QUASI REGULAR GROUPS OF FINITE COMMUTATIVE NILPOTENT ALGEBRAS

Norman Henry Eggert, Jr.
Let $J$ be a finite commutative nilpotent algebra over a field $F$ of characteristic $p$. $J$ forms an abelian group under the "circle" operation, defined by $a \circ b = a + b + ab$. This group is called the quasi regular group of $J$.

Our main purpose is to investigate the relationship between the structure of $J$ as an algebra, and the structure of its quasi regular group.

In particular, the structure of the quasi regular group is described in terms of certain subalgebras of $J$. These subalgebras are, for fixed $j$, the $p^j$ powers of elements in $J$. They are denoted by $J^{(j)}$.

It is conjectured that the dimension of $J^{(j)}$ is greater than or equal to $p$ times the dimension of $J^{(j+1)}$. If this is true, then Theorems 1.1 and 2.1 completely describe the possibilities for the quasi regular group of $J$. Paragraph 2 considers some special cases of the conjecture.

1. The quasi regular group of $J$. Let $J$ be a finite commutative nilpotent algebra over a field $F$ with $p^k$ elements. Denote by $J^{(j)}$ the set of $p^j$th powers of elements in $J$, $j = 0, 1, \ldots$. The $J^{(j)}$ form a descending chain of subalgebras of $J$. If $t$ is the minimum exponent such that $x^{p^t} = 0$ for all $x \in J$ then $J^{(t-1)} \neq (0)$ and $J^{(t)} = (0)$. The constant $t$ will be called the height of $J$. Let the dimension of $J^{(j)}$ be $r_j$ and set $s_h = r_{h-1} + r_{h+1} - 2r_h$, $h = 1, \ldots, t$.

We denote by $G(p, u; s_1, \ldots, s_t)$ the group which is the direct sum of $us_h$, $h = 1, \ldots, t$, copies of the cyclic group of order $p^h$.

**Theorem 1.1.** The quasi regular group of $J$ is isomorphic to $G(p, u; s_1, \ldots, s_t)$.

**Proof.** Since the $p^j$th power of $x \in J$ with respect to the operation "o" is $x^{p^j}$, the number of cyclic summands of order greater than $p^h$ is the dimension of the quotient group $J^{(h)}/J^{(h+1)}$ over the integers modulo $p$, that is $u(r_h - r_{h+1})$ [1, page 27]. Hence the number of cyclic summands of order $p^h$ in the quasi regular group $J$ is $u(r_{h-1} + r_{h+1} - 2r_h)$, $h = 1, \ldots, t$.

2. The possibilities for the quasi regular group of $J$. Given certain $p$-groups, finite commutative nilpotent algebras can be con-
structured with these groups as their quasi regular groups.

**Theorem 2.1.** Let $a_i$ be arbitrary nonnegative integers for $i = 1, \cdots, t, a_t \neq 0$. Then there exists a finite commutative nilpotent algebra $J$ over a field $F$ of order $p^w$ where:

1. $r_i = 0$ and $r_{i-1} = pr_i + a_i, i = 1, \cdots, t$.
2. the quasi regular group of $J$ is $G(p, u; s_1, \cdots, s_t)$ where $s_h = r_{h-1} + r_{h+1} - 2r_h$.

**Proof.** Let $J_j$ be the Jacobson radical of $F[X]/(X^n)$, where $n = p^{j-1} + 1$. If $x = X + (X^n)$ then a basis for $J_j$ over $F$ is \{x, x^2, \cdots, x^{n-1}\}. Thus the dimension of $J_j^{(i)}$ is $p^{j-i-1}$ for $i < j$. Let $J$ be the direct sum of $a_j$ copies of $J_j$ for $j = 1, \cdots, t$. Then $r_i = \dim J^{(i)} = \sum_{j=i+1}^t a_j p^{j-i-1}, i < t, r_i = \dim J^{(i)} = 0$. A simple calculation gives $r_{i-1} - pr_i = a_i$. By using Theorem 1.1, the proof is complete.

