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# COMPLETIONS OF DEDEKIND PRIME RINGS AS SECOND ENDOMORPHISM RINGS

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The purpose of this paper is to show that if M is a maximal two-sided ideal of a Dedekind prime ring R and P is any maximal right ideal containing M, then the M-adic completion  $\overline{R}$  of R can be realized as the second endomorphism ring of E=E(R/P), the R-injective hull of R/P; that is, as end ( $_{K}E$ ) where K=end ( $E_{R}$ ). The ring K turns out to be a complete, local, principal ideal domain.

This paper was motivated by a result of Matlis [6] which says that if P is a prime ideal of a commutative Noetherian ring R, then the P-adic completion of the localization of Rat P can be realized as the ring of endomorphisms of E = E(R/P), the R-injective hull of R/P.

Since  $\overline{R}$  is a full matrix ring over a complete local domain L [4], we are able to approach the problem by considering first the case that R is a complete local domain, then by means of the Morita theorems we pass to the case  $\overline{R}=R$ , and finally pass to the general case.

1. Introduction. A prime ring R is called a Dedekind prime ring if it is Noetherian, hereditary, and a maximal order in its classical quotient ring Q (see [3]). A ring R is called local if the nonunits of R form an ideal.

If R is a Dedekind prime ring with a nonzero prime ideal M, then M is a maximal two-sided ideal and  $\cap M^n = 0$  (see Robson [7]). Let  $\overline{R} = \overline{R}_M$  be the completion of R at M in the sense of Goldie [3]. In this situation combining results of Goldie ([3], Theorem 4.5) and Gwynne and Robson ([4], Theorem 2.3) yields the following theorem.

THEOREM 1.1. Let R be a Dedekind prime ring with a maximal ideal M. Then (i)  $\overline{R}$  has a unique maximal two-sided ideal  $\overline{M}$ ,  $\overline{M}$  is the Jacobson radical of  $\overline{R}$ , and  $R \cap \overline{M}^p = M^p$ .

(ii)  $\overline{R}$  is a full  $k \times k$  matrix ring over a domain L which has a unique maximal ideal N, and L/N = F where F is a division ring. Also  $R/M^p \simeq \overline{R}/\overline{M}^p$  (each coset of  $\overline{M}^p$  has a representative in R).

(iii)  $\overline{R}$  is a prime principal ideal ring and L is a complete, local, principal ideal domain. The only one-sided ideals of L are the powers of N.

For the rest of this section let R, M,  $\overline{R}$ ,  $\overline{M}$ , L, and N be as in Theorem 1.1. Let x be the generator of N; then N = xL = Lx and  $N^k = x^kL = Lx^k$ .

2. The Ring L. This section will be concerned with the construction of the *L*-injective hull of  $(L/N)_L$  and with showing that Theorem 4.4 holds for *L*.

LEMMA 2.1.  $L/N^k$  can be embedded in  $L/N^{k+1}$  as a right L-module via the map  $h_k: L/N^k \to L/N^{k+1}$  defined by  $h_k$   $([u + N^k]) = [xu + N^{k+1}].$ 

*Proof.*  $h_k$  is clearly additive and right L-linear. Suppose  $h_k([u + N^k]) = [0 + N^{k+1}]$ . From the definition of  $h_k$  it follows that  $xu \in N^{k+1}$  so that  $xu = x^{k+1}u$ , for some u' in L and  $u = x^ku' \in N^k$ . Hence  $[u + N^k] = [0 + N^k]$  and  $n_k$  is a monomorphism. A similar argument shows that  $h_k$  is well-defined.

The maps  $\{h_k\}$  and the right *L*-modules  $\{(L/N^k)_L\}$  give rise to a directed system. Let  $E_L$  be the direct limit of this system. Then  $E_L$  can be considered as an ascending union of a family of submodules,  $\{(S_j)_L\}$ , which is totally ordered by inclusion and where each  $(S_j)_L$  is isomorphic to  $(L/N^i)_L$ .

