## QUASI REGULAR GROUPS OF FINITE COMMUTATIVE NILPOTENT ALGEBRAS

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Let $J$ be a finite commutative nilpotent algebra over a field $F$ of characteristic $p$. $J$ forms an abelian group under the "circle" operation, defined by $a \circ b = a + b + ab$. This group is called the quasi regular group of $J$.

Our main purpose is to investigate the relationship between the structure of $J$ as an algebra, and the structure of its quasi regular group.

In particular, the structure of the quasi regular group is described in terms of certain subalgebras of $J$. These subalgebras are, for fixed $j$, the $p^j$ powers of elements in $J$. They are denoted by $J^{(j)}$.

It is conjectured that the dimension of $J^{(j)}$ is greater than or equal to $p$ times the dimension of $J^{(j+1)}$. If this is true, then Theorems 1.1 and 2.1 completely describe the possibilities for the quasi regular group of $J$. Paragraph 2 considers some special cases of the conjecture.

1. **The quasi regular group of J.** Let $J$ be a finite commutative nilpotent algebra over a field $F$ with $p^u$ elements. Denote by $J^{(j)}$ the set of $p^j$th powers of elements in $J$, $j = 0, 1, \cdots$. The $J^{(j)}$ form a descending chain of subalgebras of $J$. If $t$ is the minimum exponent such that $x^{p^t} = 0$ for all $x \in J$ then $J^{(t-1)} \neq (0)$ and $J^{(t)} = (0)$. The constant $t$ will be called the height of $J$. Let the dimension of $J^{(j)}$ be $r_j$ and set $s_h = r_{h-1} + r_{h+1} - 2r_h$, $h = 1, \cdots, t$.

We denote by $G(p, u; s_1, \cdots, s_t)$ the group which is the direct sum of $us_h$, $h = 1, \cdots, t$, copies of the cyclic group of order $p^h$.

THEOREM 1.1. *The quasi regular group of $J$ is isomorphic to $G(p, u; s_1, \cdots, s_t)$.*

*Proof.* Since the $p$th power of $x \in J$ with respect to the operation "$o$" is $x^p$, the number of cyclic summands of order greater than $p^h$ is the dimension of the quotient group $J^{(h)}/J^{(h+1)}$ over the integers modulo $p$, that is $u(r_h - r_{h+1})$ [1, page 27]. Hence the number of cyclic summands of order $p^h$ in the quasi regular group $J$ is $u(r_{h-1} + r_{h+1} - 2r_h)$, $h = 1, \cdots, t$.

2. **The possibilities for the quasi regular group of J.** Given certain $p$-groups, finite commutative nilpotent algebras can be con-

structed with these groups as their quasi regular groups.

**THEOREM 2.1.** *Let $a_i$ be arbitrary nonnegative integers for $i = 1, \cdots, t, a_t \neq 0$. Then there exists a finite commutative nilpotent algebra $J$ over a field $F$ of order $p^u$ where:*

( i )   *$r_t = 0$ and $r_{i-1} = pr_i + a_i$, $i = 1, \cdots, t$.*

( ii )   *the quasi regular group of $J$ is $G(p, u; s_1, \cdots, s_t)$ where $s_h = r_{h-1} + r_{h+1} - 2r_h$.*

*Proof.* Let $J_j$ be the Jacobson radical of $F[X]/(X^n)$, where $n = p^{j-1} + 1$. If $x = X + (X^n)$ then a basis for $J_j$ over $F$ is $\{x, x^2, \cdots, x^{n-1}\}$. Thus the dimension of $J_j^{(i)}$ is $p^{j-i-1}$ for $i < j$. Let $J$ be the direct sum of $a_j$ copies of $J_j$ for $j = 1, \cdots, t$. Then $r_i = \dim J^{(i)} = \sum_{j=i+1}^{t} a_j p^{j-i-1}$, $i < t$, $r_t = \dim J^{(t)} = 0$. A simple calculation gives $r_{i-1} - pr_i = a_i$. By using Theorem 1.1, the proof is complete.

The author conjectures that the converse of the above theorem is also true, that is:

(C)   If $J$ is a finite commutative nilpotent algebra over $F$ then $\dim J^{(i-1)} - p \dim J^{(i)} = r_{i-1} - pr_i \geqq 0$.

This is immediate for algebras of height one, height two and $\dim J^{(1)} = 1$, and height two and $p = 2$. The following theorem establishes (C) for algebras of height two and $\dim J^{(1)} = 2$.

