

# Pacific Journal of Mathematics

**QUASI REGULAR GROUPS OF FINITE COMMUTATIVE  
NILPOTENT ALGEBRAS**

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## QUASI REGULAR GROUPS OF FINITE COMMUTATIVE NILPOTENT ALGEBRAS

N. H. EGGERT

Let  $J$  be a finite commutative nilpotent algebra over a field  $F$  of characteristic  $p$ .  $J$  forms an abelian group under the "circle" operation, defined by  $a \circ b = a + b + ab$ . This group is called the quasi regular group of  $J$ .

Our main purpose is to investigate the relationship between the structure of  $J$  as an algebra, and the structure of its quasi regular group.

In particular, the structure of the quasi regular group is described in terms of certain subalgebras of  $J$ . These subalgebras are, for fixed  $j$ , the  $p^j$  powers of elements in  $J$ . They are denoted by  $J^{(j)}$ .

It is conjectured that the dimension of  $J^{(j)}$  is greater than or equal to  $p$  times the dimension of  $J^{(j+1)}$ . If this is true, then Theorems 1.1 and 2.1 completely describe the possibilities for the quasi regular group of  $J$ . Paragraph 2 considers some special cases of the conjecture.

1. The quasi regular group of  $J$ . Let  $J$  be a finite commutative nilpotent algebra over a field  $F$  with  $p^u$  elements. Denote by  $J^{(j)}$  the set of  $p^j$ th powers of elements in  $J$ ,  $j = 0, 1, \dots$ . The  $J^{(j)}$  form a descending chain of subalgebras of  $J$ . If  $t$  is the minimum exponent such that  $x^{p^t} = 0$  for all  $x \in J$  then  $J^{(t-1)} \neq (0)$  and  $J^{(t)} = (0)$ . The constant  $t$  will be called the height of  $J$ . Let the dimension of  $J^{(j)}$  be  $r_j$  and set  $s_h = r_{h-1} + r_{h+1} - 2r_h$ ,  $h = 1, \dots, t$ .

We denote by  $G(p, u; s_1, \dots, s_t)$  the group which is the direct sum of  $u s_h$ ,  $h = 1, \dots, t$ , copies of the cyclic group of order  $p^h$ .

**THEOREM 1.1.** *The quasi regular group of  $J$  is isomorphic to  $G(p, u; s_1, \dots, s_t)$ .*

*Proof.* Since the  $p$ th power of  $x \in J$  with respect to the operation "o" is  $x^p$ , the number of cyclic summands of order greater than  $p^h$  is the dimension of the quotient group  $J^{(h)}/J^{(h+1)}$  over the integers modulo  $p$ , that is  $u(r_h - r_{h+1})$  [1, page 27]. Hence the number of cyclic summands of order  $p^h$  in the quasi regular group  $J$  is  $u(r_{h-1} + r_{h+1} - 2r_h)$ ,  $h = 1, \dots, t$ .

2. The possibilities for the quasi regular group of  $J$ . Given certain  $p$ -groups, finite commutative nilpotent algebras can be con-

structed with these groups as their quasi regular groups.

**THEOREM 2.1.** *Let  $a_i$  be arbitrary nonnegative integers for  $i = 1, \dots, t, a_i \neq 0$ . Then there exists a finite commutative nilpotent algebra  $J$  over a field  $F$  of order  $p^u$  where:*

- (i)  $r_i = 0$  and  $r_{i-1} = pr_i + a_i, i = 1, \dots, t$ .
- (ii) *the quasi regular group of  $J$  is  $G(p, u; s_1, \dots, s_t)$  where  $s_h = r_{h-1} + r_{h+1} - 2r_h$ .*

*Proof.* Let  $J_j$  be the Jacobson radical of  $F[X]/(X^n)$ , where  $n = p^{j-1} + 1$ . If  $x = X + (X^n)$  then a basis for  $J_j$  over  $F$  is  $\{x, x^2, \dots, x^{n-1}\}$ . Thus the dimension of  $J_j^{(i)}$  is  $p^{j-i-1}$  for  $i < j$ . Let  $J$  be the direct sum of  $a_j$  copies of  $J_j$  for  $j = 1, \dots, t$ . Then  $r_i = \dim J^{(i)} = \sum_{j=i+1}^t a_j p^{j-i-1}, i < t, r_t = \dim J^{(t)} = 0$ . A simple calculation gives  $r_{i-1} - pr_i = a_i$ . By using Theorem 1.1, the proof is complete.

The author conjectures that the converse of the above theorem is also true, that is:

(C) If  $J$  is a finite commutative nilpotent algebra over  $F$  then  $\dim J^{(i-1)} - p \dim J^{(i)} = r_{i-1} - pr_i \geq 0$ .

