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RINGS OF QUOTIENTS OF Φ -ALGEBRAS

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RINGS OF QUOTIENTS OF Φ -ALGEBRAS

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Let \mathcal{X} be a completely regular (Hausdorff) space. Fine, Gillman, and Lambek have studied the (generalized) rings of quotients of $C(\mathcal{X}) = C(\mathcal{X}; \mathbf{R})$, with particular emphasis on the maximal ring of quotients, $Q(\mathcal{X})$. In this note, we start with a characterization of $Q(\mathcal{X})$ that differs only slightly from one of theirs. This characterization is easily altered to fit more general circumstances, and so serves to obtain some results on non-maximal rings of quotients of $C(\mathcal{X})$, and to generalize these results to the class of Φ -algebras.

We consider only commutative rings with unit. Let A be one such, and recall that the (unitary) over-ring B of A is called a *rational extension* or *ring of quotients* of A if it satisfies the following condition: given $b \in B$, for every $0 \neq b' \in B$ there is $a \in A$ with $ba \in A$ and $b'a \neq 0$. A ring without proper rational extensions is said to be *rationally complete*. For the rings to be considered here (all are semi-prime), the condition above can be replaced by the simpler condition: for $0 \neq b \in B$, there exists $a \in A$ such that $0 \neq ba \in A$ ([1], p. 5). Accordingly, we make the following

DEFINITION. If B is an over-ring of A and $0 \neq b \in B$, say that b is *rational over* A if there is $a \in A$ with $0 \neq ba \in A$.

Let $m\beta\mathcal{X}$ denote the minimal projective extension of $\beta\mathcal{X}$ and $\tau: m\beta\mathcal{X} \rightarrow \beta\mathcal{X}$ the minimal perfect map ([2]). In [1], it is shown that $Q(\mathcal{X})$ is a dense, point-separating subalgebra of $D(m\beta\mathcal{X})$, the set of all continuous maps from $m\beta\mathcal{X}$ into the two-point compactification of the real line which are real-valued on a dense subset of $m\beta\mathcal{X}$ (see, also, [3]). Since $Q(\mathcal{X})$ contains every ring of quotients of $C(\mathcal{X})$, this leads to

PROPOSITION 1. *If B is any ring of quotients of $C(\mathcal{X})$, then there exist a compact (Hausdorff) space \mathcal{Y} and minimal perfect maps α and γ such that B is a point-separating subalgebra of $D(\mathcal{Y})$ and the following diagram commutes:*

$$\begin{array}{ccc}
 m\beta\mathcal{X} & \xrightarrow{\alpha} & \mathcal{Y} \\
 & \searrow \tau & \downarrow \gamma \\
 & & \beta\mathcal{X} .
 \end{array}$$

\mathcal{Y} is the obvious identification space, and the proof consists of a routine argument to show that the quotient map α is closed, whence \mathcal{Y} is Hausdorff. Since $C(\mathcal{X}) \subseteq B$, the existence of γ follows immediately. (Note that, although $D(m\beta\mathcal{X})$ is an algebra, $D(\mathcal{Y})$ for other spaces \mathcal{Y} is, in general, only a partial algebra.)

For our purposes, it is convenient to view $C(\mathcal{X})$ as a subalgebra of $D(\beta\mathcal{X})$. This allows us to decree that all spaces are compact (Hausdorff).

Let us say that any space \mathcal{Y} that is situated in a commutative diagram of the form

$$\begin{array}{ccc}
 m\mathcal{X} & \xrightarrow{\alpha} & \mathcal{Y} \\
 & \searrow \tau & \downarrow \gamma \\
 & & \mathcal{X}
 \end{array}
 ,$$

where all maps are minimal perfect, is *near to* \mathcal{X} . (Of course, the existence of γ automatically guarantees the existence of α .) Note that we have already adopted the convention of identifying $f \in D(\mathcal{X})$ with its image $f \circ \gamma$ in $D(\mathcal{Y})$ whenever convenient. With this convention, if A is a subalgebra of $D(\mathcal{Y})$ and $f \in D(\mathcal{Y})$ then we may consider f as an element of an over-ring of $A - D(m\mathcal{X}) -$, even if there is no subalgebra of $D(\mathcal{Y})$ containing both A and f .