LEMMA 2.2. Consider  $(L/N^{p+t+1})_L$ . Take  $a \in N^p/N^{p+t+1}$  and  $d \in N^p \setminus N^{p+1}$ . The equation yd = a has a solution in  $(L/N^{p+t+1})_L$ .

*Proof.*  $a \in N^p/N^{p+t+1}$  so that  $a = [x^pv + N^{p+t+1}]$ .  $d \in N^p \setminus N^{p+1}$  so that  $d = x^p u$  where u is a unit in L. In L,  $x^p v u^{-1} = w x^p$  since

$$egin{aligned} N^p &= x^p L = L x^p. & ext{Let } y = [w + N^{p+t+1}]. \ yd \ &= [w + N^{p+t+1}]d = [wd + N^{p+t+1}] = [wx^p u + N^{p+t+1}] \ &= [x^p v u^{-1} u + N^{p+t+1}] = [x^p v N^{p+t+1}] = a. \end{aligned}$$

PROPOSITION 2.3.  $E_L$  is isomorphic to the L-injective hull of the simple right L-module  $(L/N)_L$ .

**Proof.**  $E_L$  contains a copy of  $(L/N)_L$ , namely  $S_1$ . Thus it is enough to show that E is an essential injective extension of  $S_1$ .  $S_1$ is essential in E for if  $a \in E$ ,  $a \in S_k$  for some integer k. Let t be the first such integer: then  $a \in S_t \setminus S_{t-1}$ , a is a generator for  $S_t$ , and  $aL = S_t$ . Thus  $aL \cap S_1 = S_1$  and  $S_1$  is essential. Since L is a principal ideal domain, it is a hereditary two-sided order in its quotient division ring. In order to prove  $E_L$  is injective it is sufficient by a result of Levy ([5], Theorem 3.4) to show that it is L-divisible. Take  $a \in E$  and  $0 \neq d \in L$ .  $a \in S_t$  for some t and  $d \in N^p \setminus N^{p+1}$  for some p. yd = a has a solution in  $S_{p+t+1}$ , and hence in *E*, by Lemma 2.2. *E* is thus an essential injective extension of  $S_1$  and hence is its injective hull.

Let  $K = \operatorname{end}_{L}(E)$  and let K act on E by left multiplication; E then becomes a left K-module. Let  $H = \operatorname{end}_{K}(E)$ ; in similar manner E then becomes a right H-module. Ed = E (since E is L-divisible) for all nonzero d in L; thus E is a faithful right L-module. Hence L may be considered as a unital subring of H.

LEMMA 2.4. The  $S_k$ 's are the only proper L-submodules of  $E_L$ .

*Proof.* Suppose  $M_L$  is a submodule of E with generating set  $\{m_i\}$ . Since  $E = \bigcup S_k$ , each  $m_i$  is in some  $S_k$ . Let  $k_i$  be the first k for which  $m_i \in S_k$ . Then  $m_i \in S_{k_i} \setminus S_{k_i-1}$  and  $m_i L = S_{k_i}$ .  $M = \Sigma m_i L = \Sigma S_{k_i}$  so that if  $\{k_i\}$  is bounded,  $M = S_{k_i}$  where  $k_i = \max\{k_i\}$ , and if  $\{k_i\}$  is not bounded, then  $M = E_L$ .

LEMMA 2.5. If  $a \in S_n$  and if  $b \in S_{n-1}$ , then there is a  $q \in K$  such that q(b) = a.

*Proof.* Assume that t is the first integer for which  $b \in S_{n+t}$ . Then  $\operatorname{ann}_L(b) = N^{n+t}$  which is contained in  $N^n$  which in turn is contained in  $\operatorname{ann}_L(a)$ . Thus the map  $\overline{q}: bL \to aL$  defined by  $\overline{q}(bd) = ad$  is well defined.  $E_L$  is L-injective so that  $\overline{q}$  can be extended to an endomorphism q of E.  $q \in K$ .

PROPOSITION 2.6. Each  $S_n$  is a cyclic left K-submodule of  $_{\kappa}E$ , the composition length of  $_{\kappa}(S_n)$  is n, and the  $S_n$ 's are the only proper K-submodules of E.

