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# ON THE CONJUGATING REPRESENTATION OF A FINITE GROUP

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**A natural permutation representation for any finite group is the conjugating representation $T$: for each $g \in G$, $T(g)$ is the permutation on the set $\{x \mid x \in G\}$ given by $T(g)(x) = gxg^{-1}$. Frame, Solomon and Gamba have studied some of its properties. This paper considers the question of which complex irreducible representations occur as components of $T$, in particular the conjecture that any such representation whose kernel contains the center of $G$ is a component of $T$. This conjecture is verified for a few special cases and a number of related results are obtained, especially with respect to the one-dimensional components of $T$.**

In §2 we see that the conjecture does hold for groups of "central type" which were studied by DeMeyer and Janusz in [4]. In § 3 we obtain further information with respect to the linear characters of $G$; it is shown that if $G/H$ is a cyclic group then the number of irreducible characters of $G$ which are induced from irreducible characters of $H$ is the same as the number of conjugacy classes of $G$ having the property that the centralizers of their elements belong to $H$. This number is precisely the multiplicity in the conjugating representation of a linear character of $G$ whose kernel is $H$.

NOTATION. $G$ is a finite group with conjugacy classes $C_1, C_2, \cdots, C_k$. $\chi^1, \chi^2, \cdots, \chi^k$ are the irreducible complex characters of $G$. $\{g_1, g_2 \cdots, g_k\}$ will be a set of representatives of the conjugacy classes with $g_j \in C_j$ for $j = 1, 2, \cdots, k$. We let $T$ denote the conjugating representation of $G$ defined above and $\theta$ will be the character of $G$ corresponding to $T$. The transitivity classes (orbits) under $T$ are then $C_1, \cdots, C_k$ and restricting $T$ to the set $C_i$ gives the corresponding transitive permutation representation $T^i$ where $i = 1, 2, \cdots, k$. Let $\varphi^i$ be the character of $T^i$ for each $i$, so that $\theta = \sum_{i=1}^{k} \varphi^i$.

If $\eta$ and $\lambda$ are two complex-valued characters on $G$, then $(\eta, \lambda)$ will denote the usual "inner product" given by

$$(\eta, \lambda) = |G|^{-1} \sum_{g \in G} \eta(g) \overline{\lambda(g)}$$

where $\overline{\lambda(g)}$ is the complex conjugate of $\lambda(g)$, and $|G|$ is the order of $G$. $Z$ will denote the center of the group $G$. The kernel of $\lambda$, denoted $\operatorname{Ker} \lambda$, is to mean the kernel of a representation affording the character $\lambda$. The underlying field is always assumed to be the complex numbers. If $g \in G$ then $C(g)$ denotes the centralizer of $g$ in $G$. For general background material the reader is referred to [3] and [5].

## 1. General properties of the conjugating representation.

LEMMA 1.1. *$\theta = \sum_{i=1}^{k} a_i \chi^i$ where $a_i = \sum_{j=1}^{k} \chi^i(g_j)$.*

*Proof.* This was proved by Solomon in [11]. See also Theorem 6.5 in [5]. This lemma is also noted without proof in [7, p. 192].

We use lemma 1.1 to give a new proof of the following theorem due to Frame (see [6]).

THEOREM 1.2.

$$\theta = \sum_{i=1}^{k} \chi^i \overline{\chi^i} .$$

*Proof.*

$$\text{Let } \sum_{i=1}^{k} \chi^i \overline{\chi^i} = \sum_{j=1}^{k} b_j \chi^j .$$

