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Let $x(t)$ denote the number of jumps occurring in the time interval $[0, t)$ and $v_k(t) = P\{x(t) = k\}$. The generating function of $v_k(t)$ is given by

$$\exp\{\lambda t[\phi(x) - 1]\}, \phi(x) = \sum_{k=1}^{\infty} p_k x^k, \sum_{k=1}^{\infty} p_k = 1.$$

Lay off to the right of the origin successive intervals of length $z^j/j^\alpha, j = 1, 2, \dots$. Explicitly the end points are

$$t_1(z) = 0$$

$$t_j(z) = \sum_{k=1}^{j-1} z^k/k^\alpha, j = 2, 3, \dots, \alpha > 0,$$

and

$$t_\infty(z) = \sum_{k=1}^{\infty} z^k/k^\alpha.$$

Following Shepp and Lloyd L_r , the length of the r th longest cycle and S_r , the length of the r th shortest cycle have been defined for our choice of $x(t)$ and $t_j, j = 1, 2, \dots$. This paper obtains the asymptotics for the m th moments of L_r and S_r suitably normalized by a new technique of generating functions. It is further shown that the results of Shepp and Lloyd are particular cases of these more general results.

Here we consider a problem involving a random permutation which is closely linked with the cycle structure of the permutation. Let S_n be the $n!$ permutation operators on n numbered places. Let $\alpha(\pi) = \{\alpha_1(\pi), \alpha_2(\pi), \dots, \alpha_n(\pi)\}$ be the cycle class of $\pi \in S_n$. In this permutation π , there are $\alpha_1(\pi)$ cycles of length one, $\alpha_2(\pi)$ cycles of length two, etc. Usually the elements of S_n are assigned a probability $1/n!$ each. John Riordan has considered a model where he has assigned the probability

$$1.1 \quad P\{\alpha_1 = a_1, \alpha_2 = a_2, \dots, \alpha_n = a_n\} = \prod_{j=1}^n (1/j)^{a_j}/a_j! \text{ if } \sum_{j=1}^n ja_j = n,$$

$$= 0 \text{ otherwise,}$$

for the cycle class $\alpha(\pi)$, the a 's being nonnegative integers. Here α 's would be independent if it were not for the condition $\sum ja_j = n$. Shepp and Lloyd has considered a sequence $\alpha = \{\alpha_1, \alpha_2, \dots\}$ of mutually independent nonnegative integral valued random variables where for $j = 1, 2, \dots$ the random variable α_j follows the Poisson distribution

with mean z^j/j , $0 < z < 1$, z being same for all values of j . Accordingly

$$1.2 \quad P_z\{\alpha_1 = a_1, \alpha_2 = a_2, \dots\} = (1 - z)z^{\sum_{j=1}^{\infty} j a_j} \prod_{j=1}^{\infty} (1/j)^{a_j}/a_j!, \\ a_j > 0, j = 1, 2, \dots$$

From this it can be seen that the probability distribution of the random variable $\nu(\alpha) = \sum_{j=1}^{\infty} j a_j$ is

$$1.3 \quad P\{\nu(\alpha) = n\} = (1 - z)z^n, n = 0, 1, 2, \dots$$

Also

$$1.4 \quad P_z\{\alpha_1 = a_1, \alpha_2 = a_2, \dots | \nu(\alpha) = n\} = \prod_{j=1}^n (1/j)^{a_j}/a_j!, \sum_{j=1}^{\infty} j a_j = n \\ = 0 \text{ otherwise.}$$

Thus Shepp and Lloyd were able to recover 1.1 assumed in the model. In this paper, for the cycle class $\alpha(\pi)$ we have assigned the probability

$$1.5 \quad P_z(\alpha_1 = a_1, \alpha_2 = a_2, \dots, \alpha_n = a_n) = I/II, 0 < z < 1, \sum_{j=1}^n j a_j = n \\ = 0 \text{ otherwise.}$$

