DIFFERENTIAL SIMPLICITY AND COMPLETE INTEGRAL CLOSURE

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Let $R$ be an integral domain containing the rational numbers, and let $R'$ denote the complete integral closure of $R$. It is shown that if $R$ is differentiably simple, then $R$ need not be equal to $R'$, even when $R$ is Noetherian, and then the relationship between $R$ and $R'$ is studied.

Let $\mathcal{D}$ be any set of derivations of $R$. Seidenberg has shown that the conductor $C = \{x \in R \mid xR' \subseteq R\}$ is a $\mathcal{D}$-ideal of $R$, so that when $R$ is $\mathcal{D}$-simple and $C \neq 0$, then $R = R'$. We investigate here the situation when $C = 0$.

The first observation that one must make is that it is no longer true that $R = R'$ when $R$ is differentiably simple, even when $R$ is Noetherian. We show this in Example 2.2 where we construct a 1-dimensional local domain containing the rational numbers which is differentiably simple but not integrally closed. This counterexample answers a conjecture of Posner [4, p. 1421] and also answers affirmatively a question of Vasconcelos [6, p. 230].

Thus, it is not a redundant task to study the relationship between a differentiably simple ring $R$ and its complete integral closure. An important tool in this study is the technique of § 3 which associates to any prime ideal $P$ of $R$ containing no $D$-ideal a rank-1, discrete valuation ring centered on $P$; by means of this, we show in Theorem 3.2 that over such a prime ideal $P$ of $R$ there lies a unique prime ideal of $R'$. When $R$ is a Noetherian $\mathcal{D}$-simple ring with $\{P_a\}_{a \in A}$ as set of minimal prime ideals, Theorem 3.3 asserts that $R' = \bigcap_{a \in A} \{R_a \mid R_a$ is the valuation ring associated with the minimal prime ideal $P_a\}$; Corollary 3.5 asserts that $R'$ is the largest $\mathcal{D}$-simple over-ring of $R$ having a prime ideal lying over every minimal prime ideal of $R$.

1. Preliminaries. Our notation and terminology adhere to that of Zariski-Samuel [7] and [8]. Throughout the paper we use $R$ to denote a commutative ring with 1, $K$ to denote the total quotient ring of $R$, and $A$ to denote an ideal of $R$; $A$ is proper if $A \neq R$. A derivation $D$ of $R$ is a map of $R$ into $R$ such that

$$D(a + b) = D(a) + D(b) \quad \text{and} \quad D(ab) = aD(b) + bD(a)$$

for all $a, b \in R$.

Such a derivation can be uniquely extended to $K$, and we shall
also denote the extended derivation by $D$. $D$ is said to be regular on a subring $S$ of $K$ if $D(S) \subset S$. If $\mathcal{D}$ is a family of derivations of $R$, $A$ is called a $\mathcal{D}$-ideal if $D(A) \subset A$ for every $D \in \mathcal{D}$; when $\mathcal{D} = \{D\}$, we merely say $D$-ideal. If $R$ has no $\mathcal{D}$-ideal different from $(0)$ and $(1)$, $R$ is said to be $\mathcal{D}$-simple. We use $D^{(n)}(x)$ to denote $x$, and for $n \geq 1$ $D^{(n)}(x)$ to denote $D(D^{(n-1)}(x))$, i.e. the $n$th derivative of $x$; by induction one proves Leibnitz’s rule:

$$D^{(n)}(ab) = \sum_{i=0}^{n} C_i^n D^{(n-i)}(a)D^{(i)}(b).$$

We assume henceforth that $\mathcal{D}$ is a family of derivations of $R$ and that $D \in \mathcal{D}$. Let $\varphi: R \to S$ be a homomorphism onto; then

$$D'(\varphi(r)) = \varphi(D(r))$$
defines a derivation $D'$ on $S$ if and only if the kernel $I$ of $\varphi$ is a $D$-ideal. Suppose that $I$ is a $\mathcal{D}$-ideal, and write $\mathcal{D}'$ to denote the set of derivations of $S$ thus induced by $\mathcal{D}$; if $A$ is a $\mathcal{D}$-ideal of $R$, then $\varphi(A)$ is a $\mathcal{D}'$-ideal of $S$, and conversely if $B$ is a $\mathcal{D}'$-ideal of $S$, then $\varphi^{-1}(B)$ is a $\mathcal{D}$-ideal of $R$ containing $I$. Thus, in particular, if $A$ is a maximal proper $\mathcal{D}$-ideal of $R$, then $R/A$ is $\mathcal{D}'$-simple.