*Proof.* If  $q \in K, q(S_n)$  is an *L*-submodule of *E* of composition length less than or equal to *n* and hence must be contained in  $S_n$  by Lemma 2.4; hence each  $S_n$  is a left *K*-submodule. Each  $_{\kappa}(S_n)$  is cyclic via Lemma 2.5; in fact, any *L* generator of  $S_n$  will be a *K* generator of  $S_n$ . This implies that  $_{\kappa}(S_1)$  is simple and inductively that the composition length of  $_{\kappa}(S_n)$  is *n*. The proof of Lemma 2.4 shows that these are the only *K*-submodules of *E*.

LEMMA 2.7. Let  $H_i$  be the annihilator of  $S_i$  in H. Then  $H_i$  is a two-sided ideal of H,  $H_{i+1}$  is properly contained in  $H_i$ , and  $\cap H_i = 0$ .

*Proof.*  $H_i$  is clearly a right ideal of H. If  $h \in H$ , then  $(S_i)h$  is a K-submodule of E of composition length less than or equal to i. By Proposition 2.6 it must be that  $(S_i)h \subset S_i$  so that each  $S_i$  is H-invariant. As a result  $H_i$  is a left ideal and hence an ideal. The inclusions are

proper, for  $H_i \cap L = N^i$  and  $N_i \neq N^{i+1}$ . Since  $E = \bigcup S_i$ , anything in  $\cap H_i$  would annihilate all of E and hence be zero.

PROPOSITION 2.8. H = L. That is, L is the second endomorphism ring of  $E_L$ .

*Proof.* Take  $f \in H$ . By Proposition 2.6 there is a nonzero  $y \in S_1$  such that  $S_1 = Ky = yL$ . Hence there is a  $p_1 \in L$  such that  $yf = yp_1$ . Also, if  $z \in S_1$ , z = ky for some  $k \in K$  and

$$z(f - p_1) = (ky) (f - p_1) = k0 = 0$$
. Hence  $f - p_1 \in \operatorname{ann}_{H}(S_1) = H_1$ .

Inductively suppose that there is a  $p_i \in L$  such that  $f - p_i \in H_i$ . Now take  $y \in S_{i+1} \setminus S_i$ .  $y(f - p_i) \in S_{i+1}$  so that there is a  $d \in L$  such that  $y(f - p_i) = yd$ . If  $z \in S_{i+1}$ , z = ky for some  $k \in K$ . Then  $z(f - p_i) = (ky)(f - p_i) = k(y(f - p_i) = k(yd) = (ky)d = zd$  and hence  $f - p_i - d$  is in  $H_{i+1}$ . Let  $p_{i+1} = p_i + d$ ; then  $f - p_{i+1} \in H_{i+1}$ .

The sequence  $\{p_i\}$  is Cauchy in L, for  $p_n - p_m = (p_n - f) + (f - p_m)$ an element of  $H_n + H_m$ ; but  $H_n + H_m = H_n$  if  $n \leq m$ . Thus  $p_n = p_m$  is in  $H_n \cap L = N^n$ . L is complete; therefore  $\{p_i\}$  converges to some element p of L. It only remains to be shown that p = f. Take  $z \in E$ ;  $z \in S_n$ for some n.  $\{p_i\}$  converges to p so that there is a positive integer M such that  $p_m - p \in N^n$  for all m greater than M. Take m greater than M + n.  $zf = zp_m = zp$ . z was arbitrary; therefore f = p.

3. The Ring K. In this section it will be shown that K is a complete, local, principal ideal domain.

LEMMA 3.1. Let L, E, and K be as in §2. Let J denote the Jacobson radical of K and let  $A_n = \operatorname{ann}_{\kappa}(S_n)$ . Then

- (i) K is a local domain.
- (ii)  $J = A_1, J^n \subset A_n \cap A_n = 0, and \cap J^n = 0.$
- (iii) K is complete in the topology induced by the  $A_n$ 's.

