INARIANT SUBSPACES AND PROJECTIVE REPRESENTATIONS

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Let $I'$ be a subgroup of the real line $R$ with the discrete topology, and let $G$ be its compact dual group. This paper shows the existence of a (nontrivial) simply invariant subspace of $L^2(G)$ which is not of the form $\varphi H^2(G)$ provided $I'$ contains at least two rationally independent elements. The proof relies heavily on the existence of a nontrivial local projective representation of the two-dimensional torus.

Helson and Lowdenslager [4] showed the existence of a simply invariant subspace not of the form $\varphi H^2(G)$ in case $I'$ contains an infinite set of rationally linearly independent elements. We use the correspondence introduced in [4] between simply invariant subspaces and cocycles but in contrast to [4] we use nontrivial local projective multipliers to show that the appropriate cohomology group is nontrivial.

The connection between invariant subspaces and cocycles is discussed in § 2 and in § 3 we will give a quotient group argument which allows us to reduce the general problem to its specialization on the two-dimensional torus. Sections 4 and 5 relate the notion of projective representation with a cocycle and it is shown that a nontrivial projective representation gives rise to a cocycle whose corresponding subspace is not of the form $\varphi H^2(G)$.

2. Preliminaries. Let $G$ be an arbitrary locally compact Abelian group dual to $I'$ and let $A$ be a continuous one-parameter subgroup of $G$ which we also denote by $\{e^t | t \in R\}$. Haar measure in $G$ will be denoted by $dx$ and will be normalized to have total mass one in case $G$ is compact. As usual, a.e. $(x)$ means for all but a set of Haar measure zero. A (Borel) function $\varphi$ on $G$ is said to be unitary in case $\varphi(x)$ has modulus one a.e. $(x)$.

**Definition.** A function $A$ on $A \times G$ is said to be a cocycle on $G$ in case:

(2.1) $A(e^t, \cdot)$ is a unitary function for each $e^t$ in $A$,

(2.2) $A(e^t + e^u, x) = A(e^u, x)A(e^t, x - e^u)$ for all $e^t, e^u$ in $A$ and a.e. $(x)$, and

(2.3) $A$ is strongly continuous in the sense that $A(e^t, \cdot)f$ is a continuous function from $R$ into $L^2 = L^2(G)$ for $f$ in $L^2$. 

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Cocycles of the form

\begin{equation}
A(e^t, x) = \varphi(x)/\varphi(x - e^t), \quad \text{all } e^t \text{ in } A, \text{ a.e. } (x)
\end{equation}

for some unitary function \( \varphi \) are called coboundaries. We will frequently denote \( A(e^t, x) \) by \( A(t, x) \).

If \( \lambda \) is in \( \Gamma \) we let \( \chi_\lambda \) be the character on \( G \) defined by \( \chi_\lambda(x) = x(\lambda) \) for all \( x \) in \( G \); the corresponding unitary representation \( V_0 \) of \( \Gamma \) is given by

\begin{equation}
V_0(\lambda)f(x) = \chi_\lambda(x)f(x)
\end{equation}

for all \( f \) in \( L^2 \). Any bounded operator on \( L^2 \) which commutes with all the \( V_0(\lambda) \) is necessarily a multiplication by a function in \( L^\infty \). Let \( U_0 \) be the unitary representation of \( G \) defined by

\begin{equation}
U_0(x)f(y) = f(y - x)
\end{equation}

for all \( f \) in \( L^2 \).

For the remainder of this section we will let \( \Gamma \) be a subgroup of the real line \( R \). Let \( G \) be the compact Abelian group dual to the discretely topologized \( \Gamma \). A closed subspace \( \mathcal{M} \) of \( L^2 \) is said to be simply invariant in case \( V_0(\lambda)\mathcal{M} \subseteq \mathcal{M} \) if and only if \( \lambda \geq 0 \). The Hardy space \( H^2 \) consists of those functions \( f \) in \( L^2 \) whose Fourier transforms \( \hat{f}(\lambda) = \int \chi_{-\lambda}(x)f(x)dx \) vanish for \( \lambda < 0 \). Subspaces of the form \( \mathcal{M} = \varphi H^2 = \{ \varphi f : f \text{ in } H^2 \} \) where \( \varphi \) is a unitary function are simply invariant and in the case where \( G \) is a circle all simply invariant subspaces are of this form.