The author conjectures that the converse of the above theorem is also true, that is:

(C) If $J$ is a finite commutative nilpotent algebra over $F$ then $\dim J^{(i-1)} - p \dim J^{(i)} = r_{i-1} - pr_i \geq 0$.

This is immediate for algebras of height one, height two and $\dim J^{(1)} = 1$, and height two and $p = 2$. The following theorem establishes (C) for algebras of height two and $\dim J^{(1)} = 2$.

**Theorem 2.2.** Let $J$ be a commutative nilpotent algebra over a perfect field $F$ of characteristic $p$. Let $x, y$ be elements of $J$ and suppose $x^p$ and $y^p$ are linearly independent over $F$. Then the dimension of $J$ is greater than or equal to $2p$.

**Proof.** Suppose the theorem is false. That is, assume there is a finite commutative nilpotent algebra $J$ over $F$ and:

1. $x, y \in J$ and $x^p, y^p$ are independent over $F$,
2. $\dim J < 2p$.

We assume $J$ is an algebra of least dimension over $F$ which satisfies (i) and (ii). It then follows that:

(iii) $J$ is generated by $x$ and $y$, and
(iv) If $I$ is an ideal of $J$ and an algebra over $F$ then $I = (0)$ or for some $a, b \in F$, $0 \neq ax^p + by^p \in I$.

If (iv) were false then $J/I$ would satisfy (i) and (ii) and the dimension of $J/I$ would be less than the dimension of $J$.

We may assume $x^p$ is in the annihilator of $J$. This follows since, by (iv), there are elements $a, b$ in $F$ where $ax^p + by^p \neq 0$ is in the annihilator. By replacing $x$ by $x' = a'x + b'y$, where $a'^p = a$ and $b'^p = b$, conditions (i) through (iv) hold and $x'^p$ is in the annihilator.

Let $\mathbb{N}$ be the cartesian product of the nonnegative integers with...
themselves less \((0, 0)\). Let the total ordering \(<\) be defined in \(\mathbb{C}\) by: 
\((s, t) < (i, j)\) if \(s + t < i + j\) or \(s + t = i + j\) and \(s < i\).

**Lemma.** If \(x^i y^j \neq 0\) then \(i + j \leq p\).

**Proof.** Let \((n, m(0))\) be the maximum element in \(\mathbb{C}\), with respect to \(<\), such that \(x^n y^{m(0)} \neq 0\). Suppose that \(n + m(0) > p\).

Since \(x^p\) is in the annihilator of \(J\), \(n \leq p\) and \(m(0) > 0\), thus if \(n > 0\) then \(\mathcal{A} = \{(i, j) \in \mathbb{C} : i \leq n\} \cup \{j \leq m(0)\}\) has more than \(2p\) elements. The monomials \(x^i y^j\), \((i, j) \in \mathcal{A}\), are dependent, thus a nontrivial relation.

\[\sum a_{ij} x^i y^j = 0, \quad (i, j) \in \mathcal{A}\]

exists. Let \((s, t)\) be minimum such that \(a_{st} \neq 0\). Consider

\[0 = z x^{n-s} y^{m(0)-t}\]

For \((s, t) < (i, j)\) it follows that \((n, m(0)) < (i + n - s, j + m(0) - t)\). By the definition of \((n, m(0))\) we obtain \(0 = a_{st} x^p y^{m(0)}\). This is a contradiction; thus \(n = 0\).

Now define \(m(i)\) to be the maximum integer such that \(x^i y^{m(i)} \neq 0\), \(i = 0, \ldots, p\). Since \(x, \ldots, x^p, y, \ldots, y^p\) are dependent, let

\[z = \sum_{i=0}^{p} a_i x^i + \sum_{j=1}^{p} b_j y^j = 0,\]

where \(a_i \neq 0\) and \(b_i \neq 0\). There is at least one nonzero \(a_j\) since \(y, \ldots, y^p\) are independent. Likewise at least one \(b_i\) is nonzero. Thus considering \(x^{p-k} z\) and \(y^{m(0)-l} z\) we find \(x^{p-k} y^l \neq 0\) and \(x^k y^{m(0)-l} \neq 0\).