*Proof.* (i) K is local since it is the endomorphism ring of an indecomposable injective module. To prove that K is a domain it is sufficient to show that every nonzero endomorphism of  $E_L$  is an epimorphism. Let  $0 \neq k \in K$ . If  $k(E) \neq E$ ,  $k(E) = S_n$  for some n by Lemma 2.4.  $\operatorname{Ann}_L(S_n) = N^n$ ; take  $0 \neq b \in N^n$ . Since E is L-divisible, Eb = E. As a result  $S_n = k(E) = k(Eb) = k(E)b = S_nb = 0$  contradicting the fact that  $k \neq 0$ .

(ii) The radical of K, J, is the set of all endomorphisms of  $E_L$  whose kernel is essential (see [2], page 44). Since  $(S_1)_L$  is the unique minimal submodule of E, ker(k) is essential if and only if  $k(S_1) = 0$ ;

therefore  $J = A_1$  and  $JS_1 = 0$ . Inductively suppose that  $J^{n-1}S_{n-1} = 0$ .  $JS_n \subset S_{n-1}$  since it is contained in the radical of  $K(S_n)$ ,  $S_{n-1}$ . Hence  $J^n s_n = J^{n-1}(Js_n)$  which is contained in  $J^{n-1}S_{n-1}$  which is zero, hence  $J^n \subset A_n \cap A_n = 0$  since anything in  $\cap A_n$  would annihilate all of the  $S_n$ 's and hence all of E.  $\cap J^n = 0$  since  $J^n \subset A_n$ .

(iii) Let  $\{f_i\}$  be a Cauchy sequence in K with respect to the topology induced by the decreasing family  $\{A_n\}$ . Let  $x \in E$ .  $x \in S_p$  for some p. Since  $\{f_i\}$  is Cauchy, there is an integer M such that  $f_n - f_m \in A_p$  for n, m greater than M. Define  $f(x) = f_{M+1}(x)$ . It is clear that  $f \in K$  and that  $f_i \to f$  by the nature of the construction.

Pick  $j \in J \setminus A_2$ . There is such a j, for if  $y_2 \in S_2 \setminus S_1$  and if  $0 \neq y_1 \in S_1$ , then there is a  $j \in K$  such that  $j(y_2) = y_1$  by Lemma 2.5.  $j \in J \setminus A_2$ . In fact if  $s \in S_{n+1} \setminus S_n$ , then  $j^n s$  is a nonzero element of  $S_1$ . The proof is by induction. If  $s \in S_2 \setminus S_1$ , then  $s = y_2 u$  for u a unit in L. Hence  $js = jy_2 u = y_1 u \neq 0$ . Inductively suppose that  $j^{n-1}s$  is nonzero for all s in  $S_n \setminus S_{n-1}$  and take  $s \in S_{n+1} \setminus S_n$ .  $js \in S_n$  by an argument in the previous proof. The claim is that  $js \notin S_{n-1}$ . If it were, then  $j^{n-1}s = 0$ which contradicts the induction hypothesis since  $sd \in S_n \setminus S_{n-1}$  for some d in L. Hence  $js \notin S_{n-1}$  so again by the induction hypothesis  $j^n s =$  $j^{n-1}(js) \neq 0$ .

LEMMA 3.2. Let K, J, j, E, and L be as above. (i) J = jK. (ii) J = Kj. (iii)  $J^n = j^n K = Kj^n$ .

*Proof.* (i) Let  $x \in J$ . Let  $y_2 \in S_2 \setminus S_1$ .  $x(y_2) = y \in S_1$  since  $x \in J$ . Let  $j(y_2) = y_1$ ;  $y_1$  is a nonzero element of  $S_1$  since  $j \in J \setminus A_2$ . Then there is an element d in L such that  $y = y_1 d = j(y_2)d = j(y_2d)$ . By Lemma 2.5 there exists  $k_1 \in K$  such that  $k_1(y_2) = y_2d$ . If  $s \in S_2$ , then  $s = y_2c$  for some c in L.  $x(s) = x(y_2c) = X(y_2)c = uc = (jk_1(y_2))c = jk_1(y_2c) = jk_1(s)$ . This says that  $x - jk_1 \in A_2$ .