**Lemma 1.1.** Let $D$ be a derivation of $R$, $M$ a multiplicative system of $R$, and $h: R \to R_M$ the canonical homomorphism. Then, we can define a derivation on $R_M$, which we also call $D$, by

$$D(h(r)(h(m))^{-1}) = [h(m)h(D(r)) - h(r)h(D(m))](h(m^2))^{-1}.$$  

Furthermore, if $A$ is a $D$-ideal of $R$, then $h(A)R_M$ is a $D$-ideal of $R_M$, and if $B$ is a $D$-ideal of $R_M$, then $h^{-1}(B)$ is a $D$-ideal of $R$.

**Proof.** $\ker h = \{x \in R \mid xm = 0 \text{ for some } m \in M\}$ is a $D$-ideal of $R$ since $0 = D(xm) = xD(m) + mD(x) = xmD(m) + m^2D(x) = m^2D(x)$. Hence $D$ induces a derivation on $R/\ker h$, a derivation which can be then extended to $R_M$. The remainder of the lemma is straightforward.

**Lemma 1.2.** Let $\mathcal{D}$ be a family of derivations of $R$, and suppose that $R$ contains the rational numbers. Then, the radical of a $\mathcal{D}$-ideal of $R$ is a $\mathcal{D}$-ideal.

**Proof.** See [2, Lemma 1.8, p. 12].

**Corollary 1.3.** If $P$ is a minimal prime divisor of a $\mathcal{D}$-ideal
A, and P does not contain an integer ≠ 0, then P is a $\mathcal{D}$-ideal.

Proof. Localize at P and apply 1.1 and 1.2.

**Theorem 1.4.** Let A be a maximal proper $\mathcal{D}$-ideal of R, then
(i) A is primary.
(ii) If R/A has characteristic $p \neq 0$, then $\sqrt{A}$ is a maximal ideal.
(iii) If R/A has characteristic 0, then A is prime.

Proof. (i) Suppose $x, y \in R$, $x \in A$ and $xy \in A$; then, $\bigcup_{n=0}^{\infty} (A: y^n) \supset A: y > A$. But $\bigcup_{n=0}^{\infty} (A: y^n)$ is a $\mathcal{D}$-ideal; hence, by the maximality of A, $\bigcup_{n=0}^{\infty} (A: y^n) = R$ and there exists n such that $y^n \in A$.

(ii) Let P be a maximal ideal of R containing A. Consider the ideal $B = (A, \{x^p | x \in P\}) \subset P$; since R/A has characteristic p, B is a $\mathcal{D}$-ideal; hence, by the maximality of A, $B = A$ and $P = \sqrt{A}$.

(iii) Since R/A has characteristic 0, A contains no integer other than 0, hence the prime ideal $P = \sqrt{A}$ contains no integer either, and by 1.3 P is a $\mathcal{D}$-ideal. Then, by the maximality of A, $P = A$.

**Corollary 1.5.** Let R be of characteristic 0. Then R is $\mathcal{D}$-simple if R contains the rational numbers and has no prime $\mathcal{D}$-ideal different from (0) and (1). If R is $\mathcal{D}$-simple, then R is a domain.

One should note that a $\mathcal{D}$-simple ring R always contains a field, namely $F = \{x \in R \mid D(x) = 0 \text{ for all } D \in \mathcal{D}\}$; moreover, if the characteristic of R is $p \neq 0$, 1.4 shows that R is a primary ring and hence is equal to its total quotient ring; so this case will not be of interest in our further considerations, and throughout the remainder of this section we shall be dealing with a $\mathcal{D}$-simple ring of characteristic 0, which is then a domain containing the rational numbers.

**Definition 1.6.** Let R be a domain with quotient field K. An element $x \in K$ is said to be quasi-integral over R if there exists an element $d \in R$, $d \neq 0$, such that $dx^n \in R$ for all $n \geq 1$. The set $R'$ of all elements of K that are quasi-integral over R is a ring, called the complete integral closure of R. R is said to be completely integrally closed if $R = R'$. Note that if R is Noetherian, the concepts of integral dependence and quasi-integral dependence over R for elements of K become the same.

**Lemma 1.7.** Let R be a domain with quotient field K, S a ring
such that $R \subset S \subset K$, and $\mathcal{D}$ a family of derivations of $R$ regular on $S$. Then $S$ is $\mathcal{D}$-simple if $R$ is $\mathcal{D}$-simple.

