DIVISOR CLASSES IN PSEUDO GALOIS EXTENSIONS

WILLIAM CHARLES WATERHOUSE
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Let \( R \) be a Krull domain with fraction field \( K \). Let \( L \) be a finite extension of \( K \), and let \( S \) be the integral closure of \( R \) in \( L \); then \( S \) is also a Krull domain. Let \( \mathcal{P}(R, S) \) be the group of divisor classes in \( R \) becoming principal in \( S \). Suppose there is a group scheme (or Hopf algebra) acting on \( S \) with fixed ring \( R \). Then there is a cohomology group which contains \( \mathcal{P}(R, S) \) and equals it if the action is Galois at each minimal prime. This generalizes and unifies some results of Samuel.

1. Definition of the cohomology group. Let \( R, K, L \) and \( S \) be as above. Let \( H \) be a cocommutative Hopf algebra over \( R \), with \( \delta, \varepsilon, \) and \( \rho \) its comultiplication, counit, and coinverse. One calls \( S \) an \( H \)-module algebra \([9, \text{p. 207}]\) if it has an \( H \)-module structure such that \( h \cdot 1 = \varepsilon(h) \) and \( h \cdot (ss') = \sum (h_i \cdot s)(h'_i \cdot s') \) where \( \delta(h) = \sum h_i \otimes h_i' \). We say that \( R \) is the fixed ring in \( S \) if

\[ R = \{ s \in S | h \cdot s = \varepsilon(h)s \quad \text{for all} \quad h \in H \}. \]

In this case \( L \) is naturally an \( H \)-module algebra with fixed ring \( K \).

Suppose now \( S \) is an \( H \)-module algebra with fixed ring \( R \), and consider the set

\[ \{ b \in L^* | b^{-1}(h \cdot b) \in S \quad \text{for all} \quad h \in H \}. \]

This is a group under multiplication: if \( b \) and \( c \) are in it, we have

\[ (bc)^{-1}h \cdot (bc) = \sum (b^{-1}h_i \cdot b)(c^{-1}h'_i \cdot c) \]

and

\[ (h \cdot b^{-1})b = \sum h_i \cdot [b^{-1}(\rho h'_i) \cdot b]. \]

It contains \( S^* \) and \( K^* \) as subgroups. We write \( H^\circ(H, L^*/S^*) \) for its quotient by \( S^* \), and \( \mathcal{A}(H, S) \) for the quotient by \( S^*K^* \). Note that \( h \mapsto b^{-1}h \cdot b \) defines a function \( H \to S \); it is easy to check that \( b \) and \( c \) give the same function if and only if \( bc^{-1} \) is in the fixed ring \( K \), and hence we can also view \( \mathcal{A} \) as these functions modulo the functions coming from units \( b \in S^* \).

**Proposition 1.** Assume \( S \) is an \( H \)-module algebra with fixed ring \( R \). Then there is a canonical injection

\[ \mathcal{P}(R, S) \to \mathcal{A}(H, S). \]
Proof. Let $D$ be a divisorial ideal of $R$ with $\text{div} (DS)$ principal, say $= bS$. Let $P$ be a minimal prime of $R$, and choose $r \in K$ with $\text{ord}_P r = \text{ord}_P D$; then $bS_P = rS_P$. For any $h \in H$ we have

$$h \cdot b \in h \cdot rS_P = rh \cdot S_P \subseteq rS_P = bS_P,$$

and hence $b^{-1}h \cdot b \in \bigcap_P S_P = S$. The element $b$ is well determined up to multiplication by an element of $S^*$, and thus we have a map (obviously a homomorphism) from such ideals $D$ to $H(H, L^*/S^*)$. Since $\text{div} (DS) = S$ implies $D = R$, the map is injective. Divide now by $K^*$ in both places.

One can define [9] a sequence of cohomology groups $H^i(H, S^*)$. In that theory $H^1(H, S^*)$ consists of certain equivalence classes of functions $H \to S$; it maps naturally to $H^1(H, L^*)$, and the kernel comprises functions of the form $h \mapsto b^{-1}h \cdot b$. Under our hypotheses also $H^0(H, S^*) = R^*$ and $H^0(H, L^*) = K^*$. Thus our group $H^0(H, L^*/S^*)$ fits into an exact sequence, and $\mathcal{E}(H, S)$ is its image in $H^1(H, S^*)$.