Here

$$1.6 \quad I = \prod_{j=1}^{\infty} v_{a_j}(z^j/j^\alpha), \sum_{j=1}^n j a_j = n, a_{n+1} = a_{n+2} = \dots = 0, (\sum_{j=1}^{\infty} j a_j = n)$$

where $v_{a_j}(z^j/j^\alpha)$ is the coefficient of x^{a_j} in $\exp\{\lambda(z^j/j^\alpha)[\phi(x) - 1]\}$,

$$1.7 \quad \phi(x) = \sum_{k=1}^{\infty} p_k x^k \quad \text{and} \quad \sum_{k=1}^{\infty} p_k = 1.$$

On detailed computation

$$1.8 \quad v_{a_j}(z^j/j^\alpha) = e^{-\lambda z^j/j^\alpha} \sum_{n_1+2n_2+\dots+n_j=a_j} \frac{(p_1 z^j/j^\alpha)^{n_1} (p_2 z^j/j^\alpha)^{n_2} \dots}{n_1! n_2! \dots}.$$

In the special case when $\lambda = 1$, $p_1 = 1$, $p_2 = p_3 = \dots = 0$ and $\alpha = 1$, $\exp\{\lambda(z^j/j^\alpha)[\phi(x) - 1]\}$ reduces to the generating function of the Poisson process with the time parameter equals to z^j/j , which has been considered by Shepp and Lloyd. Also the generating function of II which represents the distribution of $P\{\nu(\alpha) = n\}$, where for our choice of the sequence α_j 's defined by 1.14

$$1.9 \quad \nu(\alpha) = \sum_{j=1}^{\infty} j a_j$$

is given by

$$1.10 \quad \sum_{n=1}^{\infty} P\{\nu(\alpha) = n\}x^n = \prod_{j=1}^{\infty} \exp \{ \lambda(z^j/j^\alpha)[\phi(x^j) - 1] \} .$$

On detailed computation we note that

$$1.11 \quad P\{\nu(\alpha) = n\} = \exp \left\{ -\lambda \sum_{j=1}^{\infty} z^j/j^\alpha \right\} \times \sum_{\substack{n_1+2n_2+3n_3+\dots \\ +2(n'_1+2n'_2+\dots) \\ +3(n''_1+2n''_2+\dots) \\ +\dots=n}} \left\{ \begin{aligned} & \left[\frac{\left(\frac{p_1\lambda z}{1^\alpha}\right)^{n_1} \left(\frac{p_2\lambda z^2}{2^\alpha}\right)^{n_2} \dots}{n_1! n_2! \dots} \right] \times \\ & \left[\frac{\left(\frac{p_2\lambda z}{1^\alpha}\right)^{n'_1} \left(\frac{p_2\lambda z^2}{2^\alpha}\right)^{n'_2} \dots}{n'_1! n'_2! \dots} \right] \times \\ & \left[\frac{\left(\frac{p_3\lambda z}{1^\alpha}\right)^{n''_1} \left(\frac{p_3\lambda z^2}{2^\alpha}\right)^{n''_2} \dots}{n''_1! n''_2! \dots} \right] \times \dots \end{aligned} \right\} .$$

In particular when $\lambda = 1, \alpha = 1$ and $p_1 = 1, p_2 = p_3 = \dots = 0$, the generating function of the distribution of 1.9 reduces to

$$1.12 \quad \exp \left[-\sum (z^j/j) + \sum (x^j z^j/j) \right] = (1 - z)/(1 - zx) .$$

Hence

$$P\{\nu(\alpha) = n\} = (1 - z)z^n ,$$

which is in agreement with that considered by Shepp and Lloyd. In the special case mentioned above

$$1.13 \quad \begin{aligned} I/II &= \prod_{j=1}^n (1/j)^{a_j}/a_j! & \text{if} & \quad \sum_{j=1}^n ja_j = n, \\ &= 0 & \text{otherwise.} & \end{aligned}$$

This is also in agreement with the model discussed by Shepp and Lloyd.