In order to avoid the rather special circle group we will henceforth suppose that \( \Gamma \) is dense in \( R \). The characters \( e^t \) defined by \( e^t(\lambda) = \exp(it\lambda) \) are distinct and provide a continuous one-parameter dense subgroup \( A \) of \( G \). A correspondence is exhibited in \([3, 4]\) between simply invariant subspaces \( \mathcal{M} \) (suitably normalized) and cocycles \( A \) in such a way that \( \mathcal{M} = \varphi H^2 \) if and only if \( A \) is the coboundary (2.4).

We therefore wish to construct cocycles which are not coboundaries. If \( A \) is a coboundary then \( A \) can be extended from \( A \times G \) to \( G \times G \) so that (2.4) remains valid with \( t \) replaced by an arbitrary \( y \) in \( G \) and conversely. Moreover, the multiplication operator \( A(y, \cdot) \) is the strong operator limit of a sequence \( A(t_n, \cdot) \) where \( e^{it_n} \) tends to \( y \) in \( G \); this observation will be useful later. Equivalently, \( A \) is a coboundary if and only if the unitary representation \( U(t) = A(t, \cdot)U_0(t) \) can be extended from \( A \) to a (strongly continuous) unitary representation of \( G \). A cocycle was constructed in \([4]\) (in case \( \Gamma \) is suitably large) for which the unitary representation did not extend to \( G \). However, it is conceivable that \( U(t) \) might extend to a (local) projective representation of \( G \); this idea is turned around and will be used to extract cocycles...
There is a superficial answer to our problem in case $\Gamma$ is not all of $\mathbb{R}$ for then there are trivial cocycles which are not coboundaries. For example, let $A(t, x) = \exp(-it\lambda)$ for some fixed real $\lambda$ not in $\Gamma$. If $\lambda$ were in $\Gamma$ then $A$ would be the coboundary with unitary function $\chi_\lambda$ but with $\lambda$ not in $\Gamma$ there is no unitary function $\varphi$ such that $\exp(-it\lambda) = \varphi(x)/\varphi(x-e_t)$. Conversely, if $A$ is a cocycle which is constant a.e. $(x)$ for each $t$ (the null set depending upon $t$), then $A(t, x) = \exp(-it\lambda)$, a.e. $(x)$ for some fixed $\lambda$ in $\mathbb{R}$. We will call cocycles of this form constant cocycles. Consequently the nontrivial problem [3, p. 149] is to find cocycles which are not products of constant cocycles and coboundaries.

The cocycles defined in [3] were measurable functions on $A \times G$ but we will have no need for cocycles to be product measurable. Anyway, one can pass from one version to another [3, p. 145], [2]. Also we have departed from [3] by making an insignificant sign change in our definition of cocycle.

3. Reduction to the torus. Suppose that $\Gamma_0 \subseteq \Gamma$ are subgroups of the discrete real line and let $G_0$ and $G$ be their compact dual groups. To each cocycle $A_0$ on $G_0$ we will associate a cocycle $A$ on $G$ in such a way that if $A$ is the product of a constant cocycle and a coboundary then so is $A_0$. Since the two-dimensional torus $T^2$ is dual to the group of lattice points $\mathbb{Z}^2$ and $\mathbb{Z}^2$ is isomorphic to a subgroup $\Gamma_0 \subseteq \Gamma$ of any group $\Gamma \subseteq \mathbb{R}$ with at least two independent elements it will be sufficient to construct a cocycle on $T^2$ which is not the product of a constant cocycle and a coboundary.

Define a closed subgroup $H = \{x \in G | \chi_\lambda(x) = 1 \text{ for all } \lambda \in \Gamma_0 \}$ of $G$ so that $G_0$ can be identified with $G/H$. Let $\pi$ be the usual quotient map from $G$ onto $G/H$ and let $e_t$ and $\varepsilon_t$ be the previously defined one-parameter groups $\Lambda$ and $A_0$ in $G$ and $G_0$. One can verify $\pi(e_t) = \varepsilon_t$ by noting that $\varepsilon_t$ is the restriction of $e_t$ from $\Lambda$ to $A_0$.

