RINGS OF QUOTIENTS OF $\Phi$-ALGEBRAS

DONALD GLEN JOHNSON
RINGS OF QUOTIENTS OF $\Phi$-ALGEBRAS

D. G. JOHNSON

Let $\mathcal{X}$ be a completely regular (Hausdorff) space. Fine, Gillman, and Lambek have studied the (generalized) rings of quotients of $C(\mathcal{X}) = C(\mathcal{X}; R)$, with particular emphasis on the maximal ring of quotients, $Q(\mathcal{X})$. In this note, we start with a characterization of $Q(\mathcal{X})$ that differs only slightly from one of theirs. This characterization is easily altered to fit more general circumstances, and so serves to obtain some results on non-maximal rings of quotients of $C(\mathcal{X})$, and to generalize these results to the class of $\Phi$-algebras.

We consider only commutative rings with unit. Let $A$ be one such, and recall that the (unitary) over-ring $B$ of $A$ is called a rational extension or ring of quotients of $A$ if it satisfies the following condition: given $b \in B$, for every $0 \neq b' \in B$ there is $a \in A$ with $ba \in A$ and $b'a \neq 0$. A ring without proper rational extensions is said to be rationally complete. For the rings to be considered here (all are semi-prime), the condition above can be replaced by the simpler condition: for $0 \neq b \in B$, there exists $a \in A$ such that $0 \neq ba \in A$ ([1], p. 5). Accordingly, we make the following

Definition. If $B$ is an over-ring of $A$ and $0 \neq b \in B$, say that $b$ is rational over $A$ if there is $a \in A$ with $0 \neq ba \in A$.

Let $m\beta\mathcal{X}$ denote the minimal projective extension of $\beta\mathcal{X}$ and $\tau: m\beta\mathcal{X} \rightarrow \beta\mathcal{X}$ the minimal perfect map ([2]). In [1], it is shown that $Q(\mathcal{X})$ is a dense, point-separating subalgebra of $D(m\beta\mathcal{X})$, the set of all continuous maps from $m\beta\mathcal{X}$ into the two-point compactification of the real line which are real-valued on a dense subset of $m\beta\mathcal{X}$ (see, also, [3]). Since $Q(\mathcal{X})$ contains every ring of quotients of $C(\mathcal{X})$, this leads to

Proposition 1. If $B$ is any ring of quotients of $C(\mathcal{X})$, then there exist a compact (Hausdorff) space $\mathcal{Y}$ and minimal perfect maps $\alpha$ and $\tau$ such that $B$ is a point-separating subalgebra of $D(\mathcal{Y})$ and the following diagram commutes:

$$
\begin{array}{ccc}
m\beta\mathcal{X} & \xrightarrow{\alpha} & \mathcal{Y} \\
\downarrow{\tau} & & \downarrow{\tau} \\
\beta\mathcal{X} & \xrightarrow{\tau} & \beta\mathcal{X}.
\end{array}
$$
\( \mathcal{U} \) is the obvious identification space, and the proof consists of a routine argument to show that the quotient map \( \alpha \) is closed, whence \( \mathcal{U} \) is Hausdorff. Since \( C(\mathcal{X}) \subseteq B \), the existence of \( \gamma \) follows immediately. (Note that, although \( D(m\beta \mathcal{X}) \) is an algebra, \( D(\mathcal{U}) \) for other spaces \( \mathcal{U} \) is, in general, only a partial algebra.)

For our purposes, it is convenient to view \( C(\mathcal{X}) \) as a subalgebra of \( D(\beta \mathcal{X}) \). This allows us to decree that all spaces are compact (Hausdorff).

Let us say that any space \( \mathcal{U} \) that is situated in a commutative diagram of the form

\[
\begin{array}{ccc}
m\mathcal{X} & \xrightarrow{\alpha} & \mathcal{U} \\
& \downarrow{\tau} & \downarrow{\gamma} \\
\mathcal{X} & & 
\end{array}
\]

where all maps are minimal perfect, is near to \( \mathcal{X} \). (Of course, the existence of \( \gamma \) automatically guarantees the existence of \( \alpha \).) Note that we have already adopted the convention of identifying \( f \in D(\mathcal{X}) \) with its image \( f \circ \gamma \) in \( D(\mathcal{U}) \) whenever convenient. With this convention, if \( A \) is a subalgebra of \( D(\mathcal{U}) \) and \( f \in D(\mathcal{U}) \) then we may consider \( f \) as an element of an over-ring of \( A - D(m\mathcal{X}) - \), even if there is no subalgebra of \( D(\mathcal{U}) \) containing both \( A \) and \( f \).