This is immediate for algebras of height one, height two and  $\dim J^{(1)} = 1$ , and height two and  $p = 2$ . The following theorem establishes (C) for algebras of height two and  $\dim J^{(1)} = 2$ .

**THEOREM 2.2.** *Let  $J$  be a commutative nilpotent algebra over a perfect field  $F$  of characteristic  $p$ . Let  $x, y$  be elements of  $J$  and suppose  $x^p$  and  $y^p$  are linearly independent over  $F$ . Then the dimension of  $J$  is greater than or equal to  $2p$ .*

*Proof.* Suppose the theorem is false. That is, assume there is a finite commutative nilpotent algebra  $J$  over  $F$  and:

- (i)  $x, y \in J$  and  $x^p, y^p$  are independent over  $F$ ,
- (ii)  $\dim J < 2p$ .

We assume  $J$  is an algebra of least dimension over  $F$  which satisfies (i) and (ii). It then follows that:

- (iii)  $J$  is generated by  $x$  and  $y$ , and
- (iv) If  $I$  is an ideal of  $J$  and an algebra over  $F$  then  $I = (0)$  or for some  $a, b \in F, 0 \neq ax^p + by^p \in I$ .

If (iv) were false then  $J/I$  would satisfy (i) and (ii) and the dimension of  $J/I$  would be less than the dimension of  $J$ .

We may assume  $x^p$  is in the annihilator of  $J$ . This follows since, by (iv), there are elements  $a, b$  in  $F$  where  $ax^p + by^p \neq 0$  is in the annihilator. By replacing  $x$  by  $x' = a'x + b'y$ , where  $a'^p = a$  and  $b'^p = b$ , conditions (i) through (iv) hold and  $x'^p$  is in the annihilator.

Let  $\mathcal{S}$  be the cartesian product of the nonnegative integers with

themselves less  $(0, 0)$ . Let the total ordering  $<$  be defined in  $\mathcal{C}$  by:  $(s, t) < (i, j)$  if  $s + t < i + j$  or  $s + t = i + j$  and  $s < i$ .

LEMMA. *If  $x^i y^j \neq 0$  then  $i + j \leq p$ .*

*Proof.* Let  $(n, m(0))$  be the maximum element in  $\mathcal{C}$ , with respect to  $<$ , such that  $x^n y^{m(0)} \neq 0$ . Suppose that  $n + m(0) > p$ .

Since  $x^p$  is in the annihilator of  $J$ ,  $n \leq p$  and  $m(0) > 0$ , thus if  $n > 0$  then  $\mathcal{A} = \{(i, j) \in \mathcal{C} : i \leq n, \text{ and } j \leq m(0)\}$  has more than  $2p$  elements. The monomials  $x^i y^j, (i, j) \in \mathcal{A}$ , are dependent, thus a nontrivial relation.

$$\sum a_{ij} x^i y^j = z = 0, (i, j) \in \mathcal{A}$$

exists. Let  $(s, t)$  be minimum such that  $a_{st} \neq 0$ . Consider

$$0 = z x^{n-s} y^{m(0)-t}.$$

For  $(s, t) < (i, j)$  it follows that  $(n, m(0)) < (i + n - s, j + m(0) - t)$ . By the definition of  $(n, m(0))$  we obtain  $0 = a_{st} x^n y^{m(0)}$ . This is a contradiction; thus  $n = 0$ .

Now define  $m(i)$  to be the maximum integer such that  $x^i y^{m(i)} \neq 0, i = 0, \dots, p$ . Since  $x, \dots, x^p, y, \dots, y^p$  are dependent, let

$$(1) \quad z = \sum_{i=h}^p a_i x^i + \sum_{i=l}^p b_i y^i = 0,$$

where  $a_h \neq 0$  and  $b_l \neq 0$ . There is at least one nonzero  $a_j$  since  $y, \dots, y^p$  are independent. Likewise at least one  $b_i$  is nonzero. Thus considering  $x^{p-h} z$  and  $y^{m(0)-l} z$  we find  $x^{p-h} y^l \neq 0$  and  $x^h y^{m(0)-l} \neq 0$ .