Suppose that $G$ is a group, $H = R[G]$. To make $S$ an $H$-module algebra is simply to let $G$ act as $R$-algebra automorphisms of $S$. The definition of fixed ring is then the usual one, and $H^0(H, L^*/S^*)$ is the subset of $L^*/S^*$ fixed by $G$. In addition [9, p. 211], the cohomology $H^1(H, S^*)$ is naturally isomorphic to $H^1(G, S^*)$.

Suppose on the other hand that $H$ is the polynomial ring $R[X]$, with $\delta(X) = X \otimes 1 + 1 \otimes X$, $\varepsilon(X) = 0$, and $\rho(X) = -X$. Then an $H$-module algebra structure is given by an $R$-linear derivation $D: S \to S$ (where $Ds = X \cdot s$). The fixed ring is $\{s | Ds = 0\}$. The values $b^{-1}h \cdot b$ are determined by $b^{-1}Db$, and all lie in $S$ if this one does; hence $\mathcal{E}(H, S)$ can be identified with the logarithmic derivatives $Db/b$ lying in $S$, modulo the logarithmic derivatives of elements of $S^*$. Thus it is the group introduced by Samuel in [7, p. 86], and our formalism unifies the two separate theories he presents. We could similarly take a finite set of derivations, let $H$ be an enveloping algebra for them, and get the group used in [10] and [11]. (The paper [11] contains a different connection between Samuel's group and cohomology, but it appears to be ad hoc rather than natural.)

Suppose that $H$ is finite, i.e., a finitely generated projective $R$-module; this is the most important case. Let $A = \text{Hom} (H, R)$ be the linear dual, a commutative Hopf algebra. Making $S$ an $H$-module algebra is then the same thing as giving an algebra homomorphism $\sigma: S \to A \otimes R S$ suitably compatible with the comultiplication and counit of $A$ (cf. [5, p. 33]); in geometric language, this is an action of the finite group scheme $\text{Spec } A$ on $\text{Spec } S$ over $\text{Spec } R$. In these terms

$$\mathcal{E}(H, S) = \{\sigma(b)b^{-1} | b \in L^*, \sigma(b)b^{-1} \in (A \otimes S^*)/S^* \};$$
the group $H'(H, S^*)$ is the quotient by $S^*$ of the equalizer of two homomorphisms from $(A \otimes S)^*$ to $(A \otimes A \otimes S)^*$, and so on. One could phrase all the results equally well in terms of $A$, and I have used $H$ only because it is closer to the language used in the literature.

2. Conditions for isomorphism. Assume $S$ is an $H$-module algebra with $H$ finite. We say that $S$ with this structure is Galois if the following equivalent conditions hold [5, p. 66]:

(I) $S$ is a finitely generated projective $R$-module, and the map $H \otimes_R S \to \text{End}_R S$ given by $h \otimes s \mapsto [s \mapsto s \cdot h]$ is an $R$-module isomorphism.

(II) $S$ is a faithfully flat $R$-module, and

$$(\sigma, 1 \otimes \text{id}_S) : S \otimes_R S \to A \otimes_R S$$

is an $R$-algebra isomorphism. In geometric language, this says [6, p. 27] that Spec $S$ is a principal homogeneous space for Spec $A$. It implies that $R$ is the fixed ring.

**Proposition 2.** Suppose $H$ is finite. If $L$ is Galois as an $H \otimes_R K$-module algebra, then

$$L(H, S) = H'(H, S^*) .$$

**Proof.** This will follow if we show that $H'(H, L^*) = 0$. But it is easy to see from the definition (cf. end of § 1) that this group equals $H'(H \otimes K, L^*)$, which since the structure is Galois equals [9, p. 219] the Amitsur cohomology $H'(L/K, G_m)$; this is 0 by the generalized Hilbert Theorem 90 [1, p. 96 or 6, p. 15].

**Theorem 1.** Assume $S$ is an $H$-module algebra with $H$ finite. The following are equivalent:

(i) For all minimal primes $P$ of $R$, the $H_P$-structure on $S_P$ is Galois.

(ii) $R$ is the fixed ring, and for all minimal primes $P$ of $R$ the $H_P/PH_P$-structure on $S_P/PS_P$ is Galois.