Proof. If $B$ is any $\mathcal{D}$-ideal of $S$, then $B \cap R$ is a $\mathcal{D}$-ideal of $R$, and if $B$ is different from $(0)$ then $B \cap R$ is also different from $(0)$ since $S \subset K$.

**Theorem 1.8.** Let $R$ be a domain of characteristic 0 and $R'$ its complete integral closure. Then $R'$ is $\mathcal{D}$-simple if $R$ is $\mathcal{D}$-simple.

Proof. By [5, p. 168], any $D \in \mathcal{D}$ is regular on $R'$, hence the theorem follows from 1.7.

2. Example of a 1-dimensional local ring which is $D$-simple but not integrally closed. First, in this section, we modify an idea of Akizuki in [1] to construct some 1-dimensional local ring $R$ of arbitrary characteristic such that the integral closure $\bar{R}$ is not a finite $R$-module.

**Theorem 2.1.** Let $k$ be a field of arbitrary characteristic, $Y$ an indeterminate over $k$, $\pi = a_1Y + a_2Y^3 + \cdots + a_rY^{2r-1} + \cdots$ an element of $k[[Y]]$ which is transcendental over $k[Y]$. Set

$$\theta_1 = \pi Y^{-1}, \theta_r = (\theta_{r-1} - a_{r-1})Y^{-2r-1}$$

for $r \geq 2$ (alternatively $\theta_r = a_r + a_{r+1}Y^{2r} + \cdots + a_rY^{2r-2} + \cdots$); for $r \geq 1$, set

$$t_r = (\theta_r - a_r)^2 \quad \text{and} \quad \pi_r = \pi - (a_1Y + \cdots + a_rY^{2r-1})$$

Set also $T = k[Y, \pi, t_1, t_2, \ldots, t_r, \pi]$, and $P = (Y, \pi)T$. Note that $T \subset k[[Y]]$ and that $P \subset Yk[[Y]]$. Then,

(i) For $r > 1$, $\pi = Y^{2r}(a_r^2 + t_r) + 2a_rY\pi_r$ and $P$ is a maximal ideal of $T$.

(ii) For $r \geq 1$, $\pi_r^2 = Y^{2r+1-2}a_rY\pi_r$ and $k(Y, \pi)$ is the quotient field of $T$.

(iii) The ring $R = T$ is a 1-dimensional local domain.

(iv) The integral closure $\bar{R}$ of $R$ is not a finite $R$-module.

Proof. (i) For $r > 1$, we have

$$t_{r-1} = (\theta_{r-1} - a_{r-1})^2 = (Y^{2r-1}\theta_r)^2 = Y^{2r}(a_r^2 + t_r) + 2a_rY\pi_r(\theta_r - a_r)$$

But

$$Y^{2r}(\theta_r - a_r) = Y[\pi - (a_1Y + \cdots + a_rY^{2r-1})] = Y\pi_r,$$

1 Such an element exists; take for example $\pi = a_1Y + a_2Y^3 + \cdots + a_rY^{2r-1} + \cdots$ with $a_r \neq 0$ for every $r \geq 1$. 

hence \( t_{r+1} = Y^{2r}(a_r^2 + t_r) + 2a_r Y \pi_r \). Since furthermore \( P \subset Y k[[Y]] \), \( 1 \in P \), and \( P \) is a maximal ideal of \( T \).

(ii) 
\[
\pi_r = \pi - (a_r Y + \cdots + a_r Y^{2r-1}) \\
= Y^{2r-1}(a_{r+1} Y^{2r} + \cdots + a_r Y^{2r+2r-2} + \cdots) \\
= Y^{2r-1}(\theta_r - a_r) ;
\]
thus \( \pi_r^2 = Y^{2r+1-t_r} \) and \( k(Y, \pi) \) is the quotient field of \( T \).

(iii) Let us show that \( Y \) belongs to every nonzero prime ideal of \( R \). Since \( k(Y, \pi) \) is the quotient field of \( R \) it suffices to show that \( R[Y^{-1}] = k(Y, \pi) \). Let \( \beta \in k[Y, \pi] \); then \( \beta = \sum_{i=0}^n s_i \pi^i \) with \( s_i \in k[Y] \).