**THEOREM 2.2.** *Let $J$ be a commutative nilpotent algebra over a perfect field $F$ of characteristic $p$. Let $x, y$ be elements of $J$ and suppose $x^p$ and $y^p$ are linearly independent over $F$. Then the dimension of $J$ is greater than or equal to $2p$.*

*Proof.* Suppose the theorem is false. That is, assume there is a finite commutative nilpotent algebra $J$ over $F$ and:

( i )   $x, y \in J$ and $x^p, y^p$ are independent over $F$,

( ii )   $\dim J < 2p$.

We assume $J$ is an algebra of least dimension over $F$ which satisfies (i) and (ii). It then follows that:

(iii)   $J$ is generated by $x$ and $y$, and

(iv)   If $I$ is an ideal of $J$ and an algebra over $F$ then $I = (0)$ or for some $a, b \in F$, $0 \neq ax^p + by^p \in I$.

If (iv) were false then $J/I$ would satisfy (i) and (ii) and the dimension of $J/I$ would be less than the dimension of $J$.

We may assume $x^p$ is in the annihilator of $J$. This follows since, by (iv), there are elements $a, b$ in $F$ where $ax^p + by^p \neq 0$ is in the annihilator. By replacing $x$ by $x' = a'x + b'y$, where $a'^p = a$ and $b'^p = b$, conditions (i) through (iv) hold and $x'^p$ is in the annihilator.

Let $\mathscr{C}$ be the cartesian product of the nonnegative integers with

themselves less $(0, 0)$. Let the total ordering $\prec$ be defined in $\mathscr{C}$ by: $(s, t) \prec (i, j)$ if $s + t < i + j$ or $s + t = i + j$ and $s < i$.

LEMMA. *If* $x^i y^j \neq 0$ *then* $i + j \leqq p$.

*Proof.* Let $(n, m(0))$ be the maximum element in $\mathscr{C}$, with respect to $\prec$, such that $x^n y^{m(0)} \neq 0$. Suppose that $n + m(0) > p$.

Since $x^p$ is in the annihilator of $J$, $n \leqq p$ and $m(0) > 0$, thus if $n > 0$ then $\mathscr{A} = \{(i, j) \in \mathscr{C} : i \leqq n, \text{ and } j \leqq m(0)\}$ has more than $2p$ elements. The monomials $x^i y^j$, $(i, j) \in \mathscr{A}$, are dependent, thus a nontrivial relation.

$$\Sigma a_{ij} x^i y^j = z = 0, (i, j) \in \mathscr{A}$$

exists. Let $(s, t)$ be minimum such that $a_{st} \neq 0$. Consider

$$0 = z x^{n-s} y^{m(0)-t} .$$

For $(s, t) \prec (i, j)$ it follows that $(n, m(0)) \prec (i + n - s, j + m(0) - t)$. By the definition of $(n, m(0))$ we obtain $0 = a_{st} x^n y^{m(0)}$. This is a contradiction; thus $n = 0$.

Now define $m(i)$ to be the maximum integer such that $x^i y^{m(i)} \neq 0$, $i = 0, \cdots, p$. Since $x, \cdots, x^p, y, \cdots, y^p$ are dependent, let

$$(1) \qquad\qquad z = \sum_{i=h}^{p} a_i x^i + \sum_{i=l}^{p} b_i y^i = 0 ,$$

where $a_h \neq 0$ and $b_l \neq 0$. There is at least one nonzero $a_j$ since $y, \cdots, y^p$ are independent. Likewise at least one $b_i$ is nonzero. Thus considering $x^{p-h} z$ and $y^{m(0)-l} z$ we find $x^{p-h} y^l \neq 0$ and $x^h y^{m(0)-l} \neq 0$.

