

Pacific Journal of Mathematics

QUASI REGULAR GROUPS OF FINITE COMMUTATIVE NILPOTENT ALGEBRAS

NORMAN HENRY EGGERT, JR.

QUASI REGULAR GROUPS OF FINITE COMMUTATIVE NILPOTENT ALGEBRAS

N. H. EGGERT

Let J be a finite commutative nilpotent algebra over a field F of characteristic p . J forms an abelian group under the "circle" operation, defined by $a \circ b = a + b + ab$. This group is called the quasi regular group of J .

Our main purpose is to investigate the relationship between the structure of J as an algebra, and the structure of its quasi regular group.

In particular, the structure of the quasi regular group is described in terms of certain subalgebras of J . These subalgebras are, for fixed j , the p^j powers of elements in J . They are denoted by $J^{(j)}$.

It is conjectured that the dimension of $J^{(j)}$ is greater than or equal to p times the dimension of $J^{(j+1)}$. If this is true, then Theorems 1.1 and 2.1 completely describe the possibilities for the quasi regular group of J . Paragraph 2 considers some special cases of the conjecture.

1. **The quasi regular group of J .** Let J be a finite commutative nilpotent algebra over a field F with p^n elements. Denote by $J^{(j)}$ the set of p^j th powers of elements in J , $j = 0, 1, \dots$. The $J^{(j)}$ form a descending chain of subalgebras of J . If t is the minimum exponent such that $x^{p^t} = 0$ for all $x \in J$ then $J^{(t-1)} \neq (0)$ and $J^{(t)} = (0)$. The constant t will be called the height of J . Let the dimension of $J^{(j)}$ be r_j and set $s_h = r_{h-1} + r_{h+1} - 2r_h$, $h = 1, \dots, t$.

We denote by $G(p, u; s_1, \dots, s_t)$ the group which is the direct sum of us_h , $h = 1, \dots, t$, copies of the cyclic group of order p^h .

THEOREM 1.1. *The quasi regular group of J is isomorphic to $G(p, u; s_1, \dots, s_t)$.*

Proof. Since the p th power of $x \in J$ with respect to the operation "o" is x^p , the number of cyclic summands of order greater than p^h is the dimension of the quotient group $J^{(h)}/J^{(h+1)}$ over the integers modulo p , that is $u(r_h - r_{h+1})$ [1, page 27]. Hence the number of cyclic summands of order p^h in the quasi regular group J is $u(r_{h-1} + r_{h+1} - 2r_h)$, $h = 1, \dots, t$.

2. **The possibilities for the quasi regular group of J .** Given certain p -groups, finite commutative nilpotent algebras can be con-

structed with these groups as their quasi regular groups.

THEOREM 2.1. *Let a_i be arbitrary nonnegative integers for $i = 1, \dots, t, a_i \neq 0$. Then there exists a finite commutative nilpotent algebra J over a field F of order p^u where:*

(i) $r_i = 0$ and $r_{i-1} = pr_i + a_i, i = 1, \dots, t$.

(ii) *the quasi regular group of J is $G(p, u; s_1, \dots, s_t)$ where $s_h = r_{h-1} + r_{h+1} - 2r_h$.*

Proof. Let J_j be the Jacobson radical of $F[X]/(X^n)$, where $n = p^{j-1} + 1$. If $x = X + (X^n)$ then a basis for J_j over F is $\{x, x^2, \dots, x^{n-1}\}$. Thus the dimension of $J_j^{(i)}$ is p^{j-i-1} for $i < j$. Let J be the direct sum of a_j copies of J_j for $j = 1, \dots, t$. Then $r_i = \dim J^{(i)} = \sum_{j=i+1}^t a_j p^{j-i-1}, i < t, r_t = \dim J^{(t)} = 0$. A simple calculation gives $r_{i-1} - pr_i = a_i$. By using Theorem 1.1, the proof is complete.

The author conjectures that the converse of the above theorem is also true, that is:

(C) If J is a finite commutative nilpotent algebra over F then $\dim J^{(i-1)} - p \dim J^{(i)} = r_{i-1} - pr_i \geq 0$.

This is immediate for algebras of height one, height two and $\dim J^{(1)} = 1$, and height two and $p = 2$. The following theorem establishes (C) for algebras of height two and $\dim J^{(1)} = 2$.

THEOREM 2.2. *Let J be a commutative nilpotent algebra over a perfect field F of characteristic p . Let x, y be elements of J and suppose x^p and y^p are linearly independent over F . Then the dimension of J is greater than or equal to $2p$.*

Proof. Suppose the theorem is false. That is, assume there is a finite commutative nilpotent algebra J over F and:

(i) $x, y \in J$ and x^p, y^p are independent over F ,

(ii) $\dim J < 2p$.

We assume J is an algebra of least dimension over F which satisfies (i) and (ii). It then follows that:

(iii) J is generated by x and y , and

(iv) If I is an ideal of J and an algebra over F then $I = (0)$ or for some $a, b \in F, 0 \neq ax^p + by^p \in I$.