For any integer \( r \geq 1 \), set \( f_r = \sum_{i=0}^n s_i(a_r Y + \cdots + a_r Y^{2r-i}) \); then
\[
f_{r+1} = \sum_{i=0}^n s_i Y^{2r-i} + Y^{2r-1} \beta = f_r + Y^{2r+1-t_r} \cdot \text{ some } h_{r+1} \in k[Y], \]
with \( h_{r+1} \in k[Y] \), and since \( 2r^2 - 1 > r \), we have \( f_r = b_0 + b_1 Y + \cdots + b_r Y^r + Y^{r+1}g_r \) and
\[
f_{r+1} = b_0 + b_1 Y + \cdots + b_r Y^r + b_{r+1} Y^{r+1} + Y^{r+2}g_{r+1}
\]
with \( b_0, \ldots, b_r, b_{r+1} \in k \) and \( g_r, g_{r+1} \in k[Y] \). Now, since
\[
\pi = \pi_r + (a_r Y + \cdots + a_r Y^{2r-i}) , \quad \beta = \sum_{i=0}^n s_i \pi^i = \pi \cdot \delta_r + f_r
\]
with \( \delta_r \in T \). Hence, there exists \( \beta \in k[Y, \pi] \) such that
\[
(\ast) \quad \beta = \sum_{j=0}^r b_j Y^j + \pi \cdot \delta_r + Y^{r+1}g_r.
\]

Note that \( \pi_r \in P \) and therefore that \( \pi_r \) is a nonunit in \( R \).

If \( b_0 \neq 0 \), with \( r = 1 \), the relation \((\ast)\) gives that \( \beta = b_0 + (b_1 Y + \pi \cdot \delta_1 + Y^2 g_1) \) is a unit in \( R \) and thus that \( \beta^{-1} \in R \subset R[Y^{-1}] \).

If \( b_0 = b_1 = \cdots = b_{r-1} = 0 \) and \( b_r \neq 0 \), the relation \((\ast)\) gives \( \beta = Y^{r}(b_r + Y g_r) + \pi \cdot \delta_r \) where \( w_r = b_r + Y g_r \) is a unit in \( R \); then
\[
\beta(Y^{r}w_r - \pi \cdot \delta_r) = Y^{2r}w_r^2 - \pi \cdot \delta_r^2 = Y^{2r}(w_r - Y^{2r+1-2r-t_r} \cdot \delta_r^2)
\]
where \( w_r^2 - Y^{2r+1-2r} \cdot \delta_r^2 \) is a unit in \( R \), so that \( \beta^{-1} \in R[Y^{-1}] \).

If \( b_r = 0 \) for every \( r \geq 0 \), then by the relation \((\ast)\) we have
\[
\beta \in \bigcap_{r=1}^\infty (\pi_r, Y^{r+1}) T \subset \bigcap_{r=1}^\infty Y^{r+1}k[[Y]] = (0).
\]

Thus, if \( \beta \in k[Y, \pi] \), either \( \beta^{-1} \in R[Y^{-1}] \) or \( \beta = 0 \). If \( \eta \in k(Y, \pi) \), then \( \eta = \nu \lambda^{-1} \) with \( \nu, \lambda \in k[Y, \pi] \), \( \lambda \neq 0 \), so that \( \eta \in R[Y^{-1}] \); hence \( R[Y^{-1}] = k(Y, \pi) \).
Now, 
\[ \pi^2 = (Y\theta,)^2 = [a_i Y + (\theta_i - a_i) Y] = (t_i - a_i^2) Y^2 + 2a_i Y\pi \]
so that \( Y^{-1} \in R[\pi^{-1}] \), \( k(Y, \pi) = R[Y^{-1}] \subset R[\pi^{-1}] \), and \( \pi \) belongs also to every nonzero prime ideal of \( R \). Thus \( PR = (Y, \pi)R \), which is the unique maximal ideal of \( R \) and which is contained in every nonzero prime ideal of \( R \), is the only nonzero prime ideal of \( R \). As furthermore \( PR \) is finitely generated, \( R \) is a 1-dimensional local ring.

