.* For any $0 < \delta < 1$, let $\nu_j'$ be defined as in Lemma 2.3. Then for $n > 1/\delta$, $\mu_j(A) \leqq n\nu_j'(A)$ for all $A \in \mathscr{A}_j^+$. By Sakai's Radon-Nikodym Theorem ([3], Chapt. 1, § 4, Th. 5), $\mu_j = (\nu_j')_{T_j}$ for some $T_j \in \mathscr{A}_j^+$. From Lemma 2.3

$$(\rho(\mu_1 \otimes \mu_2, \nu_1 \otimes \nu_2) - [\rho(\mu_1, \nu_1')][\rho(\mu_2, \nu_2')]) = 0 .$$

By taking $\delta$ sufficiently small and applying Lemma 2.3, we have that for any $\varepsilon > 0$

$$(\rho(\mu_1 \otimes \mu_2, \nu_1 \otimes \nu_2) - [\rho(\mu_1, \nu_1)][\rho(\mu_2, \nu_2)]) < \varepsilon$$

which completes the proof.

REMARK. We can of course remove the normalizing condition and define $d$ and $\rho$ for any positive, normal linear functionals, as was done in [1]. Since a vector $x$ induces the functional $\mu$ if and only if the vector $k^{1/2}x$ induces $k\mu$, we have that $\rho(k\mu, \nu) = k^{1/2}\rho(\mu, \nu)$ for all $k > 0$. Moreover we can still define $\mu \otimes \nu$ and the mapping $(\mu, \nu)$ to $(\mu \otimes \nu)$ is bilinear. It follows that Theorem 2.4 will hold for positive, normal, linear functionals.

**3. Application to infinite tensor products.** In ([1], Th. 4.1), the main result of that paper, the only need of the semi-finiteness restriction was to invoke the product formula. We can now appeal to Theorem 2.4 to conclude that this result holds in general. We will however present an alternate proof here, which at the same time extends the product formula to the case of infinite product states.

We begin by reviewing some basic definitions. See [2] for a complete discussion of the following concepts.

Let $I$ be an arbitrary indexing set and let $(\mathscr{A}_i)_{i \in I}$ be a family of $W^*$-algebras.

A *product* for this family is an object $(\mathscr{A}, (\alpha_i)_{i \in I})$, where $\mathscr{A}$ is a $W^*$-algebra and for each $i \in I$ $\alpha_i$ is a one-to-one homomorphism from $\mathscr{A}_i$ into $\mathscr{A}$ satisfying:

(a) $\alpha_i(\mathscr{A}_i)$ and $\alpha_j(\mathscr{A}_j)$ commute pointwise for $i \neq j$

(b) $\{\alpha_i(\mathscr{A}_i): i \in I\}$ generates $\mathscr{A}$ as a $W^*$-algebra.

We say that the products $(\mathscr{A}, (\alpha_i))$ and $(\mathscr{B}, (\beta_i))$ are *product isomorphic* if there exists an isomorphism $\phi$ from $\mathscr{A}$ onto $\mathscr{B}$ such that $\phi\alpha_i = \beta_i$ for all $i \in I$.

Let $\Lambda = \Lambda((\mathscr{A}_i))$ denote the set of all families $(\mu_i)_{i \in I}$ where $\mu_i \in \Sigma_{\mathscr{A}_i}$ for all $i \in I$. We say that $\mu$ is a *product state* of the product $(\mathscr{A}, (\alpha_i))$ if $\mu \in \Sigma_{\mathscr{A}}$, and for some $(\mu_i) \in \Lambda$ (necessarily unique),

$$\mu(\prod_{i \in F} \alpha_i(A_i)) = \prod_{i \in F} \mu_i(A_i)$$

for all finite $F \subset I$ and all $A_i \in \mathscr{A}_i$. We denote such a state by $\bigotimes_{i \in I} \mu_i$.

DEFINITION 3.1. For any $(\mu_i) \in \Lambda$ we define a product, denoted by $\bigotimes_{i \in I}(\mathscr{A}_i, \mu_i)$, as follows.

For each $i \in I$, let $\mathscr{A}_i$ be represented as a von Neumann algebra on a Hilbert space $H_i$ such that $x_i \in H_i$ induces $\mu_i$. Let

$$H = \bigotimes_{i \in I}(H_i, x_i)$$

be von Neumann's incomplete tensor product of $(H_i)$ with respect to the $C_0$-sequence $(x_i)$ [5]. For any $k \in I$ and $A_k \in H_k$ let $\bar{A}_k$ denote the unique element of $\mathscr{L}(H)$ such that $\bar{A}_k(\otimes y_i) = \otimes y'_i$ where $y'_k = A_k y_k$ and $y'_i = y_i$ for $i \neq k$. Let $\mathscr{A}$ be the von Neumann algebra on $H$ generated by the $\mathscr{A}_i$. Then the product $\bigotimes_{i \in I}(\mathscr{A}_i, \mu_i)$ is defined to be the algebra $\mathscr{A}$, together with the injections $\alpha_i$ given by $\alpha_i(A_i) = \bar{A}_i$ for all $A_i \in \mathscr{A}_i$.