Now let \( A \) be a \( \Phi \)-algebra that is closed under bounded inversion; i.e., an archimedean lattice ordered algebra with a multiplicative identity that is a weak order unit, in which \( 1/a \in A \) whenever \( 1 \leq a \in A \). Let \( \mathcal{X} = \mathcal{M}(A) \), the space of maximal ideals of \( A \) with the hull-kernel topology. It is shown in [4] that \( A \) is (isomorphic with) a point-separating subalgebra of \( D(\mathcal{X}) \). If \( \mathcal{U} \) is any space that is near to \( \mathcal{X} \), let \( A_{\mathcal{U}} = \{ f \in D(\mathcal{U}) : \text{for each nonempty open set } \mathcal{U} \text{ in } \mathcal{U}, \text{ there are a nonempty open set } \mathcal{V} \subseteq \mathcal{U} \text{ and } g \in A \text{ such that } f|_{\mathcal{V}} = g|_{\mathcal{V}} \} \). Note that \( A_{\mathcal{U}} \) is always a lattice. However, it need not be an algebra:

**Example.** Let \( \mathcal{X} = \mathcal{U} \), the one-point compactification of the countable discrete space, and let \( A = C(\mathcal{X}) \). Then \( A_{\mathcal{U}} = D(\mathcal{U}) \), which is not an algebra.

**Remark.** One readily shows that the open sets \( \mathcal{V} \) appearing in the definition of \( A_{\mathcal{U}} \) can always be shown to have the form \( \mathcal{V}^* \mathcal{X}_1 \), where \( \mathcal{X}_1 \) is open in \( \mathcal{X} \). It follows that

\[
A_{\mathcal{U}} = \{ f \in D(\mathcal{U}) : f \circ \alpha \in A_{m\mathcal{X}} \}.
\]

**Proposition 2.** (i) Every element of \( A_{\mathcal{U}} \) is rational over \( A^* \)
(and, hence, over $A$).

(ii) $A_\mathcal{V}$ contains every rational extension of $A$ and $A^*$ in $D(\mathcal{V})$.

Proof. (i) Let $0 \neq f \in A_\mathcal{V}$, and let $\mathcal{U}$ be a nonempty open set contained in $\text{cozf}$. Since $f \in A_\mathcal{V}$, there exist a nonempty open set $\mathcal{V} = \mathcal{V}^\prime \cup \mathcal{V}^\prime_1 \subseteq U$, where $\mathcal{V}^\prime_1$ is open in $\mathcal{V}^\prime$, and $h \in A^*$ such that $f|_{\mathcal{V}^\prime} = h|_{\mathcal{V}^\prime}$. Choose $0 \neq g \in A^*$ with $\text{cozf}^\prime \subseteq \mathcal{V}^\prime_1$. Then $0 \neq fg = hg \in A^*$.

(ii) Let $f \in D(\mathcal{V}) \setminus A_\mathcal{V}$. Then, there is a nonempty open set $\mathcal{U}$ such that $f$ agrees with no member of $A$ on any nonempty open subset of $\mathcal{U}$. Choose $g \in A^*$ with $\phi \neq \text{cozf} \subseteq \mathcal{U}$.

There is no $h \in A$ with $hg \neq 0$ while $fh \in A$. For, such $h$ would agree with a unit $h_1$ of $A$ on some nonempty open subset $\mathcal{V}$ of $\mathcal{U}$ (since $A$ is closed under bounded inversion), whence

$$f|_{\mathcal{V}} = (h|h_1)f|_{\mathcal{V}},$$

while $(1/h_1)hf \in A$, a contradiction. Thus, $f$ is contained in no rational extension of $A$.

