ON NONNEGATIVE MATRICES

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The following characterisation of totally indecomposable nonnegative \( n \)-square matrices is introduced: A nonnegative \( n \)-square matrix is totally indecomposable if and only if it diminishes the number of zeros of every \( n \)-dimensional nonnegative vector which is neither positive nor zero. From this characterisation it follows quite easily that:

I. The class of totally indecomposable nonnegative \( n \)-square matrices is closed with respect to matrix multiplication.

II. The \((n - 1)\)-st power of a matrix of that class is positive.

A very short proof of two equivalent versions of the König-Frobenius duality theorem on \((0,1)\)-matrices is supplied at the end.

A matrix is called nonnegative or positive according as all its elements are nonnegative or positive respectively. An \( n \)-square matrix \( A \) is said to be decomposable if there exists a permutation matrix \( P \) such that \( PAP^T = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix} \), where \( B \) and \( D \) are square matrices; otherwise it is indecomposable. \( A \) is said to be partly decomposable if there exist permutation matrices \( P, Q \) such that

\[ PAQ = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix}, \]

where \( B \) and \( D \) are square matrices; otherwise it is totally indecomposable.

Whereas the notion of indecomposable matrices first appeared in 1912 in a paper by Frobenius [2] dealing with the spectral properties of nonnegative matrices, totally indecomposable matrices were introduced fairly recently apparently by Marcus and Minc [10]. Their properties have been studied in several papers on inequalities for the permanent function.

In [11] Minc gives the following characterisation of totally indecomposable matrices:

A nonnegative \( n \)-square matrix \( A, n \geq 2 \), is totally indecomposable if and only if every \((n - 1)\)-square submatrix of \( A \) has a positive permanent.

A well-known theorem states:

**Theorem 1.** If \( A \) is an indecomposable nonnegative \( n \)-square matrix then...
An indecomposable matrix is primitive if its characteristic value of maximum modulus is unique.

Wielandt [15] states (without proof) that for primitive $n$-square matrices we have

$$A^{n^2-2n+2} > 0.$$ \(^1\)

By using solely the properties of total indecomposability we establish a different characterisation for totally indecomposable matrices from the one given by Minc. Using part of the characterisation we show that if $A$ is a totally indecomposable nonnegative $n$-square matrix then $A^{n-1} > 0$. This result is best possible as for every $n$ there exist totally indecomposable $n$-square matrices $A$ for which $A^{n^2} \geq 0$. Theorem 1 then follows as a corollary of the latter result.

We should like to point out that Theorem 2 is by no means essential for the proof of Theorem 3. Two independent proofs of Theorem 3 are given in § 4. It seems justified however to present Theorem 2 on its own merit.

We conclude with a very short proof of two equivalent versions of König's theorem on matrices.

2. Preliminaries. $|S|$ denotes the number of elements of a given set $S$. Let $M_n$ be the set of all nonnegative $n$-square matrices, let $D_n$ be the subset of $M_n$ of indecomposable matrices and let $T_n$ be the subset of $D_n$ of totally indecomposable matrices. Let $A \in M_n$ and let $p$ and $q$ be nonempty subsets of $N = \{1, \cdots, n\}$. Then $A[p \mid q]$, $A(p \mid q)$ is the $|p| \times |q|$ submatrix of $A$ consisting precisely of those elements $a_{ij}$ of $A$ for which $i \in p$ and $j \in q$, $i \in p$ and $j \in q$ respectively. $A[p \mid q]$ and $A(p \mid q)$ are defined accordingly. We can now formulate equivalent definitions for matrices in $D_n$ and $T_n$:

D. 1. $A \in D_n$ if $A[p \mid N - p] \neq 0$ for every nonempty $p \subset N$.

D. 2. $A \in T_n$ if $A[p \mid q] \neq 0$ for any nonempty subsets $p$ and $q$ of $N$ such that $|p| + |q| = n$.

Let us now establish some connections between indecomposable and totally indecomposable matrices.