Inductively suppose that there exist  $k_1, \dots, k_{n-1}$  such that

a nonzero element of  $S_1$  by the above choice of j. Also  $z(y_{n+1}) \in S_1$ since  $z \in A_n$ . Hence by the argument above there is a  $k_n \in K$  such that  $z - j^n k_n \in A_{n+1}$ . The sequence  $\{jk_1 + \cdots + j^n k_n\}$  converges to xin the  $A_n$  topology by the nature of the construction. Also, since  $J^n \subset A_n$  the sequence  $\{k_1 + \cdots + j^{n-1}k_n\}$  is Cauchy and hence by the completeness of K converges to some element k of K. Also by the construction jk = x. Since x was arbitrary in J, J = jk.

(ii) is proven by an argument similar to that of (i).

(iii) J = jK = Kj by (i) and (ii). Inductively suppose that  $J^n = j^n K = Kj^n$ . Then  $J^{n+1} = J^n J = (j^n K)(jK) = j^n (Kj)K = j^n (jK)K = j^{n+1}K$ . Similarly  $J^{n+1} = Kj^{n+1}$ .

PROPOSITION 3.3. K as above.

(i)  $J^n = A_n$  for all n.

- (ii)  $J^n$  are the only one-sided ideals of K.
- (iii) K is a complete principal ideal domain.

*Proof.* (i)  $J = A_1$  by Lemma 3.1. Inductively suppose that  $A_n = J^n$ .  $J^{n+1} \subset A_{n+1} \subset A_n = J^n$ .  $J^n/J^{n+1} = j^n K/j^{n+1}K \simeq K/jK = K/J$  which is simple. Therefore either  $A_{n+1} = J^{n+1}$  or  $A_{n+1} = J^n$ . But by the induction hypothesis  $j^n \notin A_{n+1}$  so that  $A_{n+1} = J^{n+1}$ .

(ii) It is sufficient to show that given  $x \in K$ , xK = K or that  $xK = J^p$  for some p. Take  $x \in K$  and suppose that  $xK \neq K$ , then x is not a unit and hence  $x \in J^{p+1}$  for some p. By Lemma 3.1  $x = j^p k$ , and k must be a unit; for otherwise  $k = jk_1$  for some  $k_1$  in K and  $x = j^p jk_1 \in J^{p+1}$ . As a result  $xK = j^p kK = j^p K = J^p$ . Similarly  $Kx = J^p$ .

(iii) K is a principal ideal domain by Lemma 3.2 and (ii). K is complete by (i) and Lemma 3.1.

4. The Ring R. Let  $R, M, \overline{R}$ , and L be as in Theorem 1.1. Then  $\overline{R}$  is the full  $k \times k$  matrix ring over L. Let  $e_{ij}$ ,  $i, j = 1, 2, \dots, n$ be a complete set of matrix units for  $\overline{R}$ . Let  $M_L$  be a right L-module and let  $M^* = M_1 \bigoplus \dots \bigoplus M_n$ , a direct sum of n copies of M. Let  $f_1$ be the identity map on  $M_1$ , and let  $f_i$ ,  $i = 2, \dots, n$  be an isomorphism from  $M_1$  to  $M_i$ . Then  $M^*$  can be made into an  $\overline{R}$ -module by defining  $f_i(m)e_{ij} = f_j(m)$  and  $f_i(m)e_{kj} = 0$  if  $i \neq k$ . "\*" is a category isomorphism from the category of right L-modules to the category of right  $\overline{R}$ -modules. There is also a category isomorphism  $e_{11}$  from the category of right  $\overline{R}$ -modules to the category of right L-modules defined by  $(A_R)e_{11} = Ae_{11}$ . M and  $M^*e_{11}$  are isomorphic for any right L-module M (see [1], or [5] page 137).

PROPOSITION 4.1.  $\overline{R}$  is the second endomorphism ring of the  $\overline{R}$ injective hull of the simple right  $\overline{R}$ -module.

