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JANE M. DAY

Suppose that S is a finite dimensional cancellative commutative clan with $E = \{0, 1\}$ and that H is the group of units of S. We show that if square roots exist in S/H, not necessarily uniquely, then there is a closed positive cone T in E^n for some n and a homomorphism $f: (T \cup \infty) \times H \rightarrow S$ which is onto and one-to-one on some neighborhood of the identity. $T \cup \infty$ denotes the one point compactification of T.

K. Keimel proved in (6), and Brown and Friedberg independently in (1), that if S/H is uniquely divisible, then it is isomorphic to $T \cup \infty$ for some closed positive cone T. Brown and Friedberg went on to show that if S is uniquely divisible, then S is isomorphic to the Rees quotient $((T \cup \infty) \times H)/(\infty \times H)$. What we do here is to weaken their hypothesis to assume just square roots in S/H and conclude that S is isomorphic to some quotient of such $(T \cup \infty) \times H$, which will be a Rees quotient if square roots are unique in $(S/H)\backslash 0$, but in general need not be Rees.[1] $f((T \cup \infty) \times 1)$ is a subclan of S and a local cross section at 1 for the orbits of the group action $H \times S \rightarrow S$ (which equal $\mathscr{H}$ classes here), but an example shows that it need not be a full cross section. Also, square roots exist (uniquely) in S if and only if they exist (uniquely) in S/H and H.

The proof consists essentially of showing that the ingenious constructions of (1) can still be done under the weaker hypothesis, in a sufficiently small neighborhood of H.

For basic information about semigroups, see (5), (8) or (9). The real intervals (0, 1] and [0, 1] are semigroups under usual real multiplication; as in (5), a *one parameter semigroup* is a homomorph of (0, 1], and we also define here a *closed one parameter semigroup* to be a nonconstant homomorph of [0, 1].

The Lemmas (I)-(III) are variations on standard themes so we omit proofs. (See (1), (3), (4), B-3 of (5), (6) and (7).) Throughout this paper let S be a clan with exactly two idempotents, a zero and an identity denoted by 0 and 1 respectively.

(I) *If R is a one parameter semigroup in S which is not contained in H and is not equal to* 0, *then $R \cup 0$ is a closed one parameter semigroup and an arc with endpoints* 0 *and* 1. *Let $\phi: (0, 1] \rightarrow R$ be the homomorphism that defines R; if $x = \phi(t) \in R$ and $k \geqq 0$, we write*

[1] Keimel has concurrently proved a further generalization, by a different method, assuming instead of cancellation that $x \times H \rightarrow xH$ is one-to-one for all x near H.

x^k for $\phi(t^k)$, and if $x \neq 0, 1$, each $y \in R\backslash 0$ equals x^k for unique k.

(II) *If H is normal and every element of S/H has a square root in S/H, then for each $x \in S$ there exists a closed one parameter semigroup in S intersecting Hx.*

(III) *Let T be a commutative uniquely divisible clan with group of units $H(T)$ and $E = \{0, 1\}$, and let V be a set containing a neighborhood of 1 in T such that $T\backslash V$ is an ideal. If S is commutative and ψ': $V \to S$ is a continuous function such that $\psi'(V\backslash H(T)) \cap H = \square$ and $\psi'(xy) = \psi'(x)\psi'(y)$ whenever $x, y, xy \in V$, then ψ' can be extended to a homomorphism ψ on all of T by defining $\psi(0) = 0$ and $\psi(x^n) = \psi'(x)^n$ for each $x \in V$ and positive integer n.*

The definition of independent family which follows agrees with the algebraic independence used in [1] when H is trivial and $W = S\backslash 0$, and that notion is due to Clark [2]. We include H in our definition so that we do not have to handle the case of S with trivial H separately first, and we define independence in neighborhoods of H rather than in S in order to apply the concept effectively to a clan with nonunique roots.