(iv) First, let us show that \( \theta_1 = \pi Y^{-1} \in T \). Suppose that \( \theta_1 \in T = \{ Y, \pi, t_1, \cdots, t_r, \cdots \} \); then \( \theta_1 = f(\pi, t_1, \cdots, t_r) \) where \( f \) is a polynomial in \( r + 1 \) indeterminates over \( k[Y] \). For \( r < s \), by (i), \( t_r \) can be expressed as a linear combination of 1, \( t_s \) and \( \pi \) with coefficients in \( k[Y] \), hence \( \theta_1 = f(\pi, t_1, \cdots, t_r) = F(\pi, t_r) = F(\pi, (\theta_r - a_r)^s) \) where \( F \) is a polynomial in two indeterminates over \( k[Y] \). Furthermore, by definition \( \theta_{r-1} = Y^{s-2} + a_{r-1} \), hence \( \theta_1 = Y^{s-2} + \beta \) with \( \beta \in k[Y] \) and we have

\[ (**\) \]
\[ Y^2 \beta = G(Y^s - \theta_s, (\theta_r - a_r)^s) \]
where \( G \) is a polynomial in two indeterminates over \( k[Y] \); but \( \pi \) being transcendental over \( k[Y] \), \( \theta_r \) is transcendental over \( k[Y] \) also, and the relation (**) has to be an identity, which is absurd. Thus, \( \theta_1 \in T \).

Now, let \( R^* \) be the completion of \( R \) with the \((PR)\)-adic topology; \( \{ \pi_r \}_{r \geq 0} \) is a Cauchy sequence in \( R \). Suppose that \( \pi_r \in P^2 R \) for some \( r \geq 1 \); since \( P^2 \) is a primary ideal of \( T \), we have \( \pi_r \in P^2 R \cap T = P^2 \subset YT \), and \( \pi = \pi_r + (a_i Y + \cdots + a_r Y^{r-1}) \in YT \) which is absurd since \( \theta_i \in T \). Thus, for every \( r \geq 0, \pi_r \in P^2 R \) and \( \beta = \lim \pi_r \) is \( \neq 0 \). However, we also have \( \beta = \lim \pi_r^s = \lim Y^{s-2} + a_r t_r = 0 \); hence \( R^* \) has a nonzero nilpotent element and \( R \) is not a finite \( R \)-module \([1, \text{p. 330}]\).

Example 2.2. Let \( Q \) be the rational numbers, \( (X_1, \cdots, X_r, \cdots) \) a set of indeterminates over \( Q \) and \( k = Q(X_1, \cdots, X_r, \cdots) \). Let
\[ \pi = b_1 X_1 Y + \cdots + b_r X_r Y^{r-1} + \cdots \]
be transcendental over \( k[Y] \) with \( b_i \in Q \setminus \{ 0 \} \) for every \( i \geq 1 \). Construct the rings \( T = k[Y, \pi, t_1, \cdots, t_r, \cdots] \) and \( R = T \) as in 2.1. On the quotient field \( k(Y, \pi) = Q(X_1, \cdots, X_r, \cdots; Y, \pi) \) define a derivation \( D \) by
\[
\begin{align*}
D(q) &= 0 \quad \text{for every} \quad q \in Q \\
D(Y) &= 1 \\
D(\pi) &= 3b_2 X_2 Y^2 + b_2 X_1 \\
D(X_i) &= 0
\end{align*}
\]

\(^2\) There exists such a \( \pi \) since \( k \) is countable.
\[ D(X_0) = -7b_2X_0^4 \]
\[
\vdots
\]
\[ D(X_t) = -(2^{i+1} - 1)b_{i+1}X_{t+1}Y^{2^{i+1}-2^{i-1}} \]

Then,

(i) \( D \) is regular on \( R \)

(ii) \( R \) is a 1-dimensional local \( D \)-simple ring which is not integrally closed.

**Proof.** (i) Since \( R = T_p \), it suffices to show that \( D(T) \subset R \). By definition of \( D \) we already have \( D(k) \subset R \), \( D(Y) \in R \) and \( D(\pi) \in R \); hence it remains to show that \( D(t_r) \in R \) for every \( r \geq 1 \). Differentiating \( \pi^r = Y^{2^{r+1} - 2}t_r \), we get \( 2\pi \cdot D(\pi) = Y^{2^{r+1} - 2}D(t_r) + (2^{r+1} - 2)Y^{2^{r+1} - 2}t_r \), but \( t_r \in YR \) by 2.1, hence \( D(t_r) \in R \) if and only if \( \pi \cdot D(\pi) \in Y^{2^{r+1} - 2}R \). Let us show that in fact we have \( D(\pi_r) \in Y^{2^{r+1} - 2}R \). From \( \pi_1 = \pi - b_1X_1 \) we get \( D(\pi) = D(\pi) - b_1X_1 = 3b_1X_1Y^{2}; \) by induction, if we suppose that \( D(\pi_r) = (2^r - 1)b_rX_rY^{2^{r-2}} \) and if we differentiate the relation \( \pi_r = \pi_{r-1} - b_rX_rY^{2^{r-1}} \), we get \( D(\pi_r) = (2^{r+1} - 1)b_{r+1}X_{r+1}Y^{2^{r+1} - 2} \in Y^{2^{r+1} - 2}R \). Hence \( D \) is regular on \( R \).