See [2] for an alternative method of defining $\otimes(\mathscr{A}_i, \mu_i)$ and a justification of the above definition. It is shown that the product constructed as above is unique up to product isomorphism ([2], Th. 4.7).

Note that the product state $\otimes\mu_i$ exists on $\otimes(\mathscr{A}_i, \mu_i)$. In fact in the construction above it is induced by the vector $\otimes x_i$. It will follow from the results in this section that the converse holds. That is, if a product constructed as above from an element of $\Lambda$ admits $\otimes\mu_i$ as a product state, then this product is product isomorphic to $\otimes(\mathscr{A}_i, \mu_i)$.

If $I$ is a finite set it is well known that the $\otimes(\mathscr{A}_i, \mu_i)$ are all product isomorphic for any choice of $(\mu_i) \in \Lambda$, and the resulting product is simply $\bigotimes_{i \in I}\mathscr{A}_i$, the usual $W^*$-tensor product of a finite family.

DEFINITION 3.2. Let $(\mathscr{A}, (\alpha_i))$ be a product for the family $(\mathscr{A}_i)_{i \in I}$.

Let $\phi$ be any representation of $\mathscr{A}$ and let $\mu \in \Sigma_{\mathscr{A}}$. Then for any nonvoid $J \subset I$ we let $\mathscr{A}^J =$ the $W^*$-algebra generated by

$$\{\alpha_i(A_i): i \in J, A_i \in \mathscr{A}_i\} ,$$

and we let $\phi^J$, $\mu^J$ denote respectively the restrictions of $\phi$, $\mu$ to $\mathscr{A}^J$.

LEMMA 3.3. *Let $\mathscr{A} =$ the algebra $\bigotimes_{i \in I}(\mathscr{A}_i, \mu_i)$. Let $\mu = \bigotimes \mu_i$ and let $\nu$ be any element of $\Sigma_{\mathscr{A}}$. Then there exists $J \subset I$ with finite compliment such that $\rho(\mu^J, \nu^J) > 0$.*

*Proof.* Suppose to the contrary that $\rho(\mu^J, \nu^J) = 0$ for all $J \subset I$ with finite compliment. Let $(H_i)$, $(x_i)$ and $H$ be as in Definition 3.1. It is well known that we can choose an orthonormal basis of $H$ with the property that every basis element $x'$ is of the form $\bigotimes_{i \in I} x'_i$, where for all but a finite number of $i \in I$ $x'_i = x_i$. See ([5], Lemma 4.14). Fix such a basis element $x'$ and let $J = \{i \in I: x' = x_i\}$. Obviously $x'$ induces $\mu^J$ on $\mathscr{A}^J$. Then by our assumption and Lemma 1.2,

$$\|S(\nu^J)x'\|^2 = \mu^J(S(\nu^J)) = 0 .$$

Since $\nu(S(\nu^J)) = \nu^J(S(\nu^J)) = 1$ we have $S(\nu) \leqq S(\nu^J)$, and therefore $S(\nu)x' = 0$.

Since this is true for all $x'$ in some basis of $H$ we have $S(\nu) = 0$, a contradiction.

REMARK. We next recall some elementary facts about infinite products of numbers. If $(r_i)_{i \in I}$ is family of nonnegative numbers, $\prod_{i \in I} r_i$ is said to converge if and only if for some $J \subset I$ with finite compliment, $\lim_F (\prod_{i \in F} r_i)$ as $F$ runs over the finite subsets of $J$ exists as a positive number. The value of $\prod_{i \in I} r_i$ is then defined to be $\lim_F \prod_{i \in F} r_i$ as $F$ runs over the finite subsets of $I$. It follows that $\prod_{i \in I} r_i$ converges if and only if $\sum_{i \in I} |1 - r_i| < \infty$.

THEOREM 3.4. *Suppose that the product state $\nu = \bigotimes_{i \in I} \nu_i$ exists on $\bigotimes_{i \in I}(\mathscr{A}_i, \mu_i)$. Let $\mu = \bigotimes_{i \in I} \mu_i$. Then $\prod_{i \in I} \rho(\mu_i, \nu_i)$ converges, and*

$$\rho(\mu, \nu) = \prod_{i \in I} \rho(\mu_i, \nu_i) .$$

*Proof.* Choose any $[\phi, x, y] \in Q(\mu, \nu)$ and let $F$ be any finite subset of $I$. It is evident that $[\phi^F, x, y] \in Q(\mu^F, \nu^F)$ so that $\rho(\mu^F, \nu^F) \geqq |(x | y)|$. Taking the supremum over all elements of $Q(\mu, \nu)$ we obtain

$$\rho(\mu, \nu) \leqq \rho(\mu^F, \nu^F) . \tag{3.1}$$

It is obvious that $\mu^F = \bigotimes_{i \in F} \mu_i$ and $\nu^F = \bigotimes_{i \in F} \nu_i$ on $\bigotimes_{i \in F} \mathscr{A}_i$. So by