Although $A_\mathcal{V}$ may contain many different rational extensions of $A$, it is not true that it is the union of such extensions, as is seen in the example preceding Proposition 2. However, in those spaces $\mathcal{V}$ for which $A_\mathcal{V}$ is an algebra, $A_\mathcal{V}$ is a $\Phi$-algebra and is the largest ring of quotients of $A$ that “lives on” $\mathcal{V}$. In particular, this happens when $D(\mathcal{V})$ is an algebra (e.g., when $\mathcal{V}$ is basically disconnected or an $F$-space). Hence, $A_{m_\mathcal{V}}$ is a $\Phi$-algebra, since $m_\mathcal{V}$ is extremally disconnected, and we obtain the following generalizations of results in [1].

**Theorem 1.** $A_{m_\mathcal{V}}$ is rationally complete; thus, $A_{m_\mathcal{V}} = \mathcal{Q}(A)$, the maximal ring of quotients of $A$.

**Theorem 2.** $A_{m_\mathcal{V}}$ is uniformly dense in $D(m_\mathcal{V})$.

**Theorem 3** ([1]). $D(m_\mathcal{V})$ is rationally complete.

The proofs of Theorems 1 and 3 are virtually identical, and are related to one found on p. 30 of [1]; we prove 1. To do so, we will employ the following characterization of rational completeness (see [1], p. 7).

The commutative ring $B$ is rationally complete if and only if it satisfies: for any dense ideal $I$ of $B$, every element of $\text{Hom}_B(I, B)$ is a multiplication by an element of $B$. (In the present setting, an ideal $I$ of $A_{m_\mathcal{V}}$ is dense if and only if $\cup \{\text{cozf}: f \in I\}$ is dense in $m_\mathcal{V}$.)

**Proof of Theorem 1.** Let $I$ be a dense ideal in $A$, and let
φ ∈ Hom_{A_{m^S}}(I, A_{m^S}). By Zorn's lemma, choose a family \{U_κ : κ ∈ K\} of open sets in m^X satisfying:

(i) \( U = \bigcup U_κ \) is dense in m^X;
(ii) the \( U_κ \) are pairwise disjoint;
(iii) for each κ, there is \( f_κ ∈ I \) such that \( f_κ \) is bounded away from zero on \( U_κ \) and both \( f_κ \) and \( φ(f_κ) \) agree with members of A on \( U_κ \).

Let \( f ∈ D(m^X) \) satisfy

\[
\left. f \right|_{U_κ} = \frac{\phi(f_κ)}{f_κ} \left|_{U_κ}\right.
\]

for each κ ∈ K. This is possible, since m^X is extremally disconnected, so m^X = βU.

If \( g ∈ I \) and \( x ∈ U_κ \), then

\[
f(x)g(x) = \frac{\phi(f_κ)(x)}{f_κ(x)}g(x) = \left. \frac{g\phi(f_κ)}{f_κ} \right|_{U_κ}(x) = \left. \frac{f_κφ(g)}{f_κ} \right|_{U_κ}(x) = φ(g)(x).
\]

It follows that \( φ \) is multiplication by \( f \). Clearly, \( f ∈ A_{m^X} \), and the proof is complete.

**Proof of Theorem 2.** Let \( f ∈ D(m^X), ε > 0 \). By Zorn's lemma, choose a family \{U_κ : κ ∈ K\} of open sets in m^X which satisfies:

(i) \( U = \bigcup U_κ \) is dense in m^X;
(ii) the \( U_κ \) are pairwise disjoint;
(iii) for \( x, y ∈ U_κ, |f(x) − f(y)| < ε \) (in particular, \( f \) is real-valued on \( U_κ \)).

For each κ ∈ K, choose \( x_κ ∈ U_κ \), and define \( g : U → \mathbb{R} \) by

\[
g(y) = f(x_κ) \quad \text{if} \quad y ∈ U_κ.
\]

Since m^X = βU, g can be extended to \( \hat{g} ∈ D(m^X) \). Clearly, \( \hat{g} ∈ A_{m^X} \), and

\[
|f − \hat{g}| ≤ ε.
\]

Now the analogue of Proposition 1 for \( φ \)-algebras is routinely obtained.