(iii) $R$ is the fixed ring, and for all minimal primes $P$ of $R$ the map

$$S_P/PS_P \otimes S_P/PS_P \to A_P/PA_P \otimes S_P/PS_P$$

is an isomorphism.

(iv) The map $S \otimes S \to A \otimes S$ is a pseudo-isomorphism [in the sense that its $R$-module kernel and cokernel vanish when localized to any minimal prime]. These conditions imply

(v) $R$ is the fixed ring, and the map $H \otimes S \to \text{End}_R S$ is a
pseudo-isomorphism; they are equivalent to it if we assume either $R$ Noetherian or $S$ a finitely generated $R$-module.

**Proof.** If (i) holds then $R$ is the fixed ring because $R = \bigcap R_p$. Obviously (i) is equivalent to (iv), which implies (iii); and (iii) is equivalent to (ii) since $A_p/P A_p$ is the $R_p/PR_p$-dual of $H_p/PH_p$. If we now assume (ii) we have $\dim H_p/PH_p = \dim S_p/PS_p$. We know [3, p. 147] that the latter is $\leq |L:K|$, with equality only if $S_p$ is a free $R_p$-module. But we also know that $K$ is the fixed ring in $L$, and it follows [9, p. 219] that $\dim H_p/PH_p = \dim_K H \otimes K \geq |L:K|$. Hence we conclude that $S_p$ is free. But then the map $S_p \otimes S_p \rightarrow A_p \otimes S_p$, which is an isomorphism modulo $P$, is an actual isomorphism by Nakayama’s lemma.

As for (v), we have the diagram

$$
\begin{array}{ccc}
(H \otimes S)_P & \longrightarrow & (\text{End } S)_P \\
\downarrow & & \downarrow \\
H_p \otimes S_p & \longrightarrow & \text{End } (S_p),
\end{array}
$$

where we know that the arrow on the right is injective for any $S$ and surjective if $S$ is finitely generated [4, p. 49]. If we assume (i) we have an isomorphism on the bottom, and hence we must have an isomorphism on the top; if $S$ is finitely generated we can reverse the implication.

We claim now that $(\text{End}_S S) \otimes K = \text{End}_K L$ if and only if $S$ is an $R$-lattice in $L$. Indeed, if $S$ is an $R$-lattice, then $\text{End}_S S$ is an $R$-lattice in $\text{End}_K L$ by [4, p. 45]. For the converse let $1 = s_1, s_2, \ldots, s_n$ be a basis of $L$, and consider the maps $\varphi_i: \sum \alpha_j s_j \mapsto (\alpha_i)1$. If $\text{End}_S S$ is sufficiently large there is a $0 \neq r \in R$ such that the $r \varphi_i$ map $S$ into $S$, and then $S \subseteq (1/r)(Rs_1 + \cdots + Rs_n)$.

Now assume (v) with $R$ Noetherian. The fact that $K$ is the fixed ring implies again that $\text{rank } (H) \geq |L:K|$, so by dimension count $(\text{End } S) \otimes K$ is all of $\text{End}_K L$. Then $S$ is an $R$-lattice, hence finitely generated, and the earlier argument applies.

If the conditions of the theorem hold, we say that $S$ with its $H$-structure is pseudo-Galois. One result of the proof deserves to be noted:

**Porism.** If $R$ is Noetherian and $S$ is pseudo-Galois, then $S$ is finitely generated over $R$.

**Theorem 2.** Assume that $S$ is a pseudo-Galois $H$-module algebra. Then

$$\mathcal{P}(R, S) \cong \mathcal{O}(R, S) \cong H^1(H, S^*).$$
Proof. We know (by further localization) that $L$ is Galois for $H \otimes K$, so the second isomorphism is just Proposition 2. Take now a $b \in L^*$ with $h \cdot b \in bS$ for all $h \in H$; we must prove that $bS$ comes from a divisor of $R$. This is a local statement, so we may assume that $R$ is a discrete valuation ring and $S$ is Galois. It follows then that $bS$ is mapped to itself by all elements of $\text{End}_S S$. Choose a basis $s_1, \ldots, s_n$ of $S$ and elements $r_1, \ldots, r_n$ in $K$ such that $r_is_1, \ldots, r_is_n$ is a basis of $bS$; permuting the $s_i$, we see that $bS = r_iS$. 