(ii) The only prime ideal of \( R \) which is not (0) or (1) is \( PR = (Y, \pi)R \); it is not a \( D \)-ideal since \( D(Y) = 1 \); thus by 1.5, \( R \) is \( D \)-simple. Furthermore by 2.1. \( R \) is a 1-dimensional local, not integrally closed, domain.

3. On the complete integral closure of a \( \mathcal{D} \)-simple ring. We have seen in the preliminaries that a \( \mathcal{D} \)-simple ring of characteristic \( p \neq 0 \) is equal to its total quotient ring. In this section we are concerned with rings of characteristic 0. Henceforth, \( R \) will denote a ring containing the integers.

**Theorem 3.1.** Let \( R \) be a ring, \( D \) a derivation on \( R \), \( P \) a prime ideal of \( R \) containing no \( D \)-ideal other than (0). Define \( v: R \setminus \{0\} \rightarrow \{ \text{nonnegative integers} \} \) by \( v(x) = n \) if \( D^{(i)}(x) \in P \) for \( i = 0, \ldots, n-1 \) and \( D^n(x) \notin P \). Then,

(i) \( R \) is domain.

(ii) \( v \) is rank-1-discrete valuation whose valuation ring \( R_v \) contains \( R \) and whose maximal ideal \( M_v \) lies over \( P \).

(iii) \( D \) is regular on \( R_v \) and \( R_v \) is \( D \)-simple.

**Proof.** (i) If \( n \) is any integer, \( D(n) = 0 \) and \( nR \) is a \( D \)-ideal of \( R \); hence 0 is the only integer contained in \( P \). Now, (0) is a \( D \)-ideal, hence by 1.3 any minimal prime divisor \( Q \) of (0) is a \( D \)-ideal also; then, by the hypothesis made on \( P \), we have (0) = \( Q \) and \( R \) is a domain.
(ii) Let \( x \) and \( y \) be two nonzero elements of \( R \), and let \( v(x) = n \), \( v(y) = m \), \( n \leq m \). For every \( i \) such that \( 0 \leq i \leq n - 1 \), both \( D^{(i)}(x) \) and \( D^{(i)}(y) \) belong to \( P \), hence \( D^{(i)}(x + y) \in P \) and
\[
v(x + y) \geq n = \inf \{v(x), v(y)\}.
\]

Let \( k \) be such that \( 0 \leq k \leq n + m - 1 \). For \( 0 \leq i \leq \inf \{k, n - 1\} \) we have \( D^{(i)}(x) \in P \), hence also \( C_i D^{(i)}(x) D^{(k-i)}(y) \in P \); for \( n \leq k \) and \( n \leq i \leq k \) we have \( 0 \leq k - i \leq k - n \leq m - 1 \), hence \( D^{(k-i)}(y) \in P \) and \( C_i D^{(i)}(x) D^{(k-i)}(y) \in P \); thus
\[
D^{(k)}(xy) = \sum_{i=0}^{k} C_i D^{(i)}(x) D^{(k-i)}(y) \in P.
\]

Now,
\[
D^{(m)}(xy) = \sum_{i=0}^{n+m} C_i D^{(i)}(x) D^{(n+m-i)}(y) \in P,
\]
whereas \( C_{n+m} D^{(m)}(x) D^{(m)}(y) \notin P \) since \( C_{n+m}, D^{(m)}(x), D^{(m)}(y) \in P \); thus
\[
D^{(n+m)}(xy) \notin P,
\]
and \( v \) is a valuation, rank-1-discrete since its value group is the group of integers. Furthermore, we obviously have \( R \subset R_v \) and \( M_v \cap R = P \).