*Proof.* Let E be the *L*-injective hull of the simple right *L*-module as in §2. Then  $E^*$  is the  $\overline{R}$ -injective hull of a simple right  $\overline{R}$ -module since \* is a category isomorphism.  $\overline{R}/\overline{M}$  is simple Artinian and  $\overline{M}$  is the Jacobson radical of  $\overline{R}$  so there is only one isomorphism class of simple right- $\overline{R}$ -modules. Let  $K = \operatorname{end}_{\overline{R}}(E^*)$  and take  $q \in K$ .

 $q(E^*e_{ii}) = q(E^*_{ii}e_{ii}) = q(E^*e_{ii})e_{ii}$ ; thus each  $E^*e_{ii}$  is K-invariant and  $_{\kappa}E^{*} = {}_{\kappa}F^{*}e_{11} \bigoplus {}_{\kappa}Ee_{22} \bigoplus \cdots \bigoplus {}_{\kappa}E^{*}e_{kk}$ . Each  $e_{ij}$  is a K-isomorphism so that  $E^*$  is decomposed as a direct sum of k mutually isomorphic K-modules. Thus each K-endomorphism of  $E^*$  can be given by multiplication by a matrix of homomorphisms. The remainder of the proof shows that the entries in this matrix are of the desired forms. Each  $q \in K$  restricted to  $E^*_{ii}$  is an L-endomorphism of  $E^*_{ii}$ . Each L-endomorphism of  $E^*e_{ii}$  can be extended in one and only one way to an  $\overline{R}$ -endomorphism of  $E^*$ ; namely, if  $\overline{q}$  is an L-endomorphism of  $E^*e_{ii}$ , then its unique extension q is defined by  $q(z) = \sum_{j=1}^k \overline{q}(ze_{ji})e_{ij}$ for  $z \in E^*$ . Hence  $K \simeq \operatorname{end}_L(E^*e_{ii})$  via the restriction map. By proposition 2.8 each element of  $\operatorname{end}_{\kappa}(E^*e_{ii})$  can be given by right multiplication by an element of  $e_{ii}\overline{R}e_{ii}$ . If  $h: E^*e_{ii} \to E^*e_{jj}$  is a K-homomorphism, then  $h\bar{e}_{ji}$  is a K-endomorphism of  $E^*e_{ji}$  where  $\bar{e}_{ji}$  denotes right multiplication by  $e_{ji}$ . Hence  $h\bar{e}_{ji} = \bar{e}_{ii}r\bar{e}_{ii}$  for some  $r \in \bar{R}$ . If  $z \in E^*_{ii}$ , then  $(z)h = zhe_{jj} = zhe_{ji}e_{ij} = ze_{ii}re_{ij}e_{ij} = ze_{ii}re_{ij}$  so that h is given by right multiplication by an element of  $e_{ii}\overline{R}e_{ij}$ . As a result every Kendomorphism of  $E^*$  is given by right multiplication by an element of  $\overline{R}$ .

R can be considered as a subring of  $\overline{R}$ ; as a result every  $\overline{R}$ -module is automatically an R-module. Also, if  $\overline{M}$  is the maximal two-sided ideal of  $\overline{R}$ , then  $\overline{M}^{p} \cap R = M^{p}$  and every coset of  $\overline{R}/\overline{M}^{p}$  has a representative in R (Theorem 1.1).

LEMMA 4.2.  $E^*$  as in the proof of Proposition 4.1, then  $(E^*)_R$  is the ascending union of  $\overline{R}$ -modules  $0 \subset B_1 \subset B_2 \subset \cdots$  where the composition length of  $B_n$  is n. These are the only  $\overline{R}$ -submodules of  $E^*$ . Furthermore, the  $B_i$ 's are the only R-submodules of  $E^*$  and every R-endomorphism of  $E^*$  is an  $\overline{R}$ -endomorphism. That is, the structure of  $E^*$  as an R-module is identical to its structure as an  $\overline{R}$ -module.