Theorem 2.4 which extends to any finite number of factors by an obvious induction and the associativety properties of tensor products, we obtain that

$$\rho(\mu^F, \nu^F) = \prod_{i \in F} \rho(\mu_i, \nu_i) .$$

Then from (3.1) we have

$$\rho(\mu, \nu) \leqq \inf \{\prod_{i \in F} \rho(\mu_i, \nu_i) \colon F \text{ a finite subset of } I\} . \tag{3.2}$$

Now by Lemma 3.2 choose $J \subset I$ with finite compliment such that $\rho(\mu^J, \nu^J) > 0$. By applying the above argument to the algebra $\otimes_{i \in J}(\mathscr{A}_i, \mu_i)$ we see from (3.2) that

$$0 < \inf \{\prod_{i \in F} \rho(\mu_i, \nu_i) \colon F \text{ a finite subset of } J\} . \tag{3.3}$$

Since the value of $\rho$ is $\leqq 1$, (3.3) shows that $\prod_{i \in I} \rho(\mu_i, \nu_i)$ converges and (3.2) shows that

$$\rho(\mu, \nu) \leqq \prod_{i \in I} \rho(\mu_i, \nu_i) . \tag{3.4}$$

We now prove the other direction. Let $k$ be any positive number $<1$ and choose a sequence $(k_n)$ of positive numbers $<1$ such that $\prod_{n=1}^{\infty} k_n = k$. Let $I_0 = \{i \in I \colon d(\mu_i, \nu_i) > 0\}$. From the convergence of $\prod \rho(\mu_i, \nu_i)$ and formula (1.1) we see that $I_0$ is at most countable. Let $\gamma$ be an injection from $I_0$ into the positive integers, and let $g(i) = k_{\gamma(i)}$ for $i \in I_0$, $g(i) = 1$ for $i \in I - I_0$. Then choose for each $i \in I$ an element $[\phi_i, x_i, y_i]$ of $Q(\mu_i, \nu_i)$ such that $|(x_i | y_i)| \geqq g(i)[\rho(\mu_i, \nu_i)]$. (This is certainly possible by the definition of $\rho$ and the fact that $\mu_i = \nu_i$ for $i \in I - I_0$). By multiplying the vectors by suitable scalars of absolute value 1 we may assume

$$1 \geqq (x_i | y_i) \geqq g(i)[\rho(\mu_i, \nu_i)] . \tag{3.5}$$

Evidently $\prod_{i \in I} g(i)$ converges and its value is $\geqq \prod_{n=1}^{\infty} k_n = k$. So from (3.5) $\prod_{i \in I} (x_i | y_i)$ converges and

$$\prod_{i \in I} (x_i | y_i) \geqq k \prod_{i \in I} \rho(\mu_i, \nu_i) . \tag{3.6}$$

We have then that $\sum_{i \in I} |1 - (x_i | y_i)| < \infty$ which shows that for the family of Hilbert spaces $(H_i)$, where $H_i$ is the underlying space of $\phi_i$, $(x_i)$ and $(y_i)$ are equivalent $C_0$-sequences ([5], Definition 3.3.2). There exists therefore a vector in $\otimes_{i \in I}(H_i, x_i)$ of the form $\otimes y_i$ and this obviously induces the state $\nu$. Using (3.6),

$$\rho(\mu, \nu) \geqq |(\otimes x_i | \otimes y_i)| = \prod (x_i | y_i) \geqq k \prod \rho(\mu_i, \nu_i) .$$

Since $k$ was chosen arbitrarily we have that

$$\rho(\mu, \nu) \geqq \prod \rho(\mu_i, \nu_i)$$

which together with (3.4) completes the proof.

COROLLARY 3.5. *Let* $(\mathscr{A}_i)$ *be a family of* $W^*$*-algebras and let* $(\mu_i)$ *and* $(\nu_i) \in \Lambda$. *Then the following conditions on* $(\mu_i)$ *and* $(\nu_i)$ *are equivalent*:

(a) $\sum_{i \in I} [d(\mu_i, \nu_i)]^2 < \infty$;

(b) $\bigotimes_{i \in I}(\mathscr{A}_i, \mu_i)$ *and* $\bigotimes_{i \in I}(\mathscr{A}_i, \nu_i)$ *are product isomorphic*;

(c) $\bigotimes_{i \in I}\nu_i$ *exists as a product state on* $\bigotimes_{i \in I}(\mathscr{A}_i, \mu_i)$.

*Moreover if any of these conditions hold,*

$$\rho(\mu, \nu) = \prod_{i \in I} \rho(\mu_i, \nu_i), \text{ a convergent product .}$$

*Proof.* By Theorem 3.4 and formula (1.1), (c) implies (a). It follows easily from the second part of the proof of Theorem 3.4 that (a) implies (b). This is also proved in ([1], Lemma 3.6). It is immediate that (c) implies (a) and the final statement is immediate from Theorem 3.4.

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