We will now show that, for \(k = 0, \ldots, h\), if \(i \geq k\) and \(x^i y^j \neq 0\) then \((i, j) \leq (k, m(k))\). Suppose this has been shown for \(0, \ldots, k - 1\). Since \((i + 1, m(i + 1)) < (i, m(i))\) for \(i < k\), we see that \(m(0) \geq m(i) + 2i\).

From \(x^k y^{m(0)-l} \neq 0\) and \(h < k - 1\) we have

\[(h, m(0) - l) < (k - 1, m(k - 1))\cdot\]

Therefore \(h + m(0) - l < k - 1 + m(k - 1)\) and \(l - h \geq k\). Now let \((u, v)\) be maximum such that \(u \geq k\) and \(x^u y^v \neq 0\). Since \(x^{p-k} y^l \neq 0\) and \(p - h \geq l - h \geq k\) it follows that \(u + v \geq p - h + l \geq p + k\). If \(v = 0\) then \(u = p\) and \(k = 0\). Since for \(k = 0\) our result is established, we consider \(v > 0\). If \(u > k\) then the set \(\mathcal{A} = \{(i, j) \in \mathbb{C} : k \leq i \leq u, 0 \leq j \leq v\}\) contains \((u - k + 1)(v + 1) \geq 2(u - k + v) \geq 2p\) elements. Thus there is a nontrivial relation among the \(x^i y^j\), \((i, j) \in \mathcal{A}\). As before, let \((s, t)\) be minimum such that the coefficient, \(a_{st}\), of \(x^s y^t\) is nonzero. On multiplying the relation by \(x^{u-k} y^{v-l}\) we obtain \(0 = a_{st} x^s y^t\) which is contradictory. Therefore \(u = k\) and \(v = m(k)\). By the
definition of \((u, v)\), if \(i \geq u = k\) and \(x^iy^l \neq 0\) then \((i, j) \prec (k, m(k))\).

We now have the inequality, \(m(0) \geq 2k + m(k)\), for \(k = 0, \ldots, h\). Since \(x^hy^{m(0) - l} \neq 0\), \(m(h) \geq m(0) - l\). That is \(l \geq 2h\).

Let \(bh + c = p\) where \(0 \leq c < h\). Returning to equation (1) we obtain:

\[
0 = a_n x^p = x^c (\Sigma_i a_i x^i)^b = x^c (- \Sigma_i b_i y^i)^b = x^c y^b Y, \text{ where } Y \text{ is a polynomial in } y.
\]

Hence \(x^c y^d \neq 0\). This implies \(m(0) - 2c \geq m(c) \geq bl \geq 2bh\). Therefore \(m(0) \geq 2p\) and \(y, \ldots, y^p\) are independent. This is a contradiction and the lemma is established.

Next we show that if \(m + n = p\) and \(n \neq p\) then \(x^m y^n = c_n x^p\) where \(c_n \in F\). Suppose this holds for the powers of \(y\) being \(0, \ldots, n - 1\). If \(x^m y^n = 0\) then the result is established. Thus suppose \(x^m y^n \neq 0\). There are \((m + 1)(n + 1) \geq 2p\) monomials of the form \(x^i y^j\), \(i \leq m, j \leq n\). Thus there is a nontrivial relation

\[
\sum a_{i, j} x^i y^j + a x^p = 0.
\]

Let \((s, t)\) be minimum such that the coefficient of \(x^s y^t\) is nonzero. By multiplying the relation by \(x^{-s} y^{-t}\) we obtain:

\[
0 = \sum_{i + j = s + t} a_{i, j} x^{i + m - s} y^{j + n - t} + a x^{p + m - s} y^{n - t}.
\]

Since \(x^p\) is in the annihilator of \(J\), \(x^{p + m - s} y^{n - t}\) is \(x^p\) or \(0\). Therefore \(x^m y^n = c_n x^p\).

Similarly we obtain: if \(m + n = p\) and \(m \neq p\), then \(x^m y^n = b_m y^p\).

