A NOTE ON THE MINIMALITY OF CERTAIN BITRANSFORMATION GROUPS

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Let \((T, X)\) be a transformation group with compact Hausdorff space \(X\) and topological group \(T\). Let \((X, G)\) be a transformation group with \(G\) a compact topological group. Then the triple \((T, X, G)\) is a bitransformation group if \((tx)g = t(xg)\) for all \(t \in T, x \in X, g \in G\) and the action of \(G\) on \(X\) is strongly effective, (that is \(xg = x\) if and only if \(g = \text{the identity element} e\) of \(G\)). A bitransformation group \((T, X, G)\), induces canonically the transformation group \((T, X/G)\) where \(X/G\) is the orbit space of \((X, G)\). Let \((T, X, G)\) be a bitransformation group. Suppose \((T, X/G)\) is a minimal transformation group whereas \((T, X)\) is not a minimal transformation group then what is the possible structure of \((T, X, G)\)?

In this note, it is proved that the fundamental group of \(X\) must be of certain form when \(G\) is a circle group. Use this result together with some results of Malcev, a necessary and sufficient condition is found for the minimality of certain nilflows.

**Theorem 1.** Let \((T, X, G)\) be a bitransformation group with circle group \(G\). If \((T, X/G)\) is a minimal transformation group and \((T, X)\) is not minimal, then there exists a finite group \(H\) of \(G\) such that \(X\) is a covering space of \(X/H\) and \(X/H\) admits a section over \(X/G\).

**Proof.** Let \(M\) be a minimal set in \((T, X)\). Let \(H = \{g \in G: gM = M\}\). Then \(H\) is a proper closed subgroup of \(G\). Thus \(H\) is a finite group. The natural projection \(p: X/H \to X/G\) is a principal bundle map with fiber \(G/H\). Then \(p|M/H: M/H \to X/G\) is a homeomorphism. Thus \(p\) admits a global cross section.

**Corollary.** Besides all the notation of Theorem 1, assume that \(X\) is path connected. Then \(\pi(X)\) is a isomorphic with a subgroup of \(\pi(X/G) \cdot Z\), where \(Z\) is the integer group and the dot denotes semi-direct product.

From now on, we shall assume that \(N\) is a simply connected nilpotent analytic group. A subgroup \(H\) of \(N\) is a uniform subgroup if the homogeneous space \(N/H\) is compact. Let \(\Gamma\) be a discrete uniform subgroup of \(N\). Then \(\Gamma\) is torsion-free and finitely generated [2]. For each discrete uniform subgroup \(\Gamma\) of \(N\), there is a subset
$D = \{d_1, \cdots, d_m\}$ of $\Gamma$ with the following properties:

1. there exist $m$ one-parameter groups $d_i(t)$ such that $N = \{d_i(t_1)d_2(t_2) \cdots d_m(t_m): t_1, \cdots, t_m \in \mathbb{R}, \text{reals}\}$.

2. $\Gamma = \{d_1(n_1)d_2(n_2) \cdots d_m(n_m): n_1, n_2, \cdots, n_m \in \mathbb{Z}, \text{integers}\}$.

3. If $N_i = \{d_i(t_i) \cdots d_m(t_m): t_i, t_{i+1}, \cdots, t_m \text{ any real numbers}\}$, then $N_i$ is a closed subgroup of $N$ and $N_i$ is normal in $N_{i-1}$. $D$ is called a canonical basis of $\Gamma$.

Let $F$ be a nilpotent group and $F = F^0 \supset F^1 \supset F^2 \supset \cdots \supset F^p \supset F^{p+1} = (e)$ be the descending central series. We recall that $F^i = [F, F^i]$ where $[a, b] = aba^{-1}b^{-1}: a \in F, b \in F^{i-1}$. Let $N = N^0 \supset N^1 \supset \cdots \supset N^m \supset N^{m+1} = (e)$ be the descending central series. Then $\Gamma^p \subseteq N^p \cap \Gamma \subseteq N^p$ we shall prove that.