An *independent family* in S is a finite family $\{R_1, \cdots, R_n\}$ of closed one parameter semigroups in S such that there exists a neighborhood W of H with the property that for every partition of the set $\{1, \cdots, n\}$ into two nonnull disjoint sets A and B, this is true:

$$P_{i \in A}\{R_i\} \cap (P_{i \in B}\{R_i\})H \cap W \subset H .$$

We will also describe this situation by saying that $\{R_1, \cdots, R_n)$ is *independent in W*. We adopt the convention that if $X = \square$, then $P_{i \in X}\{x_i\} = 1$, for x_i's which are elements or subsets of S. S will be called *cancellative* if $x, y, z \in S$ and $xy = xz \neq 0$ implies $y = z$.

We will make frequent use of the following facts. $F(V)$ denotes boundary of V. Any neighborhood of H in compact S contains a neighborhood V of H such that $S\backslash V$ is an ideal (A-3.1, (5)), and if V is a set such that $S\backslash V$ is an ideal, then

$$0 \notin V,\ V = VH,\ F(V) = F(V)H,$$

$S\backslash V^*$ is an ideal if nonempty, and $xy \in V$ implies $x, y \in V$. If J is a closed ideal in compact S, shrinking J to a point gives a new compact semi-group denoted S/J and called the *Rees quotient* of S by J, and the natural map $S \to S/J$ is a homomorphism.

Part (i) of the lemma below is analogous to 1.4 of (1); part (ii) shows that the homomorphisms ϕ: $S\backslash 0 \to E^n$ and β: $S\backslash 0 \to H$ constructed in (1) can still be constructed here on a sufficiently small neighborhood of H. *Dim S* means inductive dimension of S.

LEMMA. *Let S be a cancellative commutative clan with $E = \{0, 1\}$ and let W be a closed neighborhood of* 1 *such that $S \backslash W$ is an ideal.*

(i) *If $\{R_1, \cdots, R_n\}$ is an independent family in W, and if $x_1 x_2 \cdots x_n h = x'_1 x'_{2} \cdots x'_n h' \in W$, where $x_i, x'_i \in R_i$ for each i and $h, h' \in H$, then $x_i = x'_i$ for each i and $h = h'$; consequently* $\dim S \geqq n$.

(ii) *Suppose* $\dim S \leqq N$ *or* $\dim S/H \leqq N$ *and that S/H has square roots. Then there exists a maximal independent family $\{R_1, \cdots, R_n\}$ of closed one parameter semigroups in S, and a closed neighborhood U of H may be chosen so that $S \backslash U$ is an ideal and if $x \in U$, x satisfies this condition.*

($\nmid$) There exists a unique partition (A, B) of $\{1, \cdots, n\}$ and unique elements $x_i \in R_i$ and $h \in H$ such that $i \in B$ whenever $x_i = 1$ and $x(P_{i \in A}\{x_i\}) = (P_{i \in B}\{x_i\})h \in W$.

Proof. (i) Since R_i is a closed one parameter semigroup and $x_i \neq 0$, we may factor x_i or x'_i for each i and then commute and cancel in the equality given to get $0 \neq P_{i \in A}\{r_i\} = (P_{i \in B}\{r_i\})h'h^{-1}$ for some partition (A, B) of $\{1, \cdots, n\}$. These points lie in W so by independence, $r_i = 1$, hence $x_i = x'_i$, for each i, and thus $h = h'$ also. There is a closed neighborhood V of 1 such that $V^n \subset W$, and then the multiplication function $(R_1 \cap V) \times \cdots \times (R_n \cap V) \rightarrow S$ is a homeomorphism so S contains an n-cell.

(ii) If $\dim S \leqq N$, then a maximal independent family exists by (i). If $\dim S/H \leqq N$ instead, S/H is cancellative since S is, so (i) can be applied to S/H to get a maximal independent family in S/H; a closed one parameter semigroup in S projects to a closed one parameter semigroup in S/H by (I), and it is easy to see that an independent family in S projects to one in S/H, so S can have no larger independent family than S/H does.

Now choose a maximal independent family $\{R_1, \cdots, R_n\}$ in S, and choose W smaller if necessary so that the R_i's are actually independent in a neighborhood of H containing W^2.