**Corollary 1.** Suppose $L$ is a Galois field extension of $K$ with group $G$, and assume that all the minimal primes of $R$ are unramified in $S$. Then $S$ is pseudo-Galois for $R[G]$, and hence 
\[ \mathcal{P}(R, S) \cong H'(G, S^*) . \]

Proof. The fact that $S_\rho$ is Galois for $R_\rho[G]$ when there is no ramification is a well-known bit of folklore; much more general results are proved, e.g., in [2].

**Corollary 2.** Suppose $L$ over $K$ is purely inseparable of degree $p$, and $D$ is a $K$-derivation with $DS \subseteq S$. Let $H = R[X]$ as above, and let $H_0$ be the image of $H$ in $\text{End} S$. Assume $DS$ is not contained in any minimal prime of $S$. Then $S$ is pseudo-Galois for $H_0$, and hence 
\[ \mathcal{P}(R, S) \cong \mathcal{Q}(H_0, S) \cong \mathcal{Q}(H, S) . \]

Proof. The hypotheses imply readily that $D^p = \lambda D$ for some $\lambda \in R$ [8, p. 63], and we have $H_0 \cong R[X]/(X^p - \lambda X)$. Functions $h \mapsto b^{-1}h \cdot b$ are equal on $H$ if and only if they are equal on $H_0$, so the second isomorphism is trivial. To prove that $S$ is pseudo-Galois we may localize and assume that $R$ is a discrete valuation ring with maximal ideal $P$; by inseparability there is a unique maximal ideal $Q$ of $S$ lying over it. By hypothesis $S/PS$ has a nontrivial derivation $D$ over $R/P$; in particular the two cannot be equal, and so $S/PS$ either is a $p$-dimensional field extension or has the form $(R/P)[Y]/Y^p$. In either case the hypothesis $DS \not\subseteq S$ shows that $Dy$ is invertible for a generator $y$ of $S/PS$. If $D_i$ is the derivation with $D_iy = 1$, we have $D_i = (1/Dy)D$ in the image of $H_0/PH_0 \otimes S/PS$. But it is well known (and trivial) that $D_i$ and $S/PS$ generate $\text{End} S/PS$. Thus the map from $H_0/PH_0 \otimes S/PS$ is a surjection, and dimension count shows it is an isomorphism.

The isomorphism $\mathcal{P} \cong \mathcal{Q}$ could be proved for these two cases by using the idea in Theorem 2, showing from the given hypotheses that an element $b$ with $h \cdot b \in bS$ comes locally from $R$. This is essentially
what is done in [7]. But our argument brings out the general result underlying Samuel's two theorems. It also yields the extension to several derivations in [10, Th. 2.9]. In addition, the example in the next section shows that we can treat problems (with $L^p \cong K$) which cannot be handled by derivations.

3. The surface $Z^q = XY$. Let $k$ be a field of positive characteristic $p$, and let $L$ be the fraction field of $S = k[x, y]$. Let $q$ be a power of $p$, and let $K$ be the fraction field of $R = k[x^q, y^q, xy]$. As in [8, p. 65], it is easy to see that $R = S \cap K$ and so is a Krull domain; it is the affine coordinate ring of $Z^q = XY$ with $x^q = X$ and $y^q = Y$.

Let $G$ be a cyclic group of order $q$, with generator $g$. Set $A = R[G]$ and map $S \to A \otimes_R S$ by $x \mapsto g \otimes x$ and $y \mapsto g^{-1} \otimes y$. Then the dual $H = R^G$ has a basis of idempotents $e_0, e_1, \ldots, e_{q-1}$ with $e_i \cdot x^j y^j$ equal to $x^j y^j$ if $i \equiv j \pmod{q}$ and equal to 0 otherwise. As an $R$-module, $S = \bigoplus e_i S$; the fixed ring is $e_0 S = R$.