Then

$$\begin{aligned} b_j &= \left(\sum_{i=1}^{k} \chi^i \overline{\chi^i}, \chi^j\right) = \sum_{i=1}^{k} \left(\chi^i \overline{\chi^i}, \chi^j\right) \\ &= \sum_{i=1}^{k} \frac{1}{|G|} \sum_{g \in G} \chi^i(g) \overline{\chi^i(g)} \overline{\chi^j(g)} \\ &= \sum_{g \in G} \frac{1}{|G|} \left(\sum_{i=1}^{k} \chi^i(g) \overline{\chi^i(g)}\right) \overline{\chi^j}(g) \\ &= \sum_{g \in G} \frac{1}{|G|} \frac{|G|}{h(g)} \chi^{\overline{j}}(\overline{g}) = \sum_{l=1}^{k} \overline{\chi^j}(g_l) \\ &= \overline{\sum_{l=1}^{k} \chi^j(g_l)} = \bar{a}_j = a_j \quad \text{(by (1.1))} . \end{aligned}$$

(Here $h(g)$ denotes the number of elements in the conjugacy class of $g$). So

$$\theta = \sum_{j=1}^{k} a_j \chi^j = \sum_{j=1}^{k} b_j \chi^j = \sum_{i=1}^{k} \chi^i \overline{\chi^i} .$$

LEMMA 1.3. *(Frame ⋯ [6]). If $\chi^j$ appears in the decomposition of $\theta$ (i.e., if $a_j > 0$) then $Z$ is contained in* Ker $\chi^j$.

*Proof.* If $z \in Z$, then $T(z)$ corresponds to the identical transformation and $z$ must be in the kernel of each of the irreducible components of $T$.

We conjecture that the converse is also true, i.e.,

CONJECTURE 1.4. If $\chi^j$ is a complex irreducible character and $Z \subseteqq \operatorname{Ker} \chi^j$ then $a_j > 0$.

In seeking to prove this conjecture it is of interest to examine the more specific problem of finding conditions on $C_j$ and $\chi^i$ such that $\chi^i$ appears in the decomposition of $\varphi^j$, i.e., in the special conjugating representation afforded by the $j^{th}$ conjugacy class. To that end we have

LEMMA 1.5. *Let $\chi$ be a complex irreducible character of $G$.*

( i ) *$\chi$ occurs in the decomposition of $\varphi^j$ precisely $m$ times where $m$ is the multiplicity of the 1-representation of the restriction of $\chi$ to $C(g_j)$.*

(ii) *If $\chi$ is a linear character, then $\chi$ occurs in the decomposition of $\varphi^j$ at most once and occurs once precisely if $C(g_j) \subseteqq \operatorname{Ker} \chi$.*

REMARK. The above lemma is independent of the choices of representatives $g_j$ for the conjugacy classes.

*Proof.* Under the transitive permutation representation $T^j$ of $G$ defined on the set $C_j$, $C(g_j)$ is the subgroup of $G$ of elements which leave the given element $g_j$ fixed. $T^j$ may thus be regarded as the representation induced from the 1-representation on $C(g_j)$. By the Frobenius reciprocity theorem,

$$m = (\chi, \varphi^j)_G = (\chi \mid C(g_j), 1)_{C(g_j)}$$

where 1 here stands for the 1-character of $C(g_j)$.

(ii) By ( i ) if $\chi$ is linear it occurs as a component of $\varphi^j$ precisely if $\chi$ restricted to $C(g_j)$ is the 1-representation in which case $m = 1$ and $\operatorname{Ker} \chi \supseteqq C(g_j)$.

2. **Groups of central type.** In the paper of DeMeyer and Janusz ([4]), a group of central type is defined to be a group having an irreducible character $\chi$ on $G$ with $\chi(1)^2 = [G: Z]$. We see in the following theorem that conjecture (1.4) holds for these groups.