Now let A be a Φ -algebra that is closed under bounded inversion; i.e., an archimedean lattice ordered algebra with a multiplicative identity that is a weak order unit, in which $1/a \in A$ whenever $1 \leq a \in A$. Let $\mathcal{X} = \mathcal{M}(A)$, the space of maximal ideals of A with the hull-kernel topology. It is shown in [4] that A is (isomorphic with) a point-separating subalgebra of $D(\mathcal{X})$. If \mathcal{Y} is any space that is near to \mathcal{X} , let $A_{\mathcal{Y}} = \{f \in D(\mathcal{Y}) : \text{for each nonempty open set } \mathcal{U} \text{ in } \mathcal{Y}, \text{ there are a nonempty open set } \mathcal{V} \subseteq \mathcal{U} \text{ and } g \in A \text{ such that } f|_{\mathcal{V}} = g|_{\mathcal{V}}\}$. Note that $A_{\mathcal{Y}}$ is always a lattice. However, it need not be an algebra:

EXAMPLE. Let $\mathcal{X} = \mathcal{Y}$, the one-point compactification of the countable discrete space, and let $A = C(\mathcal{X})$. Then $A_{\mathcal{Y}} = D(\mathcal{Y})$, which is not an algebra.

REMARK. One readily shows that the open sets \mathcal{V} appearing in the definition of $A_{\mathcal{Y}}$ can always be shown to have the form $\gamma^{-1}[\mathcal{V}_1]$, where \mathcal{V}_1 is open in \mathcal{X} . It follows that

$$A_{\mathcal{Y}} = \{f \in D(\mathcal{Y}) : f \circ \alpha \in A_{m\mathcal{X}}\} .$$

PROPOSITION 2. (i) Every element of $A_{\mathcal{Y}}$ is rational over A^*

(and, hence, over A).

(ii) $A_{\mathcal{Z}}$ contains every rational extension of A and A^* in $D(\mathcal{Z})$.

Proof. (i) Let $0 \neq f \in A_{\mathcal{Z}}$, and let \mathcal{U} be a nonempty open set contained in $\text{coz } f$. Since $f \in A_{\mathcal{Z}}$, there exist a nonempty open set $\mathcal{V} = \gamma^{-1}[\mathcal{V}_1] \subseteq U$, where \mathcal{V}_1 is open in \mathcal{L} , and $h \in A^*$ such that $f|_{\mathcal{V}} = h|_{\mathcal{V}}$. Choose $0 \neq g \in A^*$ with $\overline{\text{coz } g} \subseteq \mathcal{V}_1$. Then $0 \neq fg = hg \in A^*$.

(ii) Let $f \in D(\mathcal{Z}) \setminus A_{\mathcal{Z}}$. Then, there is a nonempty open set \mathcal{U} such that f agrees with no member of A on any nonempty open subset of \mathcal{U} . Choose $g \in A^*$ with $\phi \neq \overline{\text{coz } g} \subseteq \mathcal{U}$.

There is no $h \in A$ with $hg \neq 0$ while $fh \in A$. For, such h would agree with a unit h_1 of A on some nonempty open subset \mathcal{V} of \mathcal{U} (since A is closed under bounded inversion), whence

$$f|_{\mathcal{V}} = (h/h_1)f|_{\mathcal{V}},$$

while $(1/h_1)hf \in A$, a contradiction. Thus, f is contained in no rational extension of A .

Although $A_{\mathcal{Z}}$ may contain many different rational extensions of A , it is not true that it is the union of such extensions, as is seen in the example preceding Proposition 2. However, in those spaces \mathcal{Z} for which $A_{\mathcal{Z}}$ is an algebra, $A_{\mathcal{Z}}$ is a ϕ -algebra and is the largest ring of quotients of A that "lives on" \mathcal{Z} . In particular, this happens when $D(\mathcal{Z})$ is an algebra (e.g., when \mathcal{Z} is basically disconnected or an F -space). Hence, $A_{m\mathcal{L}}$ is a ϕ -algebra, since $m\mathcal{L}$ is extremally disconnected, and we obtain the following generalizations of results in [1].

THEOREM 1. $A_{m\mathcal{L}}$ is rationally complete; thus, $A_{m\mathcal{L}} = \mathcal{Q}(A)$, the maximal ring of quotients of A .

THEOREM 2. $A_{m\mathcal{L}}$ is uniformly dense in $D(m\mathcal{L})$.

THEOREM 3 ([1]). $D(m\mathcal{L})$ is rationally complete.

The proofs of Theorems 1 and 3 are virtually identical, and are related to one found on p. 30 of [1]; we prove 1. To do so, we will employ the following characterization of rational completeness (see [1], p. 7).