In case \( Y = m^X \) and \( A = C(\mathcal{A}) \) one readily translates the definition of \( A_Y \) (using the fact that m^X is extremally disconnected, and hence that every dense subspace is C*-embedded) as follows:

\[
A_{m^X} = \lim \{C(\mathcal{S}) : \mathcal{S} \text{ is a dense open subset of } X\}.
\]

Thus, the Fine-Gillman-Lambek result that this direct limit is \( Q(X) \) follows from Theorem 1.
It is easily seen that any $\Phi$-algebra $A$ is a rational extension of its bounded subring $A^*$, and hence that $(A^*)_\mathcal{U} = A_\mathcal{U}$ for any space $\mathcal{U}$ near to $\mathcal{H}(A)$. Thus, if $A$ is closed under uniform convergence, then $Q(A) = Q(A^*) = Q(\mathcal{H}(A))$, since $A^* = C(\mathcal{H}(A))$. In the general case, this may fail to hold. (So, more generally, $A_\mathcal{U} \neq C(\mathcal{H}(A))_\mathcal{U}$ even when $A \subseteq C(\mathcal{H}(A))$.)

**Example.** Let $A = Q(R)$. Then (see [1], p. 34),

$$A = Q(A^*) = D(mR) = D(M(A^*)) = Q(M(A^*)).$$

For any $\Phi$-algebra $A$ and any space $\mathcal{U}$ near to $\mathcal{H} = \mathcal{H}(A)$, every subalgebra of $A_\mathcal{U}$ that contains $A$ is a ring of quotients of $A$. Of interest are those that separate points of $\mathcal{U}$; prime candidates are the maximal subalgebras of $A_\mathcal{U}$ containing $A$, which are easily seen to exist.

The results that follow are obtained using ideas and methods employed by Nanzetta in [6] (see his 2.1, 2.3, 4.1). Conversion of his arguments to the present setting is largely an exercise in careful bookkeeping, and the details are omitted.

**Theorem 4.** If $B$ is a maximal subalgebra of $A_\mathcal{U}$, then $B$ is a lattice (hence, a $\Phi$-algebra).

We will use the term "maximal subalgebra of $A_\mathcal{U}$" to denote only those that contain $A$.

**Definition.** Let $B$ be a subalgebra of $D(\mathcal{U})$. A function $f \in D(\mathcal{U})$ is said to be *locally in $B$* if each point of $\mathcal{U}$ has a neighborhood on which $f$ coincides with some member of $B$. The subalgebra $B$ is said to be *local* (in $D(\mathcal{U})$) if each member of $D(\mathcal{U})$ that is locally in $B$ is a member of $B$.

**Theorem 5.** Every maximal subalgebra of $A_\mathcal{U}$ is local.

As in [6], this fact yields the following result.

**Theorem 6.** Let $B$ be a maximal subalgebra of $A_\mathcal{U}$, and let $\mathcal{S}$ be a stationary set of $B$. If $|\mathcal{S}| > 1$, then

(i) $\mathcal{S}$ is closed;

(ii) $\mathcal{S}$ is nowhere dense;

(iii) $\mathcal{S}$ is connected.

**Corollary.** If $\mathcal{U}$ is totally disconnected, then every maximal subalgebra of $A_\mathcal{U}$ separates points of $\mathcal{U}$. (Note that this may occur
even when \( A_\mathfrak{Y} \) is not an algebra: see the example preceding Proposition 2.)

It is not known whether every space \( \mathfrak{Y} \) near to \( \mathfrak{X} \) supports (i.e., is the structure space of) a ring of quotients of \( C(\mathfrak{X}) \). Apparently, an answer to this question awaits a more systematic description of the collection of spaces near to \( \mathfrak{X} \).

Note that \( (A_\mathfrak{Y})^* \), the set of bounded elements of \( A_\mathfrak{Y} \), is always a \( \Phi \)-algebra. Hence, it is always a ring of quotients of \( A^* \)—the largest bounded ring of quotients of \( A^* \) in \( D(\mathfrak{Y}) \). As mentioned above, it is not known whether \( (A_\mathfrak{Y})^* \) always separates points of \( \mathfrak{Y} \); it clearly does so if and only if \( A_\mathfrak{Y} \) does. However, the example that follows shows that \( A_\mathfrak{Y} \) may separate points in \( \mathfrak{Y} \) even though \( \mathfrak{Y} \) supports no ring of quotients of \( A \).