If we take $\alpha = (\alpha_1, \alpha_2, \dots)$ to be a sequence of mutually independent nonnegative integral valued random variables where for $j = 1, 2, \dots$

$$1.14 \quad P_z\{\alpha_j = a_j\} = v_{a_j}(z^j/j^\alpha), a_j = 0, 1, 2, \dots ,$$

by using the Borel-Cantelli lemma, we can easily show that $\nu(\alpha) = \sum_{j=1}^{\infty} ja_j$ is finite with probability one. Hence the joint distribution $(\alpha_1, \alpha_2, \alpha_3, \dots, \nu(\alpha))$ can be written as

$$1.15 \quad \begin{aligned} P_z\{\alpha_1 = a_1, \alpha_2 = a_2, \dots, \nu(\alpha) = n\} &= \prod_{j=1}^{\infty} v_{a_j}(z^j/j^\alpha) \text{ if } \sum_{j=1}^{\infty} ja_j = n, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

From this we can see that

$$1.16 \quad P_z\{\alpha_1 = a_1, \alpha_2 = a_2, \dots, | \nu(\alpha) = n\} = I/II,$$

which we have assumed for the model.

Shepp and Lloyd have considered a Poisson process which takes place on $T = \{-\infty < t < +\infty\}$ at unit rate. That is, for any interval of length $I \subset T$, the probability that p jumps occur in I is

$$\exp[-|I|] |I|^p/p!, \quad p = 0, 1, 2, \dots$$

independently of any conditions on the process on $T - I$. They have taken the following end points for the time intervals

$$1.17 \quad \begin{aligned} t_1(z) &= 0, \\ t_j(z) &= \sum_{k=1}^{j-1} z^k/k, \quad j = 2, 3, \dots, \\ t_\infty(z) &= \sum_{k=1}^{\infty} z^k/k = \log(1-z)^{-1}, \end{aligned}$$

so that the j th interval is

$$t_j(z) < t < t_{j+1}(z), \quad j = 1, 2, \dots$$

They define $\lambda_z(t)$; $-\infty < t < \infty$, to be a function whose value is ' j ' on the j th interval, $j = 1, 2, \dots$ and is zero if $t < 0$ or $t > t_\infty(z)$. Then for each $j = 1, 2, \dots$ the interval $\{t; \lambda_z(t) = j\}$ has length z^j/j , the probability that a_j jumps of the Poisson process occur in this interval is

$$1.18 \quad \exp(-z^j/j) \cdot (z^j/j)^{a_j}/a_j!, \quad a_j = 0, 1, 2, \dots$$

and that these various events for $j = 1, 2, \dots$ are mutually independent. They have taken a sample function of the Poisson process, with jumps in the interval $[0, t_\infty(z))$, which are finite in number with probability one, occurring at times $\tau_1 \leq \tau_2 \leq \dots \leq \tau_\sigma$ (σ , random). They take the positive integers $\lambda_z(\tau_1) \leq \lambda_z(\tau_2) \leq \dots \leq \lambda_z(\tau_\sigma)$ as the lengths of the σ cycles of a permutation on $\nu = \sum_{s=1}^{\sigma} \lambda_z(\tau_s)$ places, and in this class S_ν , they choose a permutation at random with uniform distribution. For any given $r = 1, 2, \dots$ let $S_r = S_r(\alpha)$ be the length of the r th shortest cycle in a permutation of the cycle class $\alpha \cdot S_r(\alpha) = 0$ if $\sum \alpha_j < r$. If the r th jump of the Poisson process occur at ' t ', then $S_r = \lambda_z(t)$ according to their model. Hence they have obtained the distribution of S_r . Similarly they have obtained the distribution of $L_r = L_r(\alpha)$, the length of the r th longest cycle. They have given asymptotics for the distribution and to all moments of the length of the r th longest and r th shortest cycles.