THEOREM 2.1. *Let $G$ be a finite group of central type. Then every irreducible character $\psi$ with $Z \subseteqq \operatorname{Ker} \psi$ appears as a component of the conjugating character $\theta$ at least $n$ times where $n$ is the degree of $\psi$.*

*Proof.* Let $\chi$ be an irreducible character of $G$ with $\chi(1)^2 = [G\colon Z]$. By Corollary 1 in [4], $\chi(g) = 0$ for $g \notin Z$. Let $\psi$ be an irreducible character of degree $n$ with $Z \subseteqq \operatorname{Ker} \psi$. For $g \in Z$, $\psi(g)\chi(g) = n\chi(g)$. For $g \notin Z$, $\psi(g)\chi(g) = 0 = n0 = n\chi(g)$. I.e., $\psi\chi = n\chi$.

Now by [5, (6.6)] or by [3, p. 274], we have

$$(\psi, \chi\bar{\chi}) = (\chi, \psi\chi) = (\chi, n\chi) = n\,.$$

By Theorem 1.2, $\theta = \sum_{i=1}^{k} \chi^i \overline{\chi^i}$ so $\psi$ appears in $\theta$ at least $n$ times.

REMARK. The above proof can be adapted to prove the converse of Corollary 1 in [4], namely that if $\chi(g) = 0$ for $g \notin Z$ then $\chi(1)^2 = [G\colon Z]$. Professor Janusz has noted to the author that this follows more directly by observing that $1 = (\chi, \chi) = (1/|G|)\, \chi(1)^2 |Z|$.

As an example of these groups we consider the following:

THEOREM 2.2. *Let $G$ be a nilpotent group of class 2 with cyclic center. Then $G$ is of central type and every irreducible character $\psi$ with $Z \subseteqq \operatorname{Ker} \psi$ appears as a component of $\theta$.*

*Proof.* Kochendörffer has shown in [9] that a nilpotent group with cyclic center has a faithful irreducible complex character $\chi$ (i.e., see Theorem 4). By Lemma 9, p. 1482 in [8], $\chi(g) = 0$ for $g \notin Z$. By the remark preceding the theorem we see that $G$ is of central type, and the second statement follows from (2.1) (or its proof).

REMARK. We note that in the above case $G' \subseteqq Z$, so if $Z \subseteqq \operatorname{Ker} \psi$, $\psi$ is of necessity a linear character. In the next section we concentrate on the relation of linear characters to the conjugating representation.

## 3. Linear characters.

THEOREM 3.1. *Let $\lambda$ be a linear character of $G$ and $\rho$ an irreducible character of $G$. Then $\lambda\rho = \rho \Leftrightarrow \rho$ is induced from an irreducible representation on* $\operatorname{Ker} \lambda$.

*Proof.* Suppose $\rho$ is induced from an irreducible character of the normal subgroup $\operatorname{Ker} \lambda$. Then $\rho(x) = 0$ for $x \notin \operatorname{Ker} \lambda$ and since $\lambda(x) = 1$ for $x \in \operatorname{Ker} \lambda$, we have $\lambda\rho = \rho$.

Conversely, suppose that $\lambda\rho = \rho$. Let $R$ be a representation of $G$ on a complex vector space $V$ such that $R$ affords the character $\rho$. Then there exists an invertible linear transformation $S$ of $V$ such that $SR(g)S^{-1} = \lambda(g)R(g)$ for all $g$ in $G$. Thus $SR(g) = \lambda(g)R(g)S$ for all $g$ in $G$.

Now let $v$ be an eigenvector of $S$ and $\mu$ be the corresponding eigenvalue.

$$(1) \qquad SR(g)v = \lambda(g)R(g)Sv = \lambda(g)R(g)\mu v = (\lambda(g)\mu)R(g)v \,.$$