(iii) Let \( ab^{-1} \) be any element of \( R_v \) with \( a, b \in R \), \( b \neq 0 \), \( v(a) \geq v(b) \); then \( D(ab^{-1}) = [bD(a) - aD(b)]b^{-2} \). If \( v(a) > v(b) \), then \( v(D(a)) = v(a) - 1 \geq v(b) \) and \( v(D(b)) \geq v(b) - 1 \) so that
\[
v(bD(a) - aD(b)) \geq \inf \{v(b) + v(D(a)), v(a) + v(D(b))\} \geq 2v(b)
\]
and \( D(ab^{-1}) \in R_v \). If \( v(a) = v(b) = 0 \), then \( v(bD(a) - aD(b)) \geq 0 = 2v(b) \) and \( D(ab^{-1}) \in R_v \). If \( v(a) = v(b) = n > 0 \), then \( v(bD(a) - aD(b)) = 2n - 1 \), so that \( D^{(k)}(bD(a) - aD(b)) \in P \) for every \( k \leq 2n - 2 \); furthermore we have
\[
D^{(2n-1)}(bD(a)) = \sum_{i=0}^{2n-1} C_{2n-i} D^{(i)}(b) D^{(2n-i)}(a) = \alpha_1 + C_{2n-1} D^{(n)}(b) D^{(n)}(a)
\]
with \( \alpha_1 \in P \), and similarly \( D^{(2n-1)}(aD(b)) = \alpha_2 + C_{2n-1} D^{(n)}(a) D^{(n)}(b) \) with \( \alpha_2 \in P \), so that \( D^{(2n-1)}(bD(a) - aD(b)) = \alpha_1 - \alpha_2 \in P \); hence, \( v(bD(a) - aD(b)) \geq 2n \) and \( D(ab^{-1}) \in R_v \). Thus \( D \) is regular on \( R_v \). Moreover, \( R_v \) is \( D \)-simple since if \( A \neq (0) \) were a \( D \)-ideal of \( R_v \), then \( A \cap R \neq (0) \) would be a \( D \)-ideal of \( R \) contained in \( P \), which would be absurd.

**Theorem 3.2.** Let \( R \) be a domain with quotient field \( K \), \( S \) a ring such that \( R \subset S \subset K \) and \( D \) a derivation of \( R \) regular on \( S \).
Let $P$ be a prime ideal of $R$ such that $R_P$ is $D$-simple. Then,

(i) There is at most one prime ideal $Q$ of $S$ lying over $P$, $Q$ being a minimal prime ideal when $P$ is.

(ii) If $S$ is the complete integral closure $R'$ of $R$ there is exactly one prime ideal $P'$ of $R'$ lying over $P$.

Proof. (i) Let $Q$ be a prime ideal of $S$ such that $Q \cap R = P$. Being regular on $S$, $D$ is also regular on $S_Q$, and $S_Q$ is $D$-simple since $S_Q \supset R_P$. Define $v: R\{0\} \to \{\text{nonnegative integers}\}$ by $v(x) = n$ if $D^{(n)}(x) \in P$ and $D^{(n-1)}(x) \in P$, and $w: S\{0\} \to \{\text{nonnegative integers}\}$ by $w(y) = m$ if $D^{(m)}(y) \in Q$ and $D^{(m-1)}(y) \in Q$.

and $D^{(m)}(y) \in Q$. By 3.1, $v$ and $w$ extend to valuations of $K$; furthermore, for $x \in R$ we have $D^{(n)}(x) \in P$ if and only if $D^{(n)}(x) \in Q$ since $Q \cap R = P$; hence $v = w$, and $Q = M_v \cap S$ where $M_v$ is the maximal ideal of the valuation ring $R_v$ of $v$.

If $P$ is a minimal prime ideal of $R$, suppose that $Q'$ is a prime ideal of $S$ such that $0 < Q' \subset Q$. We have $0 < Q' \cap R \subset Q \cap R = P$ and $Q' \cap R = P$ by the minimality of $P$; then $Q' = Q$ since $Q$ is the only prime ideal of $S$ lying over $P$.

(ii) By [5, p. 168] every derivation of $R$ is regular on $R'$. Being a rank-1 valuation ring, $R_v$ is completely integrally closed and contains $R'$. Then, $P' = M_v \cap R'$ is a prime ideal of $R'$ lying over $P'$; of course, by (i), $P'$ is unique.

Theorem 3.3. Let $R$ be a Noetherian $D$-simple ring and $R$ its integral closure. Let $\{P_\alpha\}_{\alpha \in \Lambda}$ be the set all the minimal prime ideals of $R$. Then,

(i) For every $\alpha \in \Lambda$, there exists $D \in D$ such that $R_{P_\alpha}$ is $D$-simple, and there exists a unique prime ideal $P_\alpha$ of $\bar{R}$ lying over $P_\alpha$.