If (iv) were false then J/I would satisfy (i) and (ii) and the dimension of J/I would be less than the dimension of J .

We may assume x^p is in the annihilator of J . This follows since, by (iv), there are elements a, b in F where $ax^p + by^p \neq 0$ is in the annihilator. By replacing x by $x' = a'x + b'y$, where $a'p = a$ and $b'p = b$, conditions (i) through (iv) hold and x'^p is in the annihilator.

Let \mathcal{S} be the cartesian product of the nonnegative integers with

themselves less $(0, 0)$. Let the total ordering $<$ be defined in \mathcal{E} by: $(s, t) < (i, j)$ if $s + t < i + j$ or $s + t = i + j$ and $s < i$.

LEMMA. *If $x^i y^j \neq 0$ then $i + j \leq p$.*

Proof. Let $(n, m(0))$ be the maximum element in \mathcal{E} , with respect to $<$, such that $x^n y^{m(0)} \neq 0$. Suppose that $n + m(0) > p$.

Since x^p is in the annihilator of J , $n \leq p$ and $m(0) > 0$, thus if $n > 0$ then $\mathcal{A} = \{(i, j) \in \mathcal{E} : i \leq n, \text{ and } j \leq m(0)\}$ has more than $2p$ elements. The monomials $x^i y^j, (i, j) \in \mathcal{A}$, are dependent, thus a nontrivial relation.

$$\sum a_{i,j} x^i y^j = z = 0, (i, j) \in \mathcal{A}$$

exists. Let (s, t) be minimum such that $a_{st} \neq 0$. Consider

$$0 = z x^{n-s} y^{m(0)-t}.$$

For $(s, t) < (i, j)$ it follows that $(n, m(0)) < (i + n - s, j + m(0) - t)$. By the definition of $(n, m(0))$ we obtain $0 = a_{st} x^n y^{m(0)}$. This is a contradiction; thus $n = 0$.

Now define $m(i)$ to be the maximum integer such that $x^i y^{m(i)} \neq 0, i = 0, \dots, p$. Since $x, \dots, x^p, y, \dots, y^p$ are dependent, let

$$(1) \quad z = \sum_{i=h}^p a_i x^i + \sum_{i=l}^p b_i y^i = 0,$$

where $a_h \neq 0$ and $b_l \neq 0$. There is at least one nonzero a_j since y, \dots, y^p are independent. Likewise at least one b_i is nonzero. Thus considering $x^{p-h} z$ and $y^{m(0)-l} z$ we find $x^{p-h} y^l \neq 0$ and $x^h y^{m(0)-l} \neq 0$.

We will now show that, for $k = 0, \dots, h$, if $i \geq k$ and $x^i y^j \neq 0$ then $(i, j) \leq (k, m(k))$. Suppose this has been shown for $0, \dots, k - 1$. Since $(i + 1, m(i + 1)) < (i, m(i))$ for $i < k$, we see that $m(0) \geq m(i) + 2i$. From $x^h y^{m(0)-l} \neq 0$ and $h < k - 1$ we have

$$(h, m(0) - l) < (k - 1, m(k - 1)).$$

Therefore $h + m(0) - l < k - 1 + m(k - 1)$ and $l - h \geq k$. Now let (u, v) be maximum such that $u \geq k$ and $x^u y^v \neq 0$. Since $x^{p-h} y^l \neq 0$ and $p - h \geq l - h \geq k$ it follows that $u + v \geq p - h + l \geq p + k$. If $v = 0$ then $u = p$ and $k = 0$. Since for $k = 0$ our result is established, we consider $v > 0$. If $u > k$ then the set $\mathcal{A} = \{(i, j) \in \mathcal{E} : k \leq i \leq u, 0 \leq j \leq v\}$ contains $(u - k + 1)(v + 1) \geq 2(u - k + v) \geq 2p$ elements. Thus there is a nontrivial relation among the $x^i y^j, (i, j) \in \mathcal{A}$. As before, let (s, t) be minimum such that the coefficient, a_{st} , of $x^s y^t$ is nonzero. On multiplying the relation by $x^{u-s} y^{v-t}$ we obtain $0 = a_{st} x^u y^v$ which is contradictory. Therefore $u = k$ and $v = m(k)$. By the

definition of (u, v) , if $i \geq u = k$ and $x^i y^j \neq 0$ then $(i, j) < (k, m(k))$.

We now have the inequality, $m(0) \geq 2k + m(k)$, for $k = 0, \dots, h$. Since $x^h y^{m(0)-l} \neq 0$, $m(h) \geq m(0) - l$. That is $l \geq 2h$.

Let $bh + c = p$ where $0 \leq c < h$. Returning to equation (1) we obtain:

$$0 \neq a_i^h x^p = x^c (\sum_i a_i x^i)^b = x^c (-\sum_i b_i y^i)^b = x^c y^{bl} Y, \text{ where } Y \text{ is a polynomial in } y.$$

Hence $x^c y^{bl} \neq 0$. This implies $m(0) - 2c \geq m(c) \geq bl \geq 2bh$. Therefore $m(0) \geq 2p$ and y, \dots, y^{2p} are independent. This is a contradiction and the lemma is established.