We will now show that, for  $k = 0, \dots, h$ , if  $i \geq k$  and  $x^i y^j \neq 0$  then  $(i, j) \leq (k, m(k))$ . Suppose this has been shown for  $0, \dots, k - 1$ . Since  $(i + 1, m(i + 1)) < (i, m(i))$  for  $i < k$ , we see that  $m(0) \geq m(i) + 2i$ . From  $x^h y^{m(0)-l} \neq 0$  and  $h < k - 1$  we have

$$(h, m(0) - l) < (k - 1, m(k - 1)).$$

Therefore  $h + m(0) - l < k - 1 + m(k - 1)$  and  $l - h \geq k$ . Now let  $(u, v)$  be maximum such that  $u \geq k$  and  $x^u y^v \neq 0$ . Since  $x^{p-h} y^l \neq 0$  and  $p - h \geq l - h \geq k$  it follows that  $u + v \geq p - h + l \geq p + k$ . If  $v = 0$  then  $u = p$  and  $k = 0$ . Since for  $k = 0$  our result is established, we consider  $v > 0$ . If  $u > k$  then the set  $\mathcal{A} = \{(i, j) \in \mathcal{C} : k \leq i \leq u, 0 \leq j \leq v\}$  contains  $(u - k + 1)(v + 1) \geq 2(u - k + v) \geq 2p$  elements. Thus there is a nontrivial relation among the  $x^i y^j, (i, j) \in \mathcal{A}$ . As before, let  $(s, t)$  be minimum such that the coefficient,  $a_{st}$ , of  $x^s y^t$  is nonzero. On multiplying the relation by  $x^{u-s} y^{v-t}$  we obtain  $0 = a_{st} x^u y^v$  which is contradictory. Therefore  $u = k$  and  $v = m(k)$ . By the

definition of  $(u, v)$ , if  $i \geq u = k$  and  $x^i y^j \neq 0$  then  $(i, j) < (k, m(k))$ .

We now have the inequality,  $m(0) \geq 2k + m(k)$ , for  $k = 0, \dots, h$ . Since  $x^h y^{m(0)-l} \neq 0$ ,  $m(h) \geq m(0) - l$ . That is  $l \geq 2h$ .

Let  $bh + c = p$  where  $0 \leq c < h$ . Returning to equation (1) we obtain:

$0 \neq a_h^l x^p = x^c (\sum_i a_i x^i)^b = x^c (-\sum_i b_i y^i)^b = x^c y^{bl} Y$ , where  $Y$  is a polynomial in  $y$ .

Hence  $x^c y^{bl} \neq 0$ . This implies  $m(0) - 2c \geq m(c) \geq bl \geq 2bh$ . Therefore  $m(0) \geq 2p$  and  $y, \dots, y^{2p}$  are independent. This is a contradiction and the lemma is established.

Next we show that if  $m + n = p$  and  $n \neq p$  then  $x^m y^n = c_n x^p$  where  $c_n \in F$ . Suppose this holds for the powers of  $y$  being  $0, \dots, n - 1$ . If  $x^m y^n = 0$  then the result is established. Thus suppose  $x^m y^n \neq 0$ . There are  $(m + 1)(n + 1) \geq 2p$  monomials of the form  $x^p$  or  $x^i y^j$ ,  $i \leq m, j \leq n$ . Thus there is a nontrivial relation

$$\sum a_{ij} x^i y^j + a x^p = 0.$$

Let  $(s, t)$  be minimum such that the coefficient of  $x^s y^t$  is nonzero. By multiplying the relation by  $x^{m-s} y^{n-t}$  we obtain:

$$\begin{aligned} 0 &= \sum_{i+j=s+t} a_{ij} x^{i+m-s} y^{j+n-t} + a x^{p+m-s} y^{n-t} \\ &= \sum_{\substack{i+j=s+t \\ (i,j) \neq (s,t)}} c_{j+n-t} a_{ij} x^p + a' x^p + a_{s,t} x^m y^n. \end{aligned}$$

Since  $x^p$  is in the annihilator of  $J$ ,  $x^{p+m-s} y^{n-t}$  is  $x^p$  or 0. Therefore  $x^m y^n = c_n x^p$ .

Similarly we obtain: if  $m + n = p$  and  $m \neq p$ , then  $x^m y^n = b_m y^p$ . Since  $x^p$  and  $y^p$  are independent, if  $m + n = p$ ,  $m \neq 0$ ,  $p$  then  $x^m y^n = 0$ .

From equation (1) we may obtain, as before,  $x^{p-h} y^l \neq 0$  and  $x^h y^{p-l} \neq 0$  where  $0 < h, l \leq p$ . Assuming, without loss of generality,  $h \geq l$  we have  $h + (p - l) \geq p$  and by the lemma we have equality, that is,  $h = l$ . Since  $x^h y^{p-h} \neq 0$  we have, by the above paragraph,  $h = l = p$ . Equation (1) becomes  $0 = a_p x^p + b_p y^p$  for nonzero  $a_p$  and  $b_p$ , a contradiction. This completes the proof of Theorem 2.2.

#### REFERENCE

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