Hence $R(g)v$ is an eigenvector of $S$ with eigenvalue $\lambda(g)\mu$. Thus each distinct value of $\lambda$ gives a distinct eigenvalue of $S$. Let $h_1=1,\cdots,h_r$ be coset representatives for a coset decomposition of $G$ modulo the kernel of $\lambda$. Then $\lambda(h_1), \lambda(h_2) \cdots \lambda(h_r)$ are precisely the distinct values that $\lambda$ takes on. For $i = 1, 2, \cdots r$ let $V_i$ be the eigenspace of $V$ consisting of all eigenvectors of $S$ with eigenvalue $\lambda(h_i)\mu$. If $g \in G$ and $v_i \in V_i$, then $R(g)v_i$ is an eigenvector with value $\lambda(g)\lambda(h_i)\mu = \lambda(h_j)\mu$ for some $j$, by equation (1). $R(g)$ thus maps $V_i$ injectively into $V_j$. $R(g^{-1})$ similarly maps $V_j$ injectively into $V_i$ so both subspaces have the same dimension and $R(g)$ maps $V_i$ injectively onto $V_j$. The subspace $V_1 \oplus V_2 + \cdots \oplus V_r$ is evidently invariant under the representation $R$ and since $R$ is irreducible, $V = V_1 \oplus \cdots \oplus V_r$. Also, $\{V_1, \cdots, V_r\}$ forms a system of imprimitivity for $V$. $R(g)$ leaves $V_1$ invariant precisely if $g \in \operatorname{Ker} \lambda$. Hence by Theorem 50.2 in [3], $V_1$ affords a representation of $\operatorname{Ker} \lambda$ which induces the representation $R$ of $G$. Clearly this representation of $\operatorname{Ker} \lambda$ must be irreducible since otherwise $R$ would not be irreducible.

Let $H$ be any normal subgroup of a group $G$ and $\psi_1$ and $\psi_2$ characters of $H$. If there exists $g \in G$ such that $\psi_1(x) = \psi_2(gxg^{-1})$ for all $x \in H$, then we say that $\psi_1$ and $\psi_2$ are $G$-conjugate (see [3, p. 278, Ex. 6; also p. 343]). The irreducible characters of $H$ are thus divided up into "$G$-conjugacy classes". Let $N(\psi) =$ the "normalizer" of $\psi$ in $\{G = g \in G \mid \psi(x) = \psi(gxg^{-1})$ all $x \in G\}$.

THEOREM 3.2. *Let $G$ be a finite group, $H$ a normal subgroup such that $G/H$ is cyclic. The following four numbers are then equal:*

*$a =$ the number of conjugacy classes $C_i$ such that $C(g_i) \subsetneqq H$.*

*$b =$ the number of $G$-conjugacy classes of irreducible characters $\psi$ of $H$ such that $N(\psi) \subsetneqq H$.*

*$c =$ the multiplicity of a linear character $\lambda$ in $\theta$, where $\lambda$ is any linear character with* $\operatorname{Ker} \lambda = H$.

*$d =$ the number of distinct irreducible characters of $G$ which are induced from irreducible characters of $H$.*

*Proof.* Let $\lambda$ be any linear character of $G$ with $H = \operatorname{Ker} \lambda$; since $G/H$ is cyclic there exist linear characters satisfying this condition. By (1.5), part (ii), $\lambda$ occurs in the conjugating representation as many times as there are conjugacy classes $C_j$ with $C(g_j) \subsetneqq \operatorname{Ker} \lambda = H$. Thus $a = c$. By Theorem 1.2, $\theta = \sum \chi^i \bar{\chi}^i$. Now $(\lambda, \chi^i\bar{\chi}^i) = (\chi^i, \lambda\chi^i) = 1$

or 0 depending on whether $\chi^i = \lambda\chi^i$ or not. Hence if $\lambda$ occurs in $\theta$ $c$ times then there are precisely $c$ irreducible characters $\chi^i$ of $G$ such that $\lambda\chi^i = \chi^i$. By Theorem 3.1 these are precisely the characters of $G$ induced from irreducible characters of $H$, so $c = d$.