Next we show that if $m + n = p$ and $n \neq p$ then $x^m y^n = c_n x^p$ where $c_n \in F$. Suppose this holds for the powers of y being $0, \dots, n - 1$. If $x^m y^n = 0$ then the result is established. Thus suppose $x^m y^n \neq 0$. There are $(m + 1)(n + 1) \geq 2p$ monomials of the form x^p or $x^i y^j, i \leq m, j \leq n$. Thus there is a nontrivial relation

$$\sum a_{ij} x^i y^j + ax^p = 0.$$

Let (s, t) be minimum such that the coefficient of $x^s y^t$ is nonzero. By multiplying the relation by $x^{m-s} y^{n-t}$ we obtain:

$$\begin{aligned} 0 &= \sum_{i+j=s+t} a_{ij} x^{i+m-s} y^{j+n-t} + ax^{p+m-s} y^{n-t} \\ &= \sum_{\substack{i+j=s+t \\ (i, j) \neq (s, t)}} c_{j+n-t} a_{ij} x^p + a' x^p + a_{st} x^m y^n. \end{aligned}$$

Since x^p is in the annihilator of $J, x^{p+m-s} y^{n-t}$ is x^p or 0 . Therefore $x^m y^n = c_n x^p$.

Similarly we obtain: if $m + n = p$ and $m \neq p$, then $x^m y^n = b_n y^p$. Since x^p and y^p are independent, if $m + n = p, m \neq 0, p$ then $x^m y^n = 0$.

From equation (1) we may obtain, as before, $x^{p-h} y^l \neq 0$ and $x^h y^{p-l} \neq 0$ where $0 < h, l \leq p$. Assuming, without loss of generality, $h \geq l$ we have $h + (p - l) \geq p$ and by the lemma we have equality, that is, $h = l$. Since $x^h y^{p-h} \neq 0$ we have, by the above paragraph, $h = l = p$. Equation (1) becomes $0 = a_p x^p + b_p y^p$ for nonzero a_p and b_p , a contradiction. This completes the proof of Theorem 2.2.

REFERENCE

1. I. Kaplansky, *Infinite Abelian Groups*, Ann Arbor 1954.

Received January 26, 1968. This research was in part supported by the National Science Foundation, grant GP-1923.

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON
Stanford University
Stanford, California 94305

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

C. R. HOBBY
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLE

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

* * *

AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

E. M. Alfsen and B. Hirsberg, <i>On dominated extensions in linear subspaces of $\mathcal{C}_C(X)$</i>	567
Joby Milo Anthony, <i>Topologies for quotient fields of commutative integral domains</i>	585
V. Balakrishnan, G. Sankaranarayanan and C. Suyambulingom, <i>Ordered cycle lengths in a random permutation</i>	603
Victor Allen Belfi, <i>Nontangential homotopy equivalences</i>	615
Jane Maxwell Day, <i>Compact semigroups with square roots</i>	623
Norman Henry Eggert, Jr., <i>Quasi regular groups of finite commutative nilpotent algebras</i>	631
Paul Erdős and Ernst Gabor Straus, <i>Some number theoretic results</i>	635
George Rudolph Gordh, Jr., <i>Monotone decompositions of irreducible Hausdorff continua</i>	647
Darald Joe Hartfiel, <i>The matrix equation $AXB = X$</i>	659
James Howard Hedlund, <i>Expansive automorphisms of Banach spaces. II</i>	671
I. Martin (Irving) Isaacs, <i>The p-parts of character degrees in p-solvable groups</i>	677
Donald Glen Johnson, <i>Rings of quotients of Φ-algebras</i>	693
Norman Lloyd Johnson, <i>Transition planes constructed from semifield planes</i>	701
Anne Bramble Searle Koehler, <i>Quasi-projective and quasi-injective modules</i>	713
James J. Kuzmanovich, <i>Completions of Dedekind prime rings as second endomorphism rings</i>	721
B. T. Y. Kwee, <i>On generalized translated quasi-Cesàro summability</i>	731
Yves A. Lequain, <i>Differential simplicity and complete integral closure</i>	741
Mordechaj Lewin, <i>On nonnegative matrices</i>	753
Kevin Mor McCrimmon, <i>Speciality of quadratic Jordan algebras</i>	761
Hussain Sayid Nur, <i>Singular perturbations of differential equations in abstract spaces</i>	775
D. K. Oates, <i>A non-compact Krein-Milman theorem</i>	781
Lavon Barry Page, <i>Operators that commute with a unilateral shift on an invariant subspace</i>	787
Helga Schirmer, <i>Properties of fixed point sets on dendrites</i>	795
Saharon Shelah, <i>On the number of non-almost isomorphic models of T in a power</i>	811
Robert Moffatt Stephenson Jr., <i>Minimal first countable Hausdorff spaces</i>	819
Masamichi Takesaki, <i>The quotient algebra of a finite von Neumann algebra</i>	827
Benjamin Baxter Wells, Jr., <i>Interpolation in $C(\Omega)$</i>	833