(ii) $\{P_\alpha\}_{\alpha \in \Lambda}$ is the set of all the minimal prime ideals of $\bar{R}$.

(iii) Let $D \in D$ such that $D(P_\alpha) \not\subset P_\alpha$, $w_\alpha$ the valuation associated by 3.1, and $R_\alpha$ its valuation ring. Then $R_\alpha = \bar{R}_{P_\alpha}$ (hence, any two derivations $D$ and $D'$ such that $D(P_\alpha) \not\subset P_\alpha$ and $D'(P_\alpha) \not\subset P_\alpha$ give rise to the same valuation $w_\alpha$).

(iv) $\bar{R} = \bigcap_{\alpha \in \Lambda} R_\alpha$.

Proof. (i) Being $D$-simple, $R$ is a domain containing the rational numbers, and for any $\alpha \in \Lambda$, there exists $D \in D$ such that $D(P_\alpha) \not\subset P_\alpha$, and by 1.3, $R_{P_\alpha}$ is $D$-simple. Then, by 3.2, there exists a unique prime ideal $P_\alpha$ of $\bar{R}$ lying over $P_\alpha$. 
(ii) That every $P_\alpha$ is a minimal prime ideal of $\bar{R}$ is given by 3.2. Now, let $\bar{P}$ be a minimal prime ideal of $\bar{R}$, and let $P = \bar{P} \cap R$; let $M$ be a minimal prime ideal of $R$ contained in $P$; by [3, (10.8), p. 30] there exists a prime ideal $\bar{M}$ of $\bar{R}$ lying over $M$; since $\bar{P}$ is the only prime ideal of $\bar{R}$ lying over $P$, we have $\bar{M} \subset \bar{P}$ by [3, (10.9), p. 30], hence $\bar{M} = \bar{P}$, and $P = \bar{P} \cap R = M$ is a minimal prime ideal of $R$.

(iii) Since $R$ is Noetherian, $\bar{R}$ is a Krull ring [3, (33.10), p. 118], and $\bar{R}_{\bar{P}_a}$ is a rank-1-discrete valuation ring. As furthermore $\bar{R}_{\bar{P}_a} \subset R_a$ we get $\bar{R}_{\bar{P}_a} = R_a$.

(iv) $\bar{R}$ is a Krull ring and $\{\bar{P}_a\}_{a \in A}$ is the set of all the minimal prime ideals of $\bar{R}$; thus $\bar{R} = \bigcap_{a \in A} \bar{R}_{\bar{P}_a} = \bigcap_{a \in A} R_a$.

**Corollary 3.4.** Let $R$ be a Noetherian $\mathcal{D}$-simple ring with quotient field $K$. Let $S$ be a ring such that $R \subset S \subset K$ and such that every $D \in \mathcal{D}$ is regular on $S$. Then, the following statements are equivalent:

(i) For every minimal prime ideal $P$ of $R$ there exists a (unique) prime ideal $Q$ of $S$ lying over $P$.

(ii) $S$ is integral over $R$.

(iii) For every prime ideal $M$ of $R$ there exists a (unique) prime ideal $N$ of $S$ lying over $M$.

**Proof.** That (ii) $\Rightarrow$ (iii) is a consequence of [3, (10.7), p. 30] and 3.2; that (iii) $\Rightarrow$ (i) is obvious. Now, let $\{P_a\}_{a \in A}$ be the set of the minimal prime ideals of $R$, $\{w_a\}_{a \in A}$ the associated valuations and $\{R_a\}_{a \in A}$ the valuation rings of the $w_a$'s. For any $a \in A$, let $D \in \mathcal{D}$ be such that $D(P_a) \subset P_a$, and let $Q_a$ be a prime ideal of $S$ lying over $P_a$, $S_{Q_a}$ is $D$-simple, the valuation associated to $Q_a$ is equal to $w_a$ and $S \subset R_a$. Hence, $S \subset \bar{R} = \bigcap_{a \in A} R_a$.

**Corollary 3.5.** Let $R$ be a Noetherian $\mathcal{D}$-simple ring with quotient field $K$, and $\bar{R}$ its integral closure. Then,

(i) $\bar{R}$ is the largest $\mathcal{D}$-simple overring of $R$ in $K$ having a prime ideal lying over every prime ideal of $R$.

(ii) $\bar{R}$ is the largest $\mathcal{D}$-simple overring of $R$ in $K$ having a prime ideal lying over every minimal prime ideal of $R$.

**Proof.** Apply 3.4.

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