In this paper, instead of the Poisson process considered by Shepp and Lloyd, we consider a more general process which can have $k(k > 1)$

jumps at any moment. Let $x(t)$ denote the number of jumps in the interval $[0, t)$ and let

$$1.19 \quad v_k(t) = P\{x(t) = k\} .$$

Let p_k be the probability of having k jumps at a chosen moment, if it is certain that jumps do occur generally at that moment. It has been shown in Khintchine that

$$1.20 \quad F(t, x) = \sum_{k=0}^{\infty} v_k(t)x^k = \exp \{ \lambda t [\phi(x) - 1] \} ,$$

where $\phi(x)$ is given by (1.7) and $\lambda > 0$. In our model, we take the end points of the time intervals to be

$$1.21 \quad \begin{aligned} t_1(z) &= 0 \\ t_j(z) &= \sum_{k=1}^{j-1} z^k / k^\alpha, \quad j = 2, 3, \dots, \alpha > 0 , \end{aligned}$$

and

$$t_\infty(z) = \sum_{k=1}^{\infty} z^k / k^\alpha .$$

Here the probability that L_r , the length of the r th longest cycle is 'j' is given by

$$1.22 \quad \begin{aligned} P_z\{L_r = j\} &= \frac{\lambda}{\sum_{k=1}^r p_k} \int_{t_j}^{t_{j+1}} \left\{ \sum_{k=1}^r p_k v_{r-k}(t_\infty - t) \right\} dt, \\ &= \frac{\lambda}{P_r} \int_{t_j}^{t_{j+1}} \left\{ \sum_{k=1}^r p_k v_{r-k}(t_\infty - t) \right\} dt , \end{aligned}$$

where

$$P_r = \sum_{k=1}^r p_k .$$

Also the probability that S_r , the length of the r th shortest cycle is 'j' is given by

$$P_z\{S_r = j\} = \frac{\lambda}{P_r} \int_{t_j}^{t_{j+1}} \left\{ \sum_{k=1}^r p_k v_{r-k}(t) \right\} dt .$$

Here we use the technique of generating functions to estimate the asymptotics of $E\{L_r\}^m$ and $E\{S_r\}^m$ suitably normalized in a way different from that used by Shepp and Lloyd. While they have considered the case where the jumps occur according to Poisson law, we have considered a more general system of which Poisson process is a special case. By assuming the Poisson law for jumps they were able to recover the model based on the uniform distribution. By assuming a more general law for

jumps we obtain a generalised probability model for the cycle class of which that derived on the basis of the uniform distribution is a special case. Thus we have in this paper discussed a generalization of the one given by Shepp and Lloyd with the help of the new technique.

2. A lemma. We now prove a lemma which we use extensively.

LEMMA. *Let*

$$2.1 \quad A(z, x) = \sum_{r=1}^{\infty} a_r(z)x^r,$$

and

$$2.2 \quad A(x) = \sum_{r=1}^{\infty} a_r x^r,$$

with $a_r(z) > 0$, satisfying

$$2.3 \quad \sum_{r=1}^{\infty} a_r(z) = c, \quad 0 < z < 1,$$

c , a constant. Then for

$$2.4 \quad a_r(z) \longrightarrow a_r, \quad z \longrightarrow 1^-,$$

it is necessary and sufficient that for $0 < x < 1$

$$2.5 \quad A(z, x) \longrightarrow A(x), \quad z \longrightarrow 1^-.$$

Proof of the lemma. First let us suppose that (2.4) holds. Then for fixed x , ($0 < x < 1$) and ε , we can choose a number n_0 such that $\{x^{n_0}/(1-x)\} < \varepsilon$. Then,

$$2.6 \quad |A(z, x) - A(x)| < \sum_{r=0}^{n_0} |a_r(z) - a_r| x^r + 2c\varepsilon.$$

Now each term in the right hand side tends to zero. Hence the necessary part. Now suppose that (2.5) holds. Since $\{a_r(z)\}$ is bounded it is always possible to find a converging subsequence. If (2.4) is not true then we can extract two subsequences converging to two different sequences $\{a_r^*\}$ and $\{a_r^{**}\}$ and the corresponding subsequences of $\{A(z, x)\}$ would converge to $A^*(x) = \sum a_r^* x^r$ and $A^{**}(x) = \sum a_r^{**} x^r$ which contradicts the assumption that (2.5) holds. Hence $\{a_r^*\} = \{a_r^{**}\} = \{a_r\}$. This proves the sufficiency part.