If $A_0$ is a cocycle on $G_0$ we define a cocycle $A$ on $G$ by

$$(3.1) \quad A(e_t, x) = A_0(\varepsilon_t, \pi(x))$$

for all $(e_t, x)$ in $A \times G$.

For each $t$ in $\mathbb{R}$ the measurable function $A(e_t, )$ on $G$ is certainly unitary because $\pi^{-1}(S)$ is a null set in $G$ whenever $S$ is a null set in $G/H$. The cocycle identity (2.2) is easy enough to verify with the aid of $\pi(e_t) = \varepsilon_t$ so all that remains is the strong continuity.

Let the Haar measures $dx$ and $dx_0$ in $G$ and $G/H$ both be normalized to have total mass one. There is a normalization for the Haar measure $d\xi$ on $H$ such that
\[
\int_{g} f(x) \, dx = \int_{g/H} \left( \int_{H} f(x + \xi) \, d\xi \right) dx_o
\]
for all \( f \) in \( L^1(G) \).

Let \( f \) be in \( L^2(G) \) and put
\[
g(x_o) = \int_{H} |f(x + \xi)|^2 \, d\xi
\]
where \( x_o = \pi(x) \). A straight-forward computation with (3.2) shows that \( A(e_t, \cdot) f \) moves continuously in \( L^2(G) \) as \( t \) varies because \( A_o(e_t, \cdot) \sqrt{g} \) moves continuously in \( L^2(G/H) \).

**THEOREM.** If \( A \) is the product of a constant cocycle and a coboundary then so is \( A_o \).

**Proof.** For some constant cocycle \( C \) and some unitary function \( \varphi \) on \( G \) we have
\[
C(t)A(t, x) = \varphi(x)/\varphi(x - e_t)
\]
for each real \( t \) and almost all \( x \).

It is advantageous to normalize by choosing \( \lambda \) in \( \mathbb{R} \) such that
\[
\int \chi_s(x) \varphi(x) \, dx
\]
does not vanish and putting \( \psi = \chi_s \varphi \). The cocycle \( B = \chi_s CA \) is really the coboundary.

(3.3)
\[
B(t, x) = \psi(x)/\psi(x - e_t)
\]
and we have \( B_o = \chi_s CA_o \). Consequently it is sufficient to show that \( B_o \) is a coboundary and we will do this by arguing that \( \psi \) must be constant on cosets of \( H \).

Since \( B(t, x) = B_o(t, \pi(x)) \) it follows that \( B(t, x) = B(t, x + h) \) for all real \( t \) and all \( (x, h) \) in \( G \times H \). Now the coboundary \( B \) can be extended to \( G \times G \) and, in fact, \( B(y, \cdot) \) is a limit in \( L^2(G) \) of a sequence \( B(t_n, \cdot) \) where \( e_{t_n} \) goes to \( y \) in \( G \). Therefore, passing to a subsequence if necessary, \( B(t_n, x) \) tends to \( B(y, x) \) for almost all \( x \) and we can conclude

(3.4)
\[
B(y, x) = B(y, x + h)
\]
for all \( y \) in \( G \), \( h \) in \( H \) and almost all \( x \) in \( G \).

From (3.3) (valid now for \( t \) replaced by any element in \( G \)) and (3.4) we have

(3.5)
\[
\psi(x + \xi) = B(h, x) \psi(x + \xi - h)
\]
for every \( \xi \) in \( H \) and almost all \( x \) in \( G \). Integrating this last expression with respect to Haar measure \( d\xi \) on \( H \) we find
\[ \int_H \psi(x + \xi)d\xi = B(h, x) \int_H \psi(x + \xi)d\xi. \]

Now \( \int_H \psi(x + \xi)d\xi \) does not vanish since \( \int_G \psi(x)dx \) is not zero (consider (3.2)) and so we may conclude \( B(h, x) = 1 \) for all \( h \) in \( H \) and almost all \( x \) in \( G \).

It follows from (3.5) that \( \psi \) is constant on cosets of \( H \) and so we can define a unitary function \( \psi_0 \) on \( G/H \) by \( \psi_0(\pi(x)) = \psi(x) \). Clearly \( B_0 \) is a coboundary determined by \( \psi_0 \). That completes the proof.