To prove the uniqueness assertion of ($\nmid$), suppose that

$$x(\underset{i \in A}{P}\{x_i\}) = (\underset{i \in B}{P}\{x_i\})h \in W \quad \text{and} \quad x(\underset{i \in A'}{P}\{x'_i\}) = (\underset{i \in B'}{P}\{x'_i\})h' \in W\,,$$

as described in ($\nmid$). Then

$$(\underset{i \in A}{P}\{x_i\})(\underset{i \in B'}{P}\{x'_i\})h' = (\underset{i \in A'}{P}\{x'_i\})(\underset{i \in B}{P}\{x_i\})h \in W^2\,;$$

for each i, collect into one term the x_k's with $k = i$, on each side, and suppose there exists $j \in A \cap B'$; $j \in A$ implies that the factor on the left which is an element of R_j is not 1, and it has to equal one of the factors on the right by (i); therefore j has to be in A' or in B, because by independence an element of $(R_j \cap W^2)\backslash 1$ cannot arise from multiples of elements of R_i's for $i \neq j$. But $j \in B$ implies $j \notin A$ and $j \in A'$ implies $j \notin B'$, both contradictions. So $A \cap B'$ must be empty, similarly $A' \cap B$ is empty, hence $(A, B) = (A', B')$. Now apply (i).

Now let R be any closed one parameter semigroup in S.

$$\{R, R_1, \cdots, R_n\}$$

is not independent in any neighborhood of H (where R and R_i are each counted if $R = R_i$ for some i), so there is a particular partition (A_R, B_R) of $\{1, \cdots, n\}$ such that $T = RP \cap QH$ contains points arbitrarily near H in $S\backslash H$, where $P = P_{i \in A}\{R_i\}$ and $Q = P_{i \in B_R}\{R_i\}$. T is also a compact semigroup, so it contains a connected subsemigroup from 1 to 0 (B-4.9, (5)). $F(W)$ separates 0 and 1 in S, hence we may select $x_R \in R$ such that $x_R P \cap QH \cap F(W) \neq \square$. Every $x \geqq x_R$ in R satisfies ($\nmid$) since the complement of an ideal in R is connected and $\{x \in R \mid xP \cap QH \subset S\backslash W\}$ is an ideal of R. It follows that every $x \geqq x_R$ in RH satisfies ($\nmid$) also.

If we can find a closed neighborhood U of H such that $x_R \notin U$ for each closed one parameter semigroup R in S, then every $y \in U$ lies in some RH by (II), U may be chosen smaller so that $S\backslash U$ is an ideal, and then every $y \in U$ satisfies ($\nmid$) by the preceding remark. Suppose no such U exists, so there is a net (x_R) of the x_R's clustering at some element of H; since there exist only a finite number of partitions of $\{1, \cdots, n\}$, we may suppose that for one particular partition (A, B) and for each x_R in the net, $(A_R, B_R) = (A, B)$. Then, since $F(W) = F(W)H$, any cluster point of (a_R) is an element of

$$\underset{i \in A}{P}\{R_i\} \cap (\underset{i \in B}{P}\{R_i\})H \cap F(W) ;$$

but this set is empty (by definition if $A = \square$, and if $A \neq \square$, by independence in W).

Euclidean n-space, denoted E^n, is a semigroup under vector addition with the origin as identity. If P^* is the set of nonnegative real numbers, N the set of negative real numbers, and juxtaposition denotes scalar multiplication, a *closed positive cone* in E^n is defined to be a closed subsemigroup T of E^n such that $P^* T \subset T$ and $NT \cap T = (0, \cdots, 0)$. The one point compactification $T \cup \infty$ of a nontrivial closed positive cone T is a continuum and becomes a clan with exactly two idempotents, a zero and an identity, when addition is extended by defining $z + \infty = \infty + z = \infty$ for each $z \in T \cup \infty$, and such clans are uniquely divisible (where the "nth root" of z would be $(1/n)z$ since the operation is addition).