**Lemma 1.** $\Gamma^p$ is uniform in $N^p$ and $\Gamma \cap N^p/\Gamma^p$ is finite.

**Proof.** Let $V$ be the vector subspace of $N^p$ spanned by $\Gamma^p$. Let $D = \{d_1, \cdots, d_k, \cdots, d_m\}$ be a canonical basis of $D$ such that $\{d_1, \cdots, d_m\}, \{d_1, \cdots, d_m\}$ are canonical basis for $N^p$ and $N^p$ respectively. Then $\{d_1(t)d_2(t)^{-1}, d_3(t)^{-1}d_1(t)^{-1}: t \in \mathbb{R}\}$ is a one-parameter group containing $d_1(t)d_2(t)^{-1}d_3(t)^{-1}$ if $l \leq j$. Hence $\{d_1(t)d_2(t)^{-1}d_3(t)^{-1}\} \subseteq V$. For each fixed $t_0 \in \mathbb{R}$, $\{d_1(t)d_2(t_0)d_3(t)^{-1}d_1(t)^{-1}: t \in \mathbb{R}\}$ is an one parameter group containing $d_1(t)d_2(t_0)d_3(t)^{-1}$ if $l \leq j$. This implies that $d_1(s)d_2(t)d_3(s)^{-1}d_1(t)^{-1} \in V$ for any $s, t \in \mathbb{R}$. Thus $N^p = [N, N^p] \subseteq V$ and $N^p = V$. Hence $\Gamma^p$ is uniform in $N^p$ and $\Gamma \cap N^p/\Gamma^p$ is finite.

In order to state our next result, we recall the definition of coset transformation group. Let $T$ be a topological group and $G/H$ a coset space. Let $\phi$ be a continuous homomorphism from $T$ into $G$. Then $(T, G/H)$ is a coset transformation group (relative to $\phi$) if $tgH = \phi(t)gH$ for $t \in T, g \in G$.

**Proposition 1.** Let $(T, N/\Gamma)$ be a coset transformation group where $N$ is a simply connected nilpotent analytic group and $\Gamma$ is discrete uniform subgroup of $N$. Assume that $\dim N^q/N^{q-1} = 1$ for $q \geq 1$. Then $(T, N/\Gamma)$ is minimal if and only if $(T, N/\Gamma N')$ is minimal.

**Proof.** We shall prove this theorem by induction based the length of nilpotency of $\Gamma$. When $\Gamma$ is abelian, there is nothing to prove. Assume $(T, N/\Gamma N')$ is minimal. By induction hypothesis $(T, N/N^p/\Gamma N^p/N^p)$ is minimal. Thus $(T, N/\Gamma N^p)$ is minimal. Let $H^q = \{d_q(t_q) \cdots d_m(t_m): t_q, \cdots, t_m \in \mathbb{R}\}$. Suppose $(T, N/\Gamma H^q)$ is minimal and $(T, N/\Gamma H^{q+1})$ is not minimal. Then $(T, N/\Gamma H^{q+1}, \Gamma H^q/\Gamma H^{q+1})$ is a bitransformation group. By Corollary 1, $\Gamma/\Gamma \cap H^{q+1}$ is isomorphic with a subgroup of $\Gamma/\Gamma \cap H^q \times Z$. Then the image of $d_q(\Gamma/\Gamma \cap H^{q+1})$ under this isomorphism
must be of the form \((x, z)\) for some nonzero integer. Thus \((x, z)^a \notin (\Gamma/\Gamma \cap H^a)\) if \(a\) is a nonzero integer. On the other hand, \([((\Gamma/\Gamma \cap H^a) \times Z, (\Gamma/\Gamma \cap H^a) \times Z] \subset (\Gamma/\Gamma \cap H^a)\). This fact together with Lemma 1, we have the contradiction. Thus \((T, N/\Gamma H^{q+1})\) is minimal. By finite induction, \((T, N/\Gamma)\) is minimal.