The commutative ring B is rationally complete if and only if it satisfies: for any dense ideal I of B , every element of $\text{Hom}_B(I, B)$ is a multiplication by an element of B . (In the present setting, an ideal I of $A_{m\mathcal{L}}$ is dense if and only if $\cup\{\text{coz } f: f \in I\}$ is dense in $m\mathcal{L}$.)

Proof of Theorem 1. Let I be a dense ideal in A , and let

$\phi \in \text{Hom}_{A_{m\mathcal{L}}}(I, A_{m\mathcal{L}})$. By Zorn's lemma, choose a family $\{\mathcal{U}_\kappa: \kappa \in K\}$ of open sets in $m\mathcal{L}$ satisfying:

- (i) $\mathcal{U} = \bigcup \mathcal{U}_\kappa$ is dense in $m\mathcal{L}$;
- (ii) the \mathcal{U}_κ are pairwise disjoint;
- (iii) for each κ , there is $f_\kappa \in I$ such that f_κ is bounded away from zero on \mathcal{U}_κ and both f_κ and $\phi(f_\kappa)$ agree with members of A on \mathcal{U}_κ .

Let $f \in D(m\mathcal{L})$ satisfy

$$f \Big|_{\mathcal{U}_\kappa} = \frac{\phi(f_\kappa)}{f_\kappa} \Big|_{\mathcal{U}_\kappa}$$

for each $\kappa \in K$. This is possible, since $m\mathcal{L}$ is extremally disconnected, so $m\mathcal{L} = \beta\mathcal{U}$.

If $g \in I$ and $x \in \mathcal{U}_\kappa$, then

$$f(x)g(x) = \frac{\phi(f_\kappa)(x)}{f_\kappa(x)}g(x) = \frac{g\phi(f_\kappa)}{f_\kappa} \Big|_{\mathcal{U}_\kappa}(x) = \frac{f_\kappa\phi(g)}{f_\kappa} \Big|_{\mathcal{U}_\kappa}(x) = \phi(g)(x).$$

It follows that ϕ is multiplication by f . Clearly, $f \in A_{m\mathcal{L}}$, and the proof is complete.

Proof of Theorem 2. Let $f \in D(m\mathcal{L}), \varepsilon > 0$. By Zorn's lemma, choose a family $\{\mathcal{U}_\kappa: \kappa \in K\}$ of open sets in $m\mathcal{L}$ which satisfies:

- (i) $\mathcal{U} = \bigcup \mathcal{U}_\kappa$ is dense in $m\mathcal{L}$;
- (ii) the \mathcal{U}_κ are pairwise disjoint;
- (iii) for $x, y \in \mathcal{U}_\kappa, |f(x) - f(y)| < \varepsilon$ (in particular, f is real-valued on \mathcal{U}_κ).

For each $\kappa \in K$, choose $x_\kappa \in \mathcal{U}_\kappa$, and define $g: \mathcal{U} \rightarrow \mathbf{R}$ by

$$g(y) = f(x_\kappa) \quad \text{if } y \in \mathcal{U}_\kappa.$$

Since $m\mathcal{L} = \beta\mathcal{U}$, g can be extended to $\hat{g} \in D(m\mathcal{L})$. Clearly, $\hat{g} \in A_{m\mathcal{L}}$, and

$$|f - \hat{g}| \leq \varepsilon.$$

Now the analogue of Proposition 1 for Φ -algebras is routinely obtained.

In case $\mathcal{V} = m\mathcal{L}$ and $A = C(\mathcal{L})$ one readily translates the definition of $A_{\mathcal{V}}$ (using the fact that $m\mathcal{L}$ is extremally disconnected, and hence that every dense subspace is C^* -embedded) as follows:

$$A_{m\mathcal{L}} = \varinjlim \{C(\mathcal{S}): \mathcal{S} \text{ is a dense open subset of } \mathcal{L}\}.$$

Thus, the Fine-Gillman-Lambek result that this direct limit is $Q(\mathcal{L})$ follows from Theorem 1.

It is easily seen that any Φ -algebra A is a rational extension of its bounded subring A^* , and hence that $(A^*)_{\mathcal{V}} = A_{\mathcal{V}}$ for any space \mathcal{V} near to $\mathcal{M}(A)$. Thus, if A is closed under uniform convergence, then $\mathcal{Q}(A) = \mathcal{Q}(A^*) = \mathcal{Q}(\mathcal{M}(A))$, since $A^* = C(\mathcal{M}(A))$. In the general case, this may fail to hold. (So, more generally, $A_{\mathcal{V}} \neq C(\mathcal{M}(A))_{\mathcal{V}}$ even when $A \cong C(\mathcal{M}(A))$.)