3. The r th longest cycle. The m th raw moment of the r th longest cycle is

$$3.1 \quad E_z\{L_r\}^m = \lambda \sum_{j=1}^{\infty} \frac{j^m}{P_r} \int_{t_j}^{t_{j+1}} \sum_{k=1}^r p_k v_{r-k}(t_{\infty} - t) dt.$$

Hence

$$\begin{aligned}
 \sum_{r=1}^{\infty} P_r \omega^{r-1} E_z \{L_r\}^m &= \lambda \sum_{r=1}^{\infty} \omega^{r-1} \sum_{j=1}^{\infty} j^m \int_{t_j}^{t_{j+1}} \sum_{k=1}^r p_k v_{r-k}(t_{\infty} - t) dt, \\
 3.2 \qquad \qquad \qquad &= \lambda \sum_{j=1}^{\infty} j^m \int_{t_j}^{t_{j+1}} \sum_{r=1}^{\infty} \omega^{r-1} \left\{ \sum_{k=1}^r v_{r-k}(t_{\infty} - t) p_k \right\} dt, \\
 &= \lambda \sum_{j=1}^{\infty} j^m \int_{t_j}^{t_{j+1}} e^{\lambda[\phi(x)-1](t_{\infty}-t)} \{\phi(x)/x\} dx dt.
 \end{aligned}$$

Let $F = F(\lambda)$ denotes the left hand side of (3.2) and $F' = F(\lambda s^{1-\alpha})$.

$$\begin{aligned}
 3.3 \qquad F' &= s^{1-\alpha} \lambda \sum_{j=1}^{\infty} j^m \int_{t_j}^{t_{j+1}} e^{\lambda s^{1-\alpha}[\phi(x)-1](t_{\infty}-t)} \{\phi(x)/x\} dx dt, \\
 &= \sum_{r=1}^{\infty} P_r \omega^{r-1} E_z (L'_r)^m,
 \end{aligned}$$

where L'_r is the same as L_r with λ replaced by $\lambda s^{1-\alpha}$.

Let us now consider some analytical preliminaries regarding $t_j(z)$. With $z = e^{-s}$, $0 < s < \infty$. We have

$$3.4 \qquad t_{\infty}(e^{-s}) - t_j(e^{-s}) = \sum_{k=j}^{\infty} \{e^{-ks}/k^{\alpha}\}.$$

In the interval $\{y: ks < y < (k+1)s\}$, we have

$$\frac{e^{-ks}}{k^{\alpha} s^{\alpha}} > \frac{e^{-y}}{y^{\alpha}} > \frac{e^{-(k+1)s}}{(k+1)^{\alpha} s^{\alpha}},$$

and

$$3.5 \qquad \frac{e^{-ks} s^{1-\alpha}}{k^{\alpha}} > \int_{ks}^{(k+1)s} \frac{e^{-y}}{y^{\alpha}} dy > \frac{s^{1-\alpha} e^{-(k+1)s}}{(k+1)^{\alpha}}.$$

Summing with respect to k , we have,

$$3.6 \qquad s^{1-\alpha} \sum_{k=j}^{\infty} (e^{-ks}/k^{\alpha}) > \int_{js}^{\infty} (e^{-y}/y^{\alpha}) dy.$$

Let

$$3.7 \qquad E(\theta) = \int_{\theta}^{\infty} (e^{-y}/y^{\alpha}) dy.$$

Then from (3.6) $E(js) < s^{1-\alpha} \sum_{k=j}^{\infty} e^{-ks}/k^{\alpha}$. Also

$$\int_{(j-1)s}^{\infty} (e^{-y}/y^{\alpha}) dy > s^{1-\alpha} \sum_{k=j}^{\infty} \{e^{-ks}/k^{\alpha}\}.$$

Combining the two

$$3.8 \qquad E(js) < s^{1-\alpha} \sum_{k=j}^{\infty} \{e^{-ks}/k^{\alpha}\} < E((j-1)s).$$

Now consider the equation

$$3.9 \quad s^{1-\alpha} \sum_{k=j}^{\infty} \{e^{-ks}/k^\alpha\} = E(X) .$$

If $X_j(s)$ is the root of the equation (3.9), we have

$$3.10 \quad \text{and} \quad \begin{array}{l} \text{(i)} \quad (j-1)s < X_j(s) < js \\ \text{(ii)} \quad X_j(s) \text{ is unique.} \end{array}$$