**Example.** Let \( \mathcal{I} = \{ (x, \sin (1/x)); x \in (0, 1] \} \), let \( \mathfrak{H} \) denote the one-point compactification of \( \mathcal{I} \), and let \( \mathfrak{Y} = \mathcal{I} \cup \{ \{0\} \times [-1, 1] \} \). Let \( A \) denote the \( \Phi \)-algebra of all functions \( f \in D(\mathfrak{H}) \) that satisfy the following condition:

There is a real number \( x_0, 0 < x_0 < 1 \), and a real polynomial \( p \) such that

\[
f\left(x, \sin \frac{1}{x}\right) = p\left(\frac{1}{x}\right) \text{ for } 0 < x < x_0
\]

(cf. [4], 3.6). Then \( (A_\mathfrak{Y})^* = C(\mathfrak{Y}) \), whereas no subalgebra of \( D(\mathfrak{Y}) \) containing \( A \) separates points in \( \mathfrak{Y} \) ([6], Theorem 4.6).

In passing, it should be noted that the development here has proceeded independently of [1]. The only results from that work that have been employed in an essential way came from Chapter 1 of [1], which consists of standard facts about rings of quotients of commutative rings (see, e.g., [5]). Thus, one can rapidly and efficiently reach the high points of the theory developed in [1] along the lines suggested by this note.

**References**


Received April 13, 1970. Sincere thanks are due to A. W. Hager whose critical comments greatly improved a hastily prepared manuscript.

New Mexico State University
E. M. Alfsen and B. Hirsberg, *On dominated extensions in linear subspaces of* \( \mathcal{E}_C(X) \) ................................................................. 567

Joby Milo Anthony, *Topologies for quotient fields of commutative integral domains* ................................................................. 585

V. Balakrishnan, G. Sankaranarayanan and C. Suyambulingom, *Ordered cycle lengths in a random permutation* .............................................. 603

Victor Allen Belfi, *Nontangential homotopy equivalences* .................................................. 615

Jane Maxwell Day, *Compact semigroups with square roots* ................................................... 623

Norman Henry Eggert, Jr., *Quasi regular groups of finite commutative nilpotent algebras* ................................................................. 631

Paul Erdős and Ernst Gabor Straus, *Some number theoretic results* .............................. 635

George Rudolph Gordh, Jr., *Monotone decompositions of irreducible Hausdorff continua* ................................................................. 647

Darald Joe Hartfiel, *The matrix equation AXB = X* .................................................. 659

James Howard Hedlund, *Expansive automorphisms of Banach spaces. II* ............... 671

I. Martin (Irving) Isaacs, *The p-parts of character degrees in p-solvable groups* ................................................................. 677

Donald Glen Johnson, *Rings of quotients of \( \Phi \)-algebras* ........................................ 693

Norman Lloyd Johnson, *Transition planes constructed from semifield planes* ................................................................. 701

Anne Bramble Searle Koehler, *Quasi-projective and quasi-injective modules* ............... 713

James J. Kuzmanovich, *Completions of Dedekind prime rings as second endomorphism rings* ................................................................. 721

B. T. Y. Kwee, *On generalized translated quasi-Cesàro summability* ................. 731

Yves A. Lequain, *Differential simplicity and complete integral closure* .................. 741

Mordechai Lewin, *On nonnegative matrices* ................................................................. 753

Kevin Mor McCrimmon, *Speciality of quadratic Jordan algebras* .......................... 761

Hussain Sayid Nur, *Singular perturbations of differential equations in abstract spaces* ................................................................. 775


Lavon Barry Page, *Operators that commute with a unilateral shift on an invariant subspace* ................................................................. 787

Helga Schirmer, *Properties of fixed point sets on dendrites* .......................................... 795

Saharon Shelah, *On the number of non-almost isomorphic models of \( T \) in a power* ................................................................. 811

Robert Moffatt Stephenson Jr., *Minimal first countable Hausdorff spaces* ............. 819

Masamichi Takesaki, *The quotient algebra of a finite von Neumann algebra* ................................................................. 827

Benjamin Baxter Wells, Jr., *Interpolation in \( C(\Omega) \)* ........................................ 833