

# Pacific Journal of Mathematics

**SUBADDITIVE FUNCTIONS**

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**In a recent paper, D. W. Boyd and J. S. W. Wong ask for an example of positive subadditive function  $\phi$  for which  $\phi(t) < t$  for all  $t$  and there exist  $0 < c < d$  such that  $\sup_{t \in [c, d]} \phi(t)/t = 1$ . Our main result is that such an example does not exist.**

For a metric space  $(X, d)$ , we shall use  $\text{Ran } d$  to denote the set  $\{d(x, y) : x, y \in X\}$  and use  $cl \text{ Ran } d$  to denote the closure of  $\text{Ran } d$ . Let  $X$  be a complete metric space, let  $\psi$  be a function of  $cl \text{ Ran } d$  into  $[0, \infty)$  and let  $T$  be a function of  $X$  into itself such that  $T$  is  $\psi$ -contractive on  $X$  ( $d(T(x), T(y)) \leq \psi(d(x, y))$ ,  $x, y \in X$ ). E. Rakotch [3, Corollary to Theorem 2] shows that if there is a decreasing function  $\alpha$  on  $[0, \infty)$  such that for any  $t > 0$ ,  $\alpha(t) < 1$  and  $\psi(t) = \alpha(t)t$ , then  $T$  has a unique fixed point. In order to show that Theorem 2 in [1] actually extends the above result when  $X$  is metrically convex, D. W. Boyd and J. S. W. Wong [1, p. 464] ask for an example of positive subadditive function  $\phi$  for which  $\phi(t) < t$  for all  $t > 0$  and there exist  $0 < c < d$  such that

$$\sup_{t \in [c, d]} \phi(t)/t = 1.$$

We now show that such an example does not exist.

**THEOREM.** *Let  $\phi$  be a positive subadditive function on an interval  $(0, b)$  ( $0 < b \leq \infty$ ) such that  $\phi(t) < t$  for all  $t$  in  $(0, b)$ . Then*

$$\sup_{a \leq t < b} \phi(t)/t < 1, \quad 0 < a < b.$$

*Proof.* If  $b = \infty$ , then by subadditivity and Theorem 7.6.2 in [2],

$$\lim_{t \rightarrow \infty} \phi(t)/t = \inf_{t > 0} \phi(t)/t;$$

thus from  $\phi(t)/t < 1$ ,  $\sup_{t > a} \phi(t)/t < 1$  for large  $a$ 's. So we may assume that  $b < \infty$ . Suppose to the contrary that there exist  $c, d$  in  $(0, b)$  such that  $c < d$  and

$$\sup_{t \in [c, d]} \phi(t)/t = 1.$$

Then there exists a sequence  $\{t_n\}$  in  $[c, d]$  such that

$$(1) \quad \lim_{n \rightarrow \infty} \phi(t_n)/t_n = 1.$$

By compactness of  $[c, d]$ , we may, by taking a subsequence, assume that  $\{t_n\}$  converges to some  $t$  in  $[c, d]$ . Let  $m$  be any positive integer. Then by the subadditivity of  $\phi$ ,

$$(2) \quad \phi(t_n) \leq m\phi(t_n/m) < t_n, \quad n \geq 1.$$

From (1) and (2),

$$(3) \quad \lim_{n \rightarrow \infty} \phi(t_n/m) = t/m.$$

We now prove by induction that

$$(4) \quad \lim_{n \rightarrow \infty} \phi(jt_n/2^k) = jt/2^k, \quad k \geq 1, 0 < j < 2^k, j \text{ is odd.}$$

Assume that (4) is true for  $k \leq i$ , where  $i$  is given. Let  $j$  be any odd number in  $(0, 2^{i+1})$ . Then

$$(5) \quad \phi((j + 1)t_n/2^{i+1}) \leq \phi(jt_n/2^{i+1}) + \phi(t_n/2^{i+1}) < (j + 1)t_n/2^{i+1}.$$

By (5), the induction hypothesis and (3) (also by (1) if  $j = (2^{i+1} - 1)/2^{i+1}$ ), we have by letting  $n \rightarrow \infty$ ,

$$(j + 1)t/2^{i+1} \leq \limsup_{n \rightarrow \infty} \phi(jt_n/2^{i+1}) + t/2^{i+1} \leq (j + 1)t/2^{i+1},$$

i.e.,

$$\lim_{n \rightarrow \infty} \phi(jt_n/2^{i+1}) = jt/2^{i+1},$$

proving (4). Take any  $s$  in  $(0, t)$ . It suffices to prove that  $s \leq \phi(s)$ . Since the set

$$D = \{jt/2^k: k \geq 1, 0 < j < 2^k, j \text{ is odd}\}$$

is dense in  $(0, t)$ , there exists a strictly decreasing sequence  $\{s_n\}$  in  $D$  which converges to  $s$ . By (4), there is a sequence  $\{w_n\}$  for which

$$(6) \quad s_n - 1/n < w_n < s_n, \phi(w_n) > s_n - 1/n, \quad n \geq 1.$$

Now

$$(7) \quad \phi(w_n) \leq \phi(w_n - s) + \phi(s) < (w_n - s) + \phi(s), \quad n \geq 1.$$

From (6) and (7), we obtain  $s \leq \phi(s)$ .

From the above result, we know that the condition (24) in [1] can be dropped. We thus have an improved version of [1, Proposition].

**PROPOSITION.** Let  $(X, d)$  be a complete metrically convex metric space and let  $f: X \rightarrow X$ . Suppose that there is a function  $\psi$  of  $cl \text{ Ran } d$  into  $[0, \infty)$  such that  $\psi(t) < t$  for all  $t$  in  $cl \text{ Ran } d - \{0\}$  and  $f$  is  $\psi$ -contractive on  $X$ . Then there exists a decreasing function  $\alpha$  on  $[0, \infty)$

such that  $\alpha(t) < 1$  for all  $t > 0$  and

$$d(f(x), f(y)) \leq \alpha(d(x, y))d(x, y), \quad x, y \in X.$$

D. W. Boyd and J. S. Wong show that  $\alpha$  in the above proposition can actually be constructed as follows:

$$\alpha(t) = \sup_{s \geq t} \phi(s)/s, \quad t > 0,$$

where

$$(*) \quad \phi(t) = \sup\{d(f(x), f(y)): x, y \in X, d(x, y) = t\}, \quad t \in \text{Ran } d.$$

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