**THEOREM 2.** Let \((T, N/H)\) be a coset transformation with nilpotent analytic group \(N\) and closed uniform subgroup \(H\) such that \(\dim (N/\Gamma H_0)^q/(N/\Gamma H_0)^{q+1} = 1\). Then \((T, N/H)\) is minimal if and only if \((T, N/H[N, N])\) is minimal.

**Proof.** Let \(H_0\) be the identity component of \(H\). Then \(H_0\) is a normal subgroup of \(N\) and \(N/H_0\) is simply connected. Let \(\pi\) be the canonical projection from \(N \to N/H_0\). Then \(\pi^{-1}(\pi(\Gamma)\pi[N/H_0, N]) = H[N, N]\). Hence \(H[N, N]\) is closed uniform subgroup of \(N\). If \((T, N/H[N, N])\) is minimal, then \((T, N/H_0/H/H_0)\) is minimal by Proposition 1. But \((T, N/H)\) is isomorphic with \((T, N/H_0/H/H_0)\). Hence \((T, N/H)\) is minimal.

**EXAMPLES.** ([1, p. 52]) consider the group \(G\) of all real matrices of the form

\[
\begin{pmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix}
\]

and let \(D\) be the uniform discrete subgroup of matrices

\[
\begin{pmatrix}
1 & a & c \\
0 & 1 & b \\
0 & 0 & 1
\end{pmatrix}
\]

for all integers \(a, b, c\). Then \(M = G/D\) is a nilmanifold. Consider a one-parameter subgroup \(\varphi(t)\) of \(G\) given by

\[
\text{expt}\begin{pmatrix}
0 & \alpha & \gamma \\
0 & 0 & \beta \\
0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
1 & 2t & \lambda t + \frac{1}{2}\alpha \beta t^2 \\
0 & 1 & \beta t \\
0 & 0 & 1
\end{pmatrix}.
\]

Take a point \(Q \in M\) given by the coset

\[
\begin{pmatrix}
1 & x_0 & z_0 \\
0 & 1 & y_0 \\
0 & 0 & 1
\end{pmatrix}D
\]

\[\text{Since } \Gamma \text{ is nilpotent, the semi-direct product here is actually a direct product.}\]
the orbit $\varphi^*_f(t)$ in $M$ is
\[
\begin{pmatrix}
1 & t + x_0 & \gamma t + \frac{\alpha \beta}{2} t^2 + z_0 + \alpha + y_0 \\
0 & 1 & \beta t + y_0 \\
0 & 0 & 1
\end{pmatrix} D.
\]

Then $D[G, G]$ is the set of all the matrices
\[
\begin{pmatrix}
1 & a & z \\
0 & 1 & b \\
0 & 0 & 1
\end{pmatrix}
\]
for all integers $a, b$ and real number $z$. And $(\varphi(t), G/D[G, G])$ is isomorphic with the continuous flow on two-dimensional torus with the direction ratio $(\alpha, \beta)$.

By Theorem 2, $(\varphi(t), M)$ is minimal if and only if $(\varphi(t), G/D[G, G])$. The latter is minimal if and only if $\alpha$ and $\beta$ are rationally independent. This answers the question in [1, p. 53].

Added in proof. After this note went in print, we have the proof of the following statement. Let $G$ be a simply connected solvable analytic group and $\Gamma$ be a nilpotent uniform subgroup of $G$. Then $(T, G/P)$ is minimal if and only if $(T, G/\Gamma N)$ is minimal, here $N$ denotes the analytic subgroup of $G$ which contains $[\Gamma, \Gamma]$ as a uniform subgroup. The proof uses a stronger form of Lemma 1 (replacing the circle group by torus groups) and the nilpotency of $\Gamma$. The detail will appear later.

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