4. Projective representations and projective cocycles. Let \( G \) be a locally compact Abelian group. A strongly continuous function \( U \) from \( G \) into the unitary operators on some Hilbert space is said to be a projective representation if

\[ U(x)U(y) = \omega(x, y)U(x + y) \]

for some function \( \omega \) of modulus one and if \( U(0) = 1 \). We say that \( \omega \) is the multiplier of the representation and it is not difficult to show that it satisfies the identity \( \omega(x, y)\omega(x + y, z) = \omega(y, z)\omega(x, y + z) \) and the normalizing condition \( \omega(x, 0) = \omega(0, x) = 1 \). Moreover, \( \omega \) is continuous on \( G \times G \). Conversely, given a function \( \omega \) with these properties one can construct a projective representation \( U_\omega \) with multiplier \( \omega \). Indeed, define \( U_\omega \) on \( L^2 \) by

\[ U_\omega(x)f(y) = \omega(x, y - x)f(y - x). \]

The projective representation \( U_\omega \) is of the form

\[ U_\omega(x) = A_\omega(x, )U_0(x) \]

where \( A_\omega(x, y) = \omega(x, y - x) \) is a function of modulus one on \( G \times G \). The (projective) group property of \( U_\omega \) implies that

\[ \omega(x, y)A_\omega(x + y, z) = A_\omega(x, z)A_\omega(y, z - x) \]

and the strong continuity of \( U_\omega \) implies that \( A_\omega(x, ) \) is a strongly continuous operator valued function in \( x \).

Observe that \( A_\omega \) differs from the ordinary cocycle (§ 2) in two respects; first, \( A_\omega \) is a function on \( G \times G \) instead of merely on \( A \times G \), and, secondly, (4.4) replaces (2.2). We say that \( A_\omega \) is a projective cocycle.

We say that \( \omega \) is trivial if

\[ \omega(x, y) = p(x)p(y)/p(x + y) \]

for some continuous function \( p \) of modulus one on \( G \). In this case any projective representation \( U \) with multiplier \( \omega \) can be made into an
ordinary representation merely by multiplying $U(x)$ by $p(x)$. The product of two multipliers is again a multiplier and two multipliers whose quotient is trivial are said to be *equivalent*.

If $\omega$ and $\sigma$ are equivalent multipliers so that

\begin{equation}
\frac{\omega(x, y)}{\sigma(x, y)} = \frac{p(x)p(y)}{p(x + y)}
\end{equation}

then a direct computation will give

\begin{equation}
A_\omega(x, y)/A_\sigma(x, y) = \frac{p(x)(\varphi(y)/\varphi(y - x))}{1/p(y)}
\end{equation}

where $\varphi(y) = 1/p(y)$. In particular if $\omega$ is trivial then $A_\omega$ is $p$ times a coboundary and conversely.

Now suppose that $G$ has a continuous one-parameter subgroup $A = \{e^t | t \in \mathbb{R}\}$ and let $A_\omega$ be a projective cocycle on $G$ with $U_\omega$ the corresponding projective representation as given by (4.3). We wish to extract an ordinary cocycle $A$ from $A_\omega$ in such a way that $A$ will not be the product of a constant cocycle and a coboundary if $\omega$ is a nontrivial multiplier.

Restrict $U_\omega$ to $A$ so that it is a projective representation of the reals. It follows that (see the last paragraph of this section) $U_\omega$ is equivalent to an ordinary representation $U$ given by

\begin{equation}
U(e^t) = p(e^t)U_\omega(e^t)
\end{equation}

where

\begin{equation}
\omega(e^t, e^u) = \frac{p(e^t)p(e^u)}{p(e^t + e^u)}
\end{equation}

for some continuous function $p$ on $A$ and for all $e^t, e^u \in A$. Observe that $U$ satisfies the Weyl commutation relation

\begin{equation}
U(e^t)V_\lambda(-e^t) = \chi_{\lambda}(-e^t)V_\lambda(U(e^t))
\end{equation}

because $U_\omega$ does.

Consequently the operator $U(e^t)V_\lambda(-e^t)$ commutes with all the $V_\lambda(\lambda)$ so that

\begin{equation}
U(e^t) = A(e^t, \cdot)U_\omega(e^t), e^t \in A,
\end{equation}

for some ordinary cocycle $A$.