The map $S \otimes S = \bigoplus e_i S \otimes S \to A \otimes S$ takes $s_i \otimes t$ to $g^i \otimes s_i t$ for $s_i \in e_i S$. Thus to show that $S$ is pseudo-Galois we must show that the multiplication maps $e_i S \otimes S \to S$ are isomorphisms at each minimal prime $P$ of $R$. Since $L$ is purely inseparable over $K$, we know that $S_P$ is a local ring; the condition then is that $e_i S$ contain a unit of $S_P$, i.e., not lie in the maximal ideal. But obviously $e_i S$, which contains both $x^i$ and $y^{q-i}$, does not lie in any minimal ideal of $S = k[x, y]$. Hence $S$ is pseudo-Galois for $H$.

Take now an element $b$ with all $e_i b \in b S$; multiplying by an element of $K^*$, we may assume $b$ is a polynomial. Then $e_i b$ consists of some of its terms, and for all these to be multiples of $b$ requires that $b = e_i b$ for some $i$. All such elements are $K$-multiples of $x^i$, and these give us a cyclic group of order $q$. Since $S$ has unique factorization, all divisors of $R$ become principal, and we have proved

**Proposition 4.** Let $k$ be a field of characteristic $p$, and $q$ a power of $p$. Then the divisor class group of $k[x^q, y^q, xy]$ is cyclic of order $q$.

We can carry out the same proof assuming only that $k$ is a unique factorization domain, just as was done in [8, p. 65]. (The result could be proved there, of course, only for $q = p$.)

4. Galois extensions and the kernel of Pic. Among the divisorial ideals of $R$ are the invertible ideals, and the group $\text{Pic} R$ of invertible ideals modulo principal ideals is a subgroup of the divisor class group. Thus the kernel of the map $\text{Pic} R \to \text{Pic} S$ is a subgroup of $\mathcal{P}(R, S)$. In general it may well be smaller. In the example of §3, for instance, $\mathcal{P}(R, S)$ is generated by the inverse image of $x S$,
which [4, p. 89] is just $xS \cap R$; this is not an invertible ideal. Suppose however that $S$ is flat over $R$. Then a divisorial ideal $D$ is mapped simply to $DS$ [4, p. 20]; since $S$ is integral, it is faithfully flat over $R$, and so $DS$ principal implies $D$ invertible. Hence we have proved the following generalization of [10, Corollary 2.8]:

**Proposition 5.** Assume that $S$ is a pseudo-Galois $H$-module algebra and is flat over $R$. Then

$$\mathcal{D}(H, S) \cong \text{Ker} (\text{Pic } R \to \text{Pic } S).$$

These hypotheses are true if $S$ is Galois for $H$. In fact, they nearly imply $S$ Galois, as the following theorem shows.

**Theorem 3.** Assume $S$ is a pseudo-Galois $H$-module algebra. The following are equivalent:

1. $S$ is Galois for $H$.
2. $S$ is a projective $R$-module.

**Proof.** By definition (1) implies (2), so assume (2). In the proof of Theorem 1 we saw that $S$ is an $R$-lattice; then $S \otimes S$ and $A \otimes S$ are projective $R$-lattices, and the map between them is an isomorphism at every minimal prime $P$.

To complete the proof we just recall that if $M$ is a projective $R$-lattice in a $K$-space $V$, then $M$ is finitely generated and $M = \bigcap M_P$. Since this result seems to have been omitted from [4], we sketch the proof. Writing $M$ as a direct summand of a free module gives us linear functions $f_i : M \to R$ and elements $m_\circ \in M$ such that (*) $m = \sum f_i(m) m_\circ$ for all $m \in M$. There is a natural extension of $f_i$ to a linear function $V \to K$, and (*) then holds for all $m \in V$. Let $v_1, \ldots, v_n$ be a basis of $V$, with dual basis $v_1^*, \ldots, v_n^*$, and write $f_i = \sum a_{i, r} v_r^*$. Applying (*) to the $v_r$ shows that $a_{i, r} = 0$ for all but finitely many $i$; thus $M$ is finitely generated. If $m \in \bigcap M_P$, then $f_i(m) \in \bigcap R_P = R$, so $m \in M$.

**Corollary.** Assume $R$ Noetherian, $S$ pseudo-Galois and flat. Then $S$ is Galois.

**Proof.** We have $S$ flat by hypothesis and finitely generated by the Porism to Theorem 1; hence $S$ is projective.

**References**


Received December 16, 1969.

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.
Pacific Journal of Mathematics

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