THEOREM. *Suppose that S is a commutative cancellative clan with $E = \{0, 1\}$, such that every element of S/H has a square root in S/H.*

If $\dim S \leqq N$ *or* $\dim S/H \leqq N$, *then there is a closed positive cone* T *in* E^n *and an onto homomorphism* $f: (T \cup \infty) \times H \rightarrow S$ *which is a homeomorphism of some neighborhood of the identity onto a neighborhood of the identity in S.* f *maps* $(T \cup \infty) \times 1$ *to a subclan* T' *which is a local cross section at* 1 *for the natural projection homomorphism* $S \rightarrow S/H$.

Proof. Let W, U and $\{R_1, \cdots, R_n\}$ be as in (ii) of the Lemma and let $x_i \in R_i \cap F(U)$ for each i. These x_i's will remain fixed throughout the proof, and since $x_i \neq 0, 1$, by (I) each element of $R_i \backslash 0$ equals x_i^t for a unique nonnegative real number t. This together with (ii) of the Lemma implies that for each $x \in U$, there are a unique partition (A, B) of $\{1, \cdots, n\}$, unique real numbers $t_1, \cdots, t_n$, and unique $h \in H$ such that $x(P_{i \in A}\{x_i^{t_i}\}) = (P_{i \in B}\{x_i^{t_i}\})h \in W$ and $i \in B$ if $t_i = 0$; following the notation of (1), let $\varepsilon_i = 1$ if $i \in B$ and $\varepsilon_i = -1$ if $i \in A$, let $\phi(x) = (\varepsilon_1 t_1, \cdots, \varepsilon_n t_n)$, and let $\beta(x) = h$. Arguments just like those in (1) show that $\phi \times \beta$ is a homeomorphism, if one uses at judicious spots the facts that W is compact and that $S \backslash W$ is an ideal. Since S is commutative, ϕ and β are homomorphisms as far as they go.

Let $T = P^* \phi(U)$. We show next that $\phi(U)$ contains a neighborhood of the origin in T and that T is a closed positive cone in E^n. First, $T = P^* \phi(F(U))$ because each closed one parameter semigroup in S intersects $F(U)$, so T is closed in E^n because in general if A is closed in P^* and S is compact in E^n and does not contain the origin, then AB is closed. For this same reason, $[1, \infty)\phi(F(U))$ is closed, hence its complement in T is a neighborhood of the origin in T and also is a subset of $\phi(U)$ because $k\phi(x) = \phi(x^k)$ and $x \in U$ implies $x^k \in U$, for $k \in [0, 1)$. Since $\phi(U)$ contains a neighborhood of the origin in T and ϕ preserves multiplication on U, T is a subsemigroup of E^n. To see that $NT \cap T$ is the origin it suffices to prove that $(-1)\phi(U) \cap \phi(U)$ is, so suppose $x, x' \in U$ and $\phi(x) = (-1)\phi(x') = (t_1, \cdots, t_n)$. Then for some $h, h' \in H$, $x(P_{i \in A}\{x_i^{t_i}\}) = (P_{i \in B}\{x_i^{t_i}\})h \in W$ and $x'(P_{i \in B}\{x_i^{t_i}\}) = (P_{i \in A}\{x_i^{t_i}\})h' \in W$. Substituting from the first equation into the second and cancelling gives $x'xh^{-1} = h'$, hence $x, x' \in H$, hence $\phi(x)$ is the origin as required.

Now define $\psi: \phi(U) \rightarrow S$ by $\psi(z) = (\phi \times \beta)^{-1}(z, 1)$. ψ is a homeomorphism into and, if U is chosen small enough that ϕ is actually defined on U^2, ψ preserves multiplication on $\phi(U)$ also. T is uniquely divisible so by (III), ψ may be extended to a homomorphism of T into S. Now define $f: (T \cup \infty) \times H \rightarrow S$ by $f(z, h) = \psi(z)h$. f is a homomorphism because ψ is and S is commutative, and it is a homeomorphism of $\phi(U) \times H$ onto U because there it equals $(\phi \times \beta)^{-1}$. (We cannot use (III) to define f directly as an extension of $(\phi \times \beta)^{-1}$, because H need not be uniquely divisible.) Since the image of f is a subclan of S which contains a neighborhood of H and since S is divisible, f is onto. Therefore $T'H = S$ so $T' \to S/H$ is onto and the rest is clear.