In (3.3) put $E(\theta) = s^{1-\alpha}(t_\infty - t)$ so that

$$s^{1-\alpha} dt = \{e^{-\theta}/\theta^\alpha\} d\theta .$$

Hence

$$3.11 \quad F' = \lambda \sum_{j=1}^{\infty} j^m \int_{X_j(s)}^{X_{j+1}(s)} \{\phi(x)/x\} \frac{e^{\lambda[\phi(x)-1]E(\theta)-\theta}}{\theta^\alpha} d\theta .$$

Let

$$\mu_j = \int_{X_j(s)}^{X_{j+1}(s)} d\mu(\theta) ,$$

where

$$3.12 \quad d\mu(\theta) = \{e^{\lambda[\phi(x)-1]E(\theta)-\theta}/\theta^\alpha\} d\theta .$$

But

$$3.13 \quad (j-1)s < X_j(s) < js \quad \text{and} \quad js < X_{j+1}(s) < (j+1)s .$$

This implies that

$$X_j(s) < js < X_{j+1}(s) .$$

Thus

$$s^m F' = \frac{\lambda\phi(x)}{x} \sum_{j=1}^{\infty} (js)^m \int_{X_j(s)}^{X_{j+1}(s)} d\mu(\theta) .$$

Now

$$3.14 \quad \frac{\lambda\phi(x)}{x} \sum_{j=1}^{\infty} X_j^m(s) \mu_j \leq F' s^m \leq \frac{\lambda\phi(x)}{x} \sum_{j=1}^{\infty} X_{j+1}^m(s) \mu_j .$$

Consider

$$3.15 \quad \int_{X_1(s)}^{\infty} \theta^m d\mu(\theta) = \sum_{j=1}^{\infty} \int_{X_j(s)}^{X_{j+1}(s)} \theta^m d\mu(\theta) .$$

We have

$$3.16 \quad \sum_{j=1}^{\infty} X_j^m(s) \mu_j \leq \int_{X_1(s)}^{\infty} \theta^m d\mu(\theta) \leq \sum_{j=1}^{\infty} X_{j+1}^m(s) \mu_j .$$

i.e.,

$$I_1 \leq I \leq I_2 \quad (\text{say}) ,$$

where

$$I_1 = \sum_{j=1}^{\infty} X_j^m(s) \mu_j, I_2 = \sum_{j=1}^{\infty} X_{j+1}^m(s) \mu_j$$

and

$$I = \int_{X_1(s)}^{\infty} \theta^m d\mu(\theta) .$$

I_1 and I_2 are the Darboux sums for the Stieltjes integral based on the above meshes. Also $X_1(s) \rightarrow 0$ as $s \rightarrow 0^+$. Hence

$$\begin{aligned} 3.17 \quad s^m F' &\sim \{\phi(x)/x\} \lambda \int_0^{\infty} \theta^{m-\alpha} e^{\lambda[\phi(x)-1]E(\theta)-\theta} d\theta, s \rightarrow 0^+, m \geq \alpha, \\ &\sim \lambda \int_0^{\infty} \theta^{m-\alpha} e^{-\theta} d\theta \sum_{r=1}^{\infty} x^{r-1} \left\{ \sum_{k=1}^r v_{r-k}[E(\theta)] p_k \right\}, s \rightarrow 0^+. \end{aligned}$$

Now

$$\begin{aligned} 3.18 \quad s^m \sum_{r=1}^{\infty} P_r E_z(L'_r)^m &= \lambda s^{m+1-\alpha} \sum_{j=1}^{\infty} j^m \int_{t_j}^{t_{j+1}} dt = \lambda s^{m+1-\alpha} \sum_{j=1}^{\infty} j^m (t_{j+1} - t_j) \\ &= \lambda s^{m+1-\alpha} \sum_{j=1}^{\infty} j^m \{e^{-js}/j^\alpha\} = \lambda s^{m+1-\alpha} \sum_{j=1}^{\infty} \{e^{-js}/j^{\alpha-m}\} < \infty . \end{aligned}$$