**Lemma 1.** If $A \in (D_n - T_n)$ then $A$ has a zero on its main diagonal. \(^2\)

**Proof.** Since $A \notin T_n$ there exists a zero-submatrix $A[p \mid q]$ with $|p| + |q| = n$; but since $A \in D_n$, $p \cap q \neq \emptyset$, which means that $A$ has

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\(^1\) A proof is supplied in [5].

\(^2\) Lemma 1 is part of Lemma 2.3 in [1] but the shortness of our proof seems to justify its presentation.
a zero on its main diagonal.

**Corollary 1.** If \( A \in D_n \) then \( A + I \in T_n \).

Proof obvious.

3. The main results. Let \( A = (a_{ij}) \in M_n \) and let \( v \) denote an \( n \)-dimensional vector with \( a_i(v) \) its \( i \)th entry.

Define: \( J_k = \{ j : a_{kj} = 0 \} \), \( I_k = \{ i : a_{ik} = 0 \} \),

\[
I_0(v) = \{ i : a_i(v) = 0 \}, \quad I_+(v) = \{ i : a_i(v) > 0 \}.
\]

Let \( R_n \) denote the space of \( n \)-tuples of real numbers.

Let \( X_n \) be the set of all nonnegative vectors in \( R_n \) which are neither positive nor zero. We then have the following

**Theorem 2.** A nonnegative \( n \)-square matrix \( A \) is totally indecomposable if and only if \( |I_0(Ax)| < |I_0(x)| \) for every \( x \in X_n \).

**Proof.** Let \( A \in T_n \) and \( x \in X_n \). A necessary and sufficient condition for \( a_{i_0}(Ax) = 0 \) for some \( i_0 \) is

(1) \( I_+(x) \subseteq J_{i_0} \).

If \( I_0(Ax) = \emptyset \), then there is nothing to prove, so we may assume

(2) \( I_0(Ax) \neq \emptyset \).

\( x \in X_n \) implies

(3) \( I_+(x) \neq \emptyset \).

(1), (2) and (3) imply that \( A[I_0(Ax) \mid I_+(x)] \) is a zero-submatrix of \( A \). Since \( A \in T_n \) by assumption, we have (by D. 2.)

\[
|I_0(Ax)| + |I_+(x)| < n = |I_0(x)| + |I_+(x)|
\]

and hence \( |I_0(Ax)| < |I_0(x)| \) which proves the first part of the theorem. (It is not generally true however that \( I_0(Ax) \subseteq I_0(x) \) as it may happen that \( a_i(x) > 0 \) and \( a_i(Ax) = 0 \), a situation which differs somewhat from that in the similar case for indecomposable matrices (5.2.2 in [9])).

Let now \( A \in T_n \). Then \( A \) contains a zero-submatrix \( A[I \mid J] \) such that \( I, J \neq \emptyset \) and \( |I| + |J| = n \). Choose now \( x \in R_n \) such that

(4) \( I_+(x) = J \).

Then clearly \( x \in X_n \). We have \( I_0(x) = N - I_+(x) = N - J \), and hence \( |I_0(x)| = |I| \). For \( i \in I \) we have \( J_i \supseteq J \), and hence by (4) \( I_+(x) \subseteq J_i \),
so that for \( i \in I \) according to (1) \( a_i(Ax) = 0 \) and hence \( I_0(Ax) \supseteq I \). Then \( |I_0(Ax)| \geq |I| = |I_0(x)| \). This completes the proof.

\( X_n \) in Theorem 2 may of course be replaced by its subset \( Y_n \) consisting of the \( 2^n - 2 \) zero-one vectors.

Theorem 2 admits of two simple corollaries which we present as Theorems 3 and 4.

**THEOREM 3.** If \( A \) is a totally indecomposable nonnegative \( n \)-square matrix then

\[ A^{n-1} > 0. \]

*Proof.* If for some \( j_0 \) we had \( |I_{j_0}| \geq n - 1 \) then \( A \) would be partly decomposable and hence \( |I_{j_0}| \leq n - 2 \) for \( j \in N \) and the rest follows.