In a semigroup with zero, a *nilpotent* is a nonzero element some finite power of which is zero.

COROLLARY. *Let everything be as in the theorem.*

(i) *If square roots are unique in* $(S/H)\backslash 0$ (but there could be nilpotents) *then* f *is one-to-one on the complement of* $f^{-1}(0)$, *hence* f *induces an isomorphism from the Rees quotient* $((T \cup \infty) \times H)/f^{-1}(0)$ *onto* S *and also* T' *is a full cross section for* $H \times S \to S$. *If square roots are unique in all of* S/H (so there are no nilpotents) *then* $f^{-1}(0) = \infty \times H$, *so* S *is isomorphic to* $((T \cup \infty) \times H)/(\infty \times H)$ (Theorem 2.2 of (1)).

(ii) *Square roots exist* (uniquely) *in* S *if and only if they exist* (uniquely) *in* H *and* S/H.

Proof. Let $p: S \to S/H$ be the natural map. If $f(t, h) = f(s, g) \neq 0$, then $f(t, 1)h = f(s, 1)g$ hence $pf(t, 1) = pf(s, 1)$. Uniqueness of roots in $(S/H)\backslash 0$ implies $pf(kt, 1) = pf(ks, 1)$ for all $k \geqq 1$ at least, and pf is one-to-one near the identity by the theorem, hence $kt = ks$ must be true for k sufficiently small. Therefore $t = s$ and cancelling $f(t, 1)$ now gives $h = g$ also. The rest is clear.

EXAMPLE 1. This was also discovered by D. Brown and M. Friedberg (and communicated orally to this author). It is a cancellative commutative clan S with $E = \{0, 1\}$ and trivial group of units, which has no nilpotents and is divisible but not uniquely divisible; in fact, any two distinct one parameter semigroups in S are independent near 1 and have no nondegenerate arc in common, but can intersect infinitely. Thus S is not a Rees quotient of any compactified cone. The author is indebted to Kermit Sigmon for the elegance of this description of the example.

Let T be the closed first quadrant of E^2, let D be the closed unit disc in the complex plane with usual complex multiplication, and define $g: T \cup \infty \to D$ by $g(x, y) = e^{-(x+y)+(x-y)\pi i}$ and $g(\infty) = 0$. g is a homomorphism by (III), so $S = g(T \cup \infty)$ is a clan, it has $E = \{0, 1\}$, is topologically a 2-cell, and is an egg-shaped subset of D with large end at 1 and small end at $-1/e$. S is commutative, cancellative and free of nilpotents since D is, has roots of all orders since $T \cup \infty$ does, and square roots are not unique since $\phi(1, 0) = \phi(0, 1)$ but $\phi(1/2, 0) \neq \phi(0, 1/2)$.

S can also be visualized without the aid of D: there is a congruence $\sim$ on $T \cup \infty$ such that S is isomorphic to $(T \cup \infty)/\sim$: it is the smallest congruence which identifies (0, 1) and (1, 0), and dividing by it has the effect geometrically of rolling up $T \cup \infty$ into a cone with pointed end at ∞.

EXAMPLE 2. This will show that the subclan T' of the theorem need not be a full cross section for H orbits, i.e., $\mathscr{H}$ classes. Let $T \cup \infty$ be as in the previous example, let G be the circle group with usual complex number notation, and let Q be the product semigroup $(T \cup \infty) \times G$. We will twist the $\mathscr{H}$ class of $(0, 1, 1)$ and then identity it with the $\mathscr{H}$ class of $(1, 0, 1)$. Formally, let $\sim$ be the smallest closed congruence on Q which identifies $(0, 1, 1)$ and $(1, 0, -1)$, let $S = Q/\sim$, and let $f: Q \to S$ be the natural projection. Thus if $\varDelta$ is the diagonal of $Q \times Q$, $p = [(0, 1, 1), (1, 0, -1)]$, and $q = [(1, 0, -1), (0, 1, 1)]$, then $\sim$ is the smallest closed symmetric subsemigroup of $Q \times Q$ containing $p \cup \varDelta$, and $pq \in \varDelta$ so this equals $\varDelta(\Gamma(p) \cup \Gamma(q) \cup \varDelta)$. Clearly $[(0, 1, 1), (1, 0, 1)]$ is not in the semigroup generated by $p \cup q \cup \varDelta$, and $\Gamma(p)$ and $\Gamma(q)$ have only one limit point, ∞, so this point is not in $\sim$, i.e., $f(0, 1, 1) \neq f(1, 0, 1)$. On the other hand, the $\mathscr{H}$ classes in S of these points are equal, because $H = f(0 \times 0 \times G)$ is the group of units of S and $f(0, 1, 1) = f(1, 0, 1)f(0, 0, -1)$.