Hence using the lemma

$$3.19 \quad s^m P_r E_z(L'_r)^m \sim \lambda \int_0^{\infty} \left[\sum_{k=1}^r v_{r-k}[E(\theta)] p_k \right] e^{-\theta} \theta^{m-\alpha} d\theta, s \rightarrow 0^+ .$$

Since $s \sim (1 - z)$,

$$(1 - z)^m E_z(L'_r)^m \sim (\lambda/P_r) \int_0^{\infty} \left[\sum_{k=1}^r v_{r-k}[E(\theta)] p_k \right] e^{-\theta} \theta^{m-\alpha} d\theta, z \rightarrow 1^- .$$

Taking $\lambda = 1, \alpha = 1, p_1 = 1, p_2 = p_3 \dots = 0$, we now have

$$\begin{aligned} 3.20 \quad (1 - z)^m E_z\{L_r\}^m &\sim \int_0^{\infty} v_{r-1}[E(\theta)] e^{-\theta} \theta^{m-1} d\theta, z \rightarrow 1^-, \\ &\sim \int_0^{\infty} e^{-E(\theta)-\theta} [E(\theta)]^{r-1} \{\theta^{m-1}/(r-1)!\} d\theta, z \rightarrow 1^- . \end{aligned}$$

This is in agreement with Shepp and Lloyd.

4. The r th shortest cycle. Let S_r be the length of the r th shortest cycle. Then

$$4.1 \quad P\{S_r = j\} = (\lambda/P_r) \int_{t_j}^{t_{j+1}} \sum_{k=1}^r p_k v_{r-k}(t) dt .$$

Let

$$F_1 = F_1(\lambda) = \sum_{r=1}^{\infty} P_r \alpha^{r-1} E_z\{S_r\}^m .$$

Then

$$\begin{aligned}
 4.2 \quad F_1 &= \lambda \sum_{r=1}^{\infty} x^{r-1} \sum_{j=1}^{\infty} j^m \int_{t_j}^{t_{j+1}} \sum_{k=1}^r p_k v_{r-k}(t) dt, \\
 &= \lambda \sum_{j=1}^{\infty} j^m \int_{t_j}^{t_{j+1}} e^{\lambda[\phi(x)-1]t} \{\phi(x)/x\} dt.
 \end{aligned}$$

Also

$$F'_1 = F_1(\lambda s^{1-\alpha}) = \sum P_r x^{r-1} E_z(S'_r)^m,$$

where S'_r is the same as S_r with λ replaced by $\lambda s^{1-\alpha}$. Put $(t_{\infty} - t)s^{1-\alpha} = E(\theta)$ in F'_1 .

$$4.3 \quad F'_1 = \lambda \sum_{j=1}^{\infty} j^m \int_{X_j(s)}^{X_{j+1}(s)} \{\phi(x)/(x\theta^\alpha)\} e^{\lambda[s^{1-\alpha}t_{\infty}-E(\theta)][\phi(x)-1]-\theta} d\theta.$$

Let

$$\mu_j = \int_{X_j(s)}^{X_{j+1}(s)} d\mu(\theta),$$

where

$$4.4 \quad d\mu(\theta) = \{\phi(x)/x\theta^\alpha\} e^{\lambda[s^{1-\alpha}t_{\infty}-E(\theta)][\phi(x)-1]-\theta} d\theta.$$

Hence

$$4.5 \quad s^m F'_1 = \lambda \sum_{j=1}^{\infty} (js)^m \int_{X_j(s)}^{X_{j+1}(s)} \{\phi(x)/x\theta^\alpha\} e^{\lambda[s^{1-\alpha}t_{\infty}-E(\theta)][\phi(x)-1]-\theta} d\theta.$$

Since $(j-1)s < X_j(s) < js < X_{j+1}(s) < (j+1)s$,

$$4.6 \quad \lambda \sum_{j=1}^{\infty} X_j^m(s) \mu_j < F'_1 s^m < \lambda \sum_{j=1}^{\infty} X_{j+1}^m(s) \mu_j.$$