Let $\psi$ be an irreducible character of $H$. Then $\psi^G$ is irreducible precisely if $\psi^g \neq \psi$ for $g \notin H$(by (45.5) in [3]). This means that $\psi^g = \psi$ implies that $g \in H$; i.e., $N(\psi) \subseteqq H$. By Exercise 5, p. 278, in [3] two conjugate characters induce the same character of $G$. Now (45.6) in [3] applied in the case that $H_1 = H_2$ shows that two non-conjugate irreducible characters can't induce the same irreducible character of $G$. Hence $b = d$.

We describe the constant from (3.2) in still another way. $G/H$ may be considered as a group operating by conjugation on the set of conjugacy classes $D_1, \cdots, D_t$ of $H$. Let $\{D_1, \cdots, D_m\}$ be an orbit under this operation; i.e., $\bigcup_{i=1}^m D_i$ is a conjugacy class of $G$ contained in $H$. In the following lemma, the phrase "$G/H$ operates regularly on the orbit $\{D_1, \cdots, D_m\}$" means $G/H$ permutes the set transitively and no element except the identity leaves any element fixed. The following lemma shows that $a$ of (3.2) equals the number of orbits on which $G/H$ acts regularly.

LEMMA 3.3. *$G/H$ operates regularly on the orbit $\{D_1, \cdots, D_m\}$ if and only if $C(x) \subseteqq H$ where $x \in \bigcup_{i=1}^m D_i$.*

(Note this is independent of the choice of $x$.)

*Proof.* Say $G/H$ is regular on $\{D_1, \cdots, D_m\}$ and $g \in C(x)$. If $x \in D_i$ then $gH$ operating on the orbit $\{D_1, \cdots, D_m\}$ fixes $D_i$ so $gH = H$ and $g \in H$, i.e., $C(x) \subseteqq H$.

Conversely, say $C(x) \subseteqq H$. Let $g \in G$ and suppose $gH$ fixes $D_1$ (for example). Let $x \in D_1$, $gxg^{-1} = x'$ with $x' \in D_1$. Then there exists $h \in H$ such that $hxh^{-1} = x'$. Hence $h^{-1}g \in C(x) \subseteqq H$ and so $g \in H$. Thus $gH = H$ and $G/H$ operates regularly on the orbit.

REMARK. Using a lemma of Brauer (Lemma 1, § 6 in [1]) an alternate proof can be given to show that $a = b$ in Theorem 3.2. For $G/H$ is cyclic and if $r = |G/H|$ then the number of orbits of length $r$ under the action of $G/H$ on the conjugacy classes of $H$ equals the number of orbits of length $r$ in the action of $G/H$ on the characters of $H$. The latter number is in fact $b$ while the former equals $a$ by Lemma 3.3. (See [10, Proposition 1.5] for a similar use of another lemma of Brauer).

We now verify that conjecture (1.4) holds in one more special case.

THEOREM 3.4. *Let $G$ be a finite group and $p$ a prime such that $p \mid G$ but $p^2 \nmid G$. Let $\lambda$ be a linear character $G$ taking on exactly $p$ values. If $Z \subseteqq \operatorname{Ker} \lambda$ then $\lambda$ occurs as a component of the conjugating character $\theta$.*

*Proof.* By the Schur-Zassenhaus theorem ([3, (7.5)]) we may regard $G$ as the semidirect product of Ker $\lambda$ and a cyclic group $P$ or order $p$. The elements $\neq 1$ of $P$ induce nontrivial automorphisms of Ker $\lambda$ by conjugation (since $Z \subseteqq$ Ker $\lambda$). Theorem II, p. 89 of Burnside's book ([2]) states: "An isomorphism of a group $G$ whose order contains a prime factor which does not occur in the order of $G$ must interchange some of the conjugate sets of $G$". Thus if $\alpha$ is one of the nontrivial automorphisms of $P$, since it is of order $p$, there must be $p$ classes $D_1, \cdots, D_p$ of conjugacy classes of Ker $\lambda$ regularly permuted in a cycle. By (3.3) and (3.2) the multiplicity of $\lambda$ in $\theta$ is at least 1.

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