f is a homeomorphism on $[0, 1) \times [0, 1) \times G$, which is a neighborhood of the identity, and we will show below that S is cancellative, so this is exactly the situation of the theorem. However, if T' denotes $f((T \cup \infty) \times 1)$, $T' \to S/H$ is not one-to-one.

Interestingly, there actually is a full cross section semigroup for the H orbits of this clan S; the problem in the above lies in the definition of f—that is, in the choice of the independent closed one parameter semigroups in S:

$$R_1 = f([0, \infty] \times 0 \times 1) \quad \text{and} \quad R_2 = f(0 \times [0, \infty] \times 1)$$

are independent but do not themselves intersect in some of the H orbits which they both go through. Rechoosing f so that R_2 actually does intersect R_1 at the levels where $Q \to S$ collapses two H orbits to one yields a subclan T'' of S which is isomorphic to S/H. In detail, define $g: Q \to Q$ by $g(x, y, e^{i\theta}) = (x, y, e^{i(\theta + \pi y)})$, let $f' = fg$, and let $T'' = f'((T \cup \infty) \times 1)$. To see that $T'' \to S/H$ is one-to-one, suppose

$$fg(x, y, 1) = fg(x', y', 1)fg(0, 0, e^{i\theta}) \neq 0 .$$

We will prove $e^{i\theta} = 1$. In $g(x, y, 1) = g(x', y', e^{i\theta})$ then we are done because g is one-to-one, so suppose $g(x, y, 1) \neq g(x', y', e^{i\theta})$. f identifies these points and not to 0 so for some n, $((g(x, y, 1), g(x', y', e^{i\theta})) \in \varDelta p^n$. An arbitrary point of $\varDelta p^n$ is of the form $((s, n + t, e^{i\phi}), (n + s, t, e^{i(\phi + n\pi)}))$ for some s, t and ϕ, so we conclude $x' = x + n$, $y = y' + n$, $e^{i\pi y} = e^{i\phi}$, and $e^{i(\theta+\pi y')} = e^{i(\phi+n\pi)}$. These imply $e^{i(\theta+\pi y')} = e^{i\pi y'}$, so $e^{i\theta} = 1$ as asserted. From this it follows at once that $T'' \to S/H$ is one-to-one and in fact that S is isomorphic to $(T'' \times H)/(\infty \times H)$.

Now it is easy to show S cancellative, for it suffices to prove that T'' is, so suppose $fg(x, y, 1)fg(s, t, 1) = fg(x', y', 1)fg(s, t, 1)$. It follows that $x + s + n = x' + s$ and $y + t = y' + t + n$ for some n, hence $x + n = x'$ and $y = y' + n$. $fg(x, y, 1) = fg(x', y', 1)$ now is clear.

It seems at least possible that the technique used here for re-choosing f might work in general, so that there is always a full cross section semigroup for $S \to S/H$ when S is a homomorph of the direct product of H and a closed positive cone.

It also seems reasonable to conjecture that the theorem is still true with only H normal and S/H commutative, instead of S commutative. Under these weaker conditions ϕ and β still exist, but β need not be a homomorphism unless the R_i's commute with one another and with H; using Theorem VI of (5), it is possible to choose a maximal independent set in the centralizer of H, but the problem of choosing the R_j's to commute with one another also remains unsolved.

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