From (4.8) and (4.11) we see that

\begin{equation}
A(e^t, x) = p(e^t)A_\omega(e^t, x)
\end{equation}

for all $e^t \in A$ and a.e. $(x)$.

We say that $A$ is the cocycle induced by $A_\omega$; it is uniquely determined up to a constant cocycle factor. If $A$ is the product of a constant cocycle $e^{it\lambda}$ and a coboundary $\varphi(x)/\varphi(x - e^t)$ then (4.12) and (4.7) imply that $\omega$ is trivial.
This analysis will have to be refined to yield the desired result on the torus $T^2$ for $T^2$ has no nontrivial multipliers. However, there are $\frac{1}{2}n(n - 1) + 1$ inequivalent local multipliers on $T^n$ or $R^n$ as shown by Bargmann [1] and local multipliers are sufficient for our purposes. Notice, in particular, that $R$ has no nontrivial local projective representations.

5. Local multipliers and cocycles on $T^2$. A local projective multiplier $\omega$ on the torus $T^2$ is a continuous function on some neighborhood $\mathcal{N} \times \mathcal{N}$ of the identity in $T^2 \times T^2$ which satisfies the same functional equation and normalizing condition as a multiplier whenever $x, y$ and $x + y$ belong to $\mathcal{N}$. Unfortunately (4.3) cannot be used to define a local projective representation $U_\omega$, or, equivalently, a local projective cocycle $A_\omega$. We must resort to an ad hoc construction of $U_\omega$ starting from a specific nontrivial local projective multiplier $\omega$. We can then extract a cocycle from $U_\omega$ in much the same manner as in § 4 and it is a matter of detail to prove that $A$ is not the product of a constant cocycle and a coboundary.

Let $T^2$ be realized as the square $[-\pi, \pi] \times [-\pi, \pi]$ with the opposite edges identified and let $\mathcal{N}$ be the open neighborhood $(-\pi, \pi) \times (-\pi, \pi)$ of the identity. For a one-parameter subgroup $A$ we will take the familiar winding line with irrational slope $\alpha$.

Define $\omega$ on $\mathcal{N} \times \mathcal{N}$ by

$$\omega(x, y) = \exp i((x_2 - \alpha x_1)y_1 - (y_2 - \alpha y_1)x_1)$$

(5.1)

where $x = (x_1, x_2), y = (y_1, y_2)$ with $-\pi < x_i, y_i < \pi$. This is the canonical example of a nontrivial local projective multiplier on $T^2$ [1].

Since the complement of $\mathcal{N}$ is a null set we can regard $\omega(x, \cdot)$ as a unitary function on $T^2$ for each fixed $x \in \mathcal{N}$. Now put $A_\omega(x, y) = \omega(x, y - x)$ whenever $x \in \mathcal{N}$ and $y \in \mathcal{N} + x$. Then $A_\omega(x, \cdot)$ is a unitary function on $T^2$ for each fixed $x \in \mathcal{N}$ (the exceptional null set depends upon $x$). For $x \in \mathcal{N}$ we define the unitary operator $U_\omega(x)$ by

$$U_\omega(x) = A_\omega(x, \cdot)U_0(x).$$

(5.2)

It is easily verified that $U_\omega$ is a strongly continuous operator valued function on $\mathcal{N}$.

We will now extract a cocycle $A$ from $A_\omega$ even though $A_\omega(x, \cdot)$ is not defined for all $x$. The discussion parallels that of § 4 and will only be given in outline.

Let $A_1$ denote the connected segment of $A \cap \mathcal{N}$ (relative to the ordinary real line topology on $A$) which contains the identity and choose a proper segment $A_0$ of $A_1$ such that $0 \in A_0 \subseteq A_0 + A_0 \subseteq A_1$. 