From equation (1) we may obtain, as before, \(x^{p - h} y^l \neq 0\) and \(x^hy^{p - l} \neq 0\) where \(0 < h, l \leq p\). Assuming, without loss of generality, \(h \geq l\) we have \(h + (p - l) \geq p\) and by the lemma we have equality, that is, \(h = l\). Since \(x^h y^{p - h} \neq 0\) we have, by the above paragraph, \(h = l = p\). Equation (1) becomes \(0 = a_p x^p + b_p y^p\) for nonzero \(a_p\) and \(b_p\), a contradiction. This completes the proof of Theorem 2.2.

**Reference**


Received January 26, 1968. This research was in part supported by the National Science Foundation, grant GP-1923.

**Montana State University**
PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON
Stanford University
Stanford, California 94305

C. R. HOBBY
University of Washington
Seattle, Washington 98105

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH  B. H. NEUMANN  F. WOLFE  K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  STANFORD UNIVERSITY
CALIFORNIA INSTITUTE OF TECHNOLOGY  UNIVERSITY OF TOKYO
UNIVERSITY OF CALIFORNIA  UNIVERSITY OF UTAH
MONTANA STATE UNIVERSITY  WASHINGTON STATE UNIVERSITY
UNIVERSITY OF NEVADA  UNIVERSITY OF WASHINGTON
NEW MEXICO STATE UNIVERSITY  *
OREGON STATE UNIVERSITY  *
UNIVERSITY OF OREGON  *
OSAKA UNIVERSITY  *
UNIVERSITY OF SOUTHERN CALIFORNIA  *

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the Pacific Journal of Mathematics should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.
E. M. Alfsen and B. Hirsberg, *On dominated extensions in linear subspaces of \( C(X) \) ......................................................... 567

Joby Milo Anthony, *Topologies for quotient fields of commutative integral domains* .................................................. 585

V. Balakrishnan, G. Sankaranarayanan and C. Suyambulingom, *Ordered cycle lengths in a random permutation* ..................... 603

Victor Allen Belfi, *Nontangential homotopy equivalences* .................. 615

Jane Maxwell Day, *Compact semigroups with square roots* .................. 623

Norman Henry Eggert, Jr., *Quasi regular groups of finite commutative nilpotent algebras* ......................................................... 631

Paul Erdős and Ernst Gabor Straus, *Some number theoretic results* .... 635

George Rudolph Gordh, Jr., *Monotone decompositions of irreducible Hausdorff continua* ..................................................... 647

Darald Joe Hartfiel, *The matrix equation \( AXB = X \) .................. 659

James Howard Hedlund, *Expansive automorphisms of Banach spaces. II* .... 671

I. Martin (Irving) Isaacs, *The \( p \)-parts of character degrees in \( p \)-solvable groups* ..................................................... 677

Donald Glen Johnson, *Rings of quotients of \( \Phi \)-algebras* .................. 693

Norman Lloyd Johnson, *Transition planes constructed from semifield planes* ..................................................... 701

Anne Bramble Searle Koehler, *Quasi-projective and quasi-injective modules* ..................................................... 713

James J. Kuzmanovich, *Completions of Dedekind prime rings as second endomorphism rings* ..................................................... 721

B. T. Y. Kwee, *On generalized translated quasi-Cesàro summability* ...... 731

Yves A. Lequain, *Differential simplicity and complete integral closure* ...... 741

Mordechai Lewin, *On nonnegative matrices* .................................... 753

Kevin Mor McCrimmon, *Speciality of quadratic Jordan algebras* .......... 761

Hussain Sayid Nur, *Singular perturbations of differential equations in abstract spaces* ..................................................... 775


Lavon Barry Page, *Operators that commute with a unilateral shift on an invariant subspace* ..................................................... 787

Helga Schirmer, *Properties of fixed point sets on dendrites* .................. 795

Saharon Shelah, *On the number of non-almost isomorphic models of \( T \) in a power* ..................................................... 811

Robert Moffatt Stephenson Jr., *Minimal first countable Hausdorff spaces* .... 819

Masamichi Takesaki, *The quotient algebra of a finite von Neumann algebra* ..................................................... 827

Benjamin Baxter Wells, Jr., *Interpolation in \( C(\Omega) \) .................. 833