Also

$$\sum_{j=1}^{\infty} X_j^m(s) \mu_j < \sum_{j=1}^{\infty} \int_{X_j(s)}^{X_{j+1}(s)} \theta^m d\mu(\theta) < \sum_{j=1}^{\infty} X_{j+1}^m(s) \mu_j.$$

That is

$$4.7 \quad \sum_{j=1}^{\infty} X_j^m(s) \mu_j < \int_{X_1(s)}^{\infty} \theta^m d\mu(\theta) < \sum_{j=1}^{\infty} X_{j+1}^m(s) \mu_j.$$

Hence

$$4.8 \quad s^m F'_1 \sim \lambda \int_0^{\infty} \theta^{m-\alpha} \{\phi(x)/x\} e^{\lambda[s^{1-\alpha}t_{\infty}-E(\theta)][\phi(x)-1]-\theta} d\theta, \quad s \rightarrow 0^+, \quad m \geq \alpha.$$

Here also $s^m \sum_{r=1}^{\infty} P_r E_z(S'_r)^m = s^{m+1-\alpha} \sum_{j=1}^{\infty} j^m (t_{j+1} - t_j) < \infty$ {by (3.18)}. Thus as in 3.17 by equating the coefficient of x^{r-1} on both sides we can obtain $\lim_{s \rightarrow 0} s^m P_r E_z(S'_r)^m$.

Now let us consider the particular case of the above when $p_1 = 1$, $p_2 = p_3 = \dots = 0$, $\lambda = 1$ and $\alpha = 1$. Here

$$\begin{aligned}
 4.9 \quad s^m \sum_{r=1}^{\infty} x^{r-1} E_z(S_r)^m &\sim \int_0^{\infty} \theta^{m-1} e^{(x-1)[\log(1-z)^{-1}] - (x-1)E(\theta) - \theta} d\theta, \quad z \rightarrow 1^-, \\
 &\sim s \int_0^{\infty} \theta^{m-1} e^{-x[E(\theta) + \log s] + E(\theta) - \theta} d\theta, \quad s \rightarrow 0^+.
 \end{aligned}$$

Hence

$$s^{m-1} \sum_{r=1}^{\infty} x^{r-1} E_z(S_r)^m \sim e^{x \log(s^{-1})} \int_0^{\infty} e^{-xE(\theta) + E(\theta) - \theta} \theta^{m-1} d\theta .$$

So

$$4.10 \quad \frac{(1-z)^{m-1}}{(m-1)!} \sum_{r=1}^{\infty} x^{r-1} E_z(S_r)^m \sim \frac{1}{(m-1)!} \times \left[\int_0^{\infty} e^{E(\theta) - \theta} \theta^{m-1} \sum_{r=1}^{\infty} \frac{[-xE(\theta)]^{r-1}}{(r-1)!} d\theta \right] \times \left[\sum_{r=1}^{\infty} \frac{[x \log(1-z)^{-1}]^{r-1}}{(r-1)!} \right] .$$

Equating coefficient of x^{r-1} on both sides of 4.10

$$4.11 \quad \frac{(1-z)^{m-1}}{(m-1)} E_z(S_r)^m \sim \frac{1}{(m-1)!} \sum_{p=0}^{r-1} \left[\{[\log(1-z)^{-1}]^p / p!\} \times \left\{ \int_0^{\infty} \frac{[-E(\theta)]^{r-1-p} \theta^{m-1} e^{E(\theta) - \theta}}{(r-1-p)!} d\theta \right\} \right], s \rightarrow 0^+ \\ \sim \sum_{p=0}^{r-1} (1/p!) [\log(1-z)^{-1}]^p K(r-1-p, m), s \rightarrow 0^+ ,$$

where

$$4.12 \quad K(q, m) = \int_0^{\infty} \frac{\theta^{m-1} [-E(\theta)]^q e^{E(\theta) - \theta}}{(m-1)! q!} d\theta$$

which is in agreement with Shepp and Lloyd.

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