*Proof.* The first part follows since it was true of E and \* is a category isomorphism. Let  $B_i = S_i^*$ . A category isomorphism preserves the submodule lattice. Note that the composition length of  $(B_n)_{\overline{R}}$  is n; since  $\overline{M}$  is the radical of  $\overline{R}$ ,  $B_n\overline{M}^n = 0$ . In order to prove that the  $B_n$ 's are the only R-submodules of  $E^*$  it is sufficient to show that  $aR = a\overline{R}$  for all  $a \in E^*$ . Take  $a \in E^*$ . Clearly  $aR \subset a\overline{R}$ . Take  $\overline{r} \in \overline{R}$ .  $a \in B_n$  for some n so that  $a\overline{M}^n = 0$ . By theorem 1.1 there is an m in  $\overline{M}^n$  so that  $\overline{r} + m = r \in R$ , then  $a\overline{r} = a\overline{r} + 0 = a\overline{r} + am = a(\overline{r} + m)ar$ . Thus  $a\overline{R} \subset aR$  and  $a\overline{R} = aR$ .

Let q be an R-endomorphism of  $E^*$  and take  $a \in E^*$  and  $\overline{r} \in \overline{R}$ . It must be shown that  $q(a\overline{r}) = q(a)\overline{r}$ . Since  $a \in E^*$ ,  $a \in B_n$  for some n. The  $B_n$ 's are the only R-submodules of  $E^*$  and the composition length of  $B_n$  is n, so that  $q(B_n) \subset B_n$  and  $q(a) \in B_n$ . As above there is an  $m \in \overline{M}^n$  such that  $\overline{r} + m = r \in R$ .  $B_n \overline{M}^n = 0$ . Then

$$egin{aligned} q(aar{r}) &= q(aar{r}+0) = q(aar{r}+am) = q(a(ar{r}+m)) = q(ar) \ &= q(a)r = q(a)(ar{r}+m) = q(a)ar{r}+q(a)m \ &= q(a)ar{r}+0 = q(a)ar{r} \; . \end{aligned}$$

Thus q is an  $\overline{R}$ -endomorphism.

LEMMA 4.3  $E^*$  is the R-injective hull of  $(B_1)_R$ .

**Proof.** By Lemma 4.2  $(B_1)_R$  is an essential submodule of  $E^*_R$ .  $E^*$  is an injective  $\overline{R}$ -module since \* is a category isomorphism; in particular  $E^*$  is a divisible  $\overline{R}$ -module so that  $E^*$  is a divisible Rmodule. R is a hereditary two-sided order so that  $E^*$  is an injective R-module by [5], Theorem 3.4.

THEOREM 4.4. Let R be a Dedekind prime ring with a maximal two-sided ideal M, and let P be a maximal right ideal of R containing M. Then the R-endomorphism ring of the R-injective hull of R/P is a complete principal ideal domain.

*Proof.* Let  $R, \overline{R}, L, E_L$ , and  $E^*$  be as above. Then by Lemma 4.3  $E^*$  is the injective hull of a simple right R-module which is annihilated by M.  $(B_1)_R \simeq R \setminus P$  since both are simple modules over the simple Artinian ring R/M; thus  $E^* \simeq E(R/P)$ . By Lemma 4.2  $\operatorname{end}_R(E^*) = \operatorname{end}_{\overline{R}}(E^*)$  which is isomorphic to  $\operatorname{end}_L(E)$  since\* is a category isomorphism. Hence the result follows by Proposition 3.3.

THEOREM 4.5. (Main Theorem) Let R be a Dedekind prime ring with a nonzero prime ideal M, and let P be a maximal right ideal containing M with E(R/P) the R-injective hull of R/P. Then  $\overline{R}$ , the completion of R at M, is isomorphic to the second endomorphism ring of E(R/P).

*Proof.* Consider  $E^*$ ; as above  $E^* \simeq E(R/P)$ . By Lemma 4.2 the R and  $\overline{R}$  structures of  $E^*$  are identical. Thus  $\overline{R}$  is second endomorphism ring of E(R/P) by Proposition 4.1.

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