For $x, y \in A_0$, $U_\omega$ satisfies (4.1) so that $U_\omega$ is a local projective representation of the reals. Consequently $U_\omega$ is equivalent to a local ordinary representation $U$; this means that equations (4.8) and (4.9) hold for some continuous function $p$ on $A$, (say) and for all $e_t, e_u \in A_0$. The local representation $U$ can be extended to a representation $U$ (keeping the same notation) of $A$ [5, Th. 63] which must satisfy the Weyl commutation relation (4.10). Exactly as before we have $U(e_t) = A(e_t) U_0(e_t), e_t \in A$, for some ordinary cocycle $A$. We say that $A$ is the cocycle induced by $A_\omega$; notice that

$$A(e_t, x) = p(e_t) A_\omega(e_t, x)$$

holds only for $e_t \in A_0, \text{ a.e. } (x)$.

We will now show that $A$ is not the product of a constant cocycle $C$ and a coboundary. If, on the contrary, $A$ is such a product, then

$$A_\omega(e_t, x) = \tilde{p}(e_t)(\varphi(x)/\varphi(x - e_t))$$

holds for all $e_t \in A_0, \text{ a.e. } (x)$ where we have relabeled the continuous function $C/p$ on $A_1$ by $\tilde{p}$. In terms of the unitary operators $U_\omega$ and $U(y) = (\varphi(-y)) U_0(y), y \in T^i$, equation (5.4) becomes $U_\omega(e_t) = \tilde{p}(e_t) U(e_t)$ for all $e_t \in A_0$.

We wish to extend $p$ from $A_0$ to $A \cap N$ in such a way that (5.4) remains valid. A continuity argument will then enable us to extend $p$ from $A \cap N$ to $N$ and this will imply that $\omega$ is trivial.

To extend $p$ from $A_0$ to $A \cap N$ let $y \in A \cap N$ so that $y \in MA_0 = \{Me_t | e_t \in A_0\}$ for some integer $M > 0$. Thus $e_t = yM \in A_0$ and suppose, for the moment, that $ne_t \in N$ for all $n \leq M$. Then

$$U(y) = U(Me_t) = (U(e_t))^M = (p(e_t) U_\omega(e_t))^M = [(p(e_t))^M \prod_{k=1}^{M-1} \omega(e_t, (M - k)e_t)] U_\omega(y)$$

and we can define $p(y)$ to be the value of the expression in the brackets which obviously is independent of the representation $y = Me_t$.

This definition of $p(y)$ is valid whenever $(M - k)e_t$ is in the domain of $\omega(e_t, )$, i.e., whenever $ne_t \in N$ for all $0 \leq n \leq M$. For each $M$ there are only finitely many $y \in MA_0$ such that $ne_t \in N$ for some $0 \leq n \leq M$. For these exceptional values we can define $p(y)$ by continuity (relative to the usual real line topology on $A$) so that

$$(5.5) \quad U(y) = p(y) U_\omega(y)$$

holds for all $y \in A \cap N$; or, equivalently, so that (5.4) holds for all $e_t$ in $A \cap N$.

To extend $p$ from $A \cap N$ to $N$ we need only note that $A \cap N$ is dense in $N$. Let $y \in N$ and choose a sequence $y_n \in A \cap N$ which
converges to \( y \). Hence \( p(y_n)I = U(y_n)U_\omega(-y_n) \) tends strongly to

\[ U(y)U_\omega(-y) \]

and this limit must be of the form \( p(y)I \). Alternately, \( U(y)U_\omega(-y) \) is a multiple of the identity for each \( y \) in \( \mathcal{N} \) because it commutes with all bounded operators when \( y \) varies over a dense subset of \( \mathcal{A} \cap \mathcal{N} \). We have now constructed a continuous function \( p \) on \( \mathcal{N} \) such that (5.5) holds for all \( y \) in \( \mathcal{N} \). Since \( U_\omega \) is a nontrivial local projective representation of \( \mathcal{N} \) this is a contradiction. Hence the induced cocycle \( A \) cannot be the product of a constant cocycle and a coboundary. That completes the proof.

An interesting question remains. If \( A \) is a cocycle on \( T^2 \) can one find a local projective cocycle \( A_\omega \) which induces \( A \)? An affirmative answer should enable one to settle some of the open function theoretic questions on \( T^2 \).

The author gratefully acknowledges useful conversations with Professors F. Forelli, J. E. Gilbert and H. Helson.

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Received April 22, 1969.

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**Missoula, Montana**
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PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Kokusai Bunken Insatsuisha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.
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