We will now show that, for $k = 0, \cdots, h$, if $i \geqq k$ and $x^i y^j \neq 0$ then $(i, j) \leqq (k, m(k))$. Suppose this has been shown for $0, \cdots, k - 1$. Since $(i + 1, m(i + 1)) \prec (i, m(i))$ for $i < k$, we see that $m(0) \geqq m(i) + 2i$. From $x^h y^{m(0)-l} \neq 0$ and $h < k - 1$ we have

$$(h, m(0) - l) \prec (k - 1, m(k - 1)) .$$

Therefore $h + m(0) - l < k - 1 + m(k - 1)$ and $l - h \geqq k$. Now let $(u, v)$ be maximum such that $u \geqq k$ and $x^u y^v \neq 0$. Since $x^{p-h} y^l \neq 0$ and $p - h \geqq l - h \geqq k$ it follows that $u + v \geqq p - h + l \geqq p + k$. If $v = 0$ then $u = p$ and $k = 0$. Since for $k = 0$ our result is established, we consider $v > 0$. If $u > k$ then the set $\mathscr{A} = \{(i, j) \in \mathscr{C} : k \leqq i \leqq u, 0 \leqq j \leqq v\}$ contains $(u - k + 1)(v + 1) \geqq 2(u - k + v) \geqq 2p$ elements. Thus there is a nontrivial relation among the $x^i y^j$, $(i, j) \in \mathscr{A}$. As before, let $(s, t)$ be minimum such that the coefficient, $a_{st}$, of $x^r y^t$ is nonzero. On multiplying the relation by $x^{u-s} y^{v-t}$ we obtain $0 = a_{st} x^u y^v$ which is contradictory. Therefore $u = k$ and $v = m(k)$. By the

definition of $(u, v)$, if $i \geq u = k$ and $x^i y^j \neq 0$ then $(i, j) \prec (k, m(k))$.

We now have the inequality, $m(0) \geq 2k + m(k)$, for $k = 0, \cdots, h$. Since $x^h y^{m(0)-l} \neq 0$, $m(h) \geq m(0) - l$. That is $l \geq 2h$.

Let $bh + c = p$ where $0 \leq c < h$. Returning to equation (1) we obtain:

$$0 \neq a_h^b x^p = x^c (\Sigma_i a_i x^i)^b = x^c (-\Sigma_i b_i y^i)^b = x^c y^{bl} Y, \text{ where } Y \text{ is a poly-}$$

nominal in $y$.

Hence $x^c y^{bl} \neq 0$. This implies $m(0) - 2c \geq m(c) \geq bl \geq 2bh$. Therefore $m(0) \geq 2p$ and $y, \cdots, y^{2p}$ are independent. This is a contradiction and the lemma is established.

Next we show that if $m + n = p$ and $n \neq p$ then $x^m y^n = c_n x^p$ where $c_n \in F$. Suppose this holds for the powers of $y$ being $0, \cdots, n - 1$. If $x^m y^n = 0$ then the result is established. Thus suppose $x^m y^n \neq 0$. There are $(m + 1)(n + 1) \geq 2p$ monomials of the form $x^p$ or $x^i y^j$, $i \leq m, j \leq n$. Thus there is a nontrivial relation

$$\sum a_{ij} x^i y^j + ax^p = 0 .$$

Let $(s, t)$ be minimum such that the coefficient of $x^s y^t$ is nonzero. By multiplying the relation by $x^{m-s} y^{n-t}$ we obtain:

$$
\begin{aligned}
0 &= \sum_{i+j=s+t} a_{ij} x^{i+m-s} y^{j+n-t} + ax^{p+m-s} y^{n-t} \\
&= \sum_{\substack{i+j=s+t \\ (i,j) \neq (s,t)}} c_{j+n-t} a_{ij} x^p + a' x^p + a_{st} x^m y^n .
\end{aligned}
$$

Since $x^p$ is in the annihilator of $J$, $x^{p+m-s} y^{n-t}$ is $x^p$ or $0$. Therefore $x^m y^n = c_n x^p$.

Similarly we obtain: if $m + n = p$ and $m \neq p$, then $x^m y^n = b_m y^p$. Since $x^p$ and $y^p$ are independent, if $m + n = p$, $m \neq 0$, $p$ then $x^m y^n = 0$.

From equation (1) we may obtain, as before, $x^{p-h} y^l \neq 0$ and $x^h y^{p-l} \neq 0$ where $0 < h, l \leq p$. Assuming, without loss of generality, $h \geq l$ we have $h + (p - l) \geq p$ and by the lemma we have equality, that is, $h = l$. Since $x^h y^{p-h} \neq 0$ we have, by the above paragraph, $h = l = p$. Equation (1) becomes $0 = a_p x^p + b_p y^p$ for nonzero $a_p$ and $b_p$, a contradiction.    This completes the proof of Theorem 2.2.

## REFERENCE

1. I. Kaplansky, *Infinite Abelian Groups*, Ann Arbor 1954.

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