EXAMPLE. Let $A = \mathcal{Q}(\mathbf{R})$. Then (see [1], p. 34),

$$A = \mathcal{Q}(A^*) \neq D(m\mathbf{R}) = D(M(A^*)) = \mathcal{Q}(M(A^*)).$$

For any Φ -algebra A and any space \mathcal{V} near to $\mathcal{X} = \mathcal{M}(A)$, every subalgebra of $A_{\mathcal{V}}$ that contains A is a ring of quotients of A . Of interest are those that separate points of \mathcal{V} ; prime candidates are the maximal subalgebras of $A_{\mathcal{V}}$ containing A , which are easily seen to exist.

The results that follow are obtained using ideas and methods employed by Nanzetta in [6] (see his 2.1, 2.3, 4.1). Conversion of his arguments to the present setting is largely an exercise in careful bookkeeping, and the details are omitted.

THEOREM 4. *If B is a maximal subalgebra of $A_{\mathcal{V}}$, then B is a lattice (hence, a Φ -algebra).*

We will use the term “maximal subalgebra of $A_{\mathcal{V}}$ ” to denote only those that contain A .

DEFINITION. Let B be a subalgebra of $D(\mathcal{V})$. A function $f \in D(\mathcal{V})$ is said to be *locally in B* if each point of \mathcal{V} has a neighborhood on which f coincides with some member of B . The subalgebra B is said to be *local* (in $D(\mathcal{V})$) if each member of $D(\mathcal{V})$ that is locally in B is a member of B .

THEOREM 5. *Every maximal subalgebra of $A_{\mathcal{V}}$ is local.*

As in [6], this fact yields the following result.

THEOREM 6. *Let B be a maximal subalgebra of $A_{\mathcal{V}}$, and let \mathcal{S} be a stationary set of B . If $|\mathcal{S}| > 1$, then*

- (i) \mathcal{S} is closed;
- (ii) \mathcal{S} is nowhere dense;
- (iii) \mathcal{S} is connected.

COROLLARY. *If \mathcal{V} is totally disconnected, then every maximal subalgebra of $A_{\mathcal{V}}$ separates points of \mathcal{V} . (Note that this may occur*

even when $A_{\mathcal{Y}}$ is not an algebra: see the example preceding Proposition 2.)

It is not known whether every space \mathcal{Y} near to \mathcal{X} supports (i.e., is the structure space of) a ring of quotients of $C(\mathcal{X})$. Apparently, an answer to this question awaits a more systematic description of the collection of spaces near to \mathcal{X} .

Note that $(A_{\mathcal{Y}})^*$, the set of bounded elements of $A_{\mathcal{Y}}$, is always a \emptyset -algebra. Hence, it is always a ring of quotients of A^* —the largest bounded ring of quotients of A^* in $D(\mathcal{Y})$. As mentioned above, it is not known whether $(A_{\mathcal{Y}})^*$ always separates points of \mathcal{Y} ; it clearly does so if and only if $A_{\mathcal{Y}}$ does. However, the example that follows shows that $A_{\mathcal{Y}}$ may separate points in \mathcal{Y} even though \mathcal{Y} supports no ring of quotients of A .

EXAMPLE. Let $\mathcal{S} = \{(x, \sin(1/x)); x \in (0, 1]\}$, let \mathcal{X} denote the one-point compactification of \mathcal{S} , and let $\mathcal{Y} = \mathcal{S} \cup (\{0\} \times [-1, 1])$. Let A denote the \emptyset -algebra of all functions $f \in D(\mathcal{X})$ that satisfy the following condition:

There is a real number x_0 , $0 < x_0 < 1$, and a real polynomial p such that

$$f\left(x, \sin \frac{1}{x}\right) = p\left(\frac{1}{x}\right) \quad \text{for } 0 < x < x_0$$

(cf. [4], 3.6). Then $(A_{\mathcal{Y}})^* = C(\mathcal{Y})$, whereas no subalgebra of $D(\mathcal{Y})$ containing A separates points in \mathcal{Y} ([6], Theorem 4.6).

In passing, it should be noted that the development here has proceeded independently of [1]. The only results from that work that have been employed in an essential way came from Chapter 1 of [1], which consists of standard facts about rings of quotients of commutative rings (see, e.g., [5]). Thus, one can rapidly and efficiently reach the high points of the theory developed in [1] along the lines suggested by this note.

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