Theorem 1 follows from Theorem 3 as an immediate consequence of Corollary 1. For \( A = I + P \) where \( P \) is the \( n \)-square permutation matrix with ones in the superdiagonal, so that \( a_{ij} = 1 \) if \( i = j \) or \( i = j - 1 \), \( a_{ii} = 1 \) and \( a_{ij} = 0 \) otherwise, it is easy to show that \( A^{n-2} > 0 \), which shows that our result is best possible.

**THEOREM 4.** The product of any finite number of totally indecomposable nonnegative \( n \)-square matrices is totally indecomposable.

*Proof.* It is clearly sufficient to prove the statement for two matrices. Let therefore \( A, B \in T_n \). Choose an arbitrary element \( x \) of \( X_n \). We then have

\[ |I_0(ABx)| \leq |I_0(Bx)| < |I_0(x)| \quad (5) \]

by Theorem 2. Since \( x \) was arbitrary, (5) applies to all elements of \( X_n \). Again by Theorem 2 it follows that \( AB \) is totally indecomposable, which proves the theorem.

4. **Independent proofs of Theorem 3.** A lemma of Gantmacher [3] states that if \( A \in D_n \) and \( x \in X_n \), then \( I_0((A + I)x) \subseteq I_0(x) \).

The following proof of Theorem 3 assuming the lemma has been suggested by London: Let \( A \in T_n \). Using the fact that a matrix in \( T_n \) possesses a positive diagonal \( d \), put

\[ A_1 = \frac{1}{\alpha} P^r(A - \alpha P) = \frac{1}{\alpha} P^r A - I \] where \( 0 < \alpha < \min a_{ij} (a_{ij} \in d) \)

\[ ^3 D. London, oral communication. \]
and \( P = (p_{ij}) \) is an \( n \)-square permutation matrix such that \( p_{ij} = 1 \) if and only if \( a_{ij} \in d \). Then \( A \in T_n \) implies \( A_1 \in T_n \).

We have \( A = \alpha P(A_1 + I) \); since \( A_1 \in D_n \) we obtain
\[
I_0(Ax) = I_0(P(A_1 + I)x) = I_0((A_1 + I)x) \subset I_0(x),
\]
for \( x \in X_n \). Then \( I_0(A^{n-1}x) = \emptyset \), and \( A^{n-1} > 0 \).

Another proof has been kindly suggested by the referee of this paper: We show that if \( A \) is totally indecomposable, then if \( x \in X_n \), then
\[
|I_0(Ax)| < |I_0(x)|.
\]
The theorem then follows immediately.

Suppose \( |I_0(Ay)| \geq |I_0(y)| \) for some \( y \in X_n \).

Put \( |I_0(y)| = s \). There are permutation matrices \( P \) and \( Q \) such that
\[
PAy = \begin{bmatrix} 0 \\ u \end{bmatrix} \quad \text{and} \quad Q^ty = \begin{bmatrix} 0 \\ v \end{bmatrix}
\]
where \( u \) is an \((n - s)\)-dimensional nonnegative vector and \( v \) is an \((n - s)\)-dimensional positive vector: The 0’s represent \( s \) zero components in each case.

We now write \( PAQ = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \) where \( A_1 \) is \( s \times s \), \( A_2 \) is \( s \times (n - s) \), \( A_3 \) is \((n - s) \times s \) and \( A_4 \) is \((n - s) \times (n - s) \). Then \( \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} 0 \\ V \end{bmatrix} = \begin{bmatrix} 0 \\ u \end{bmatrix} \)
and so \( A_2 V = 0 \). Thus \( A_2 = 0 \) and hence \( A \not\in T_n \), a contradiction.

5. König’s Theorem. Let \( A \) be an \( m \times n \) matrix. A covering of \( A \) is a set of lines (rows or columns) containing all the positive elements of \( A \). A covering of \( A \) is a minimal covering of \( A \) if there does not exist a covering of \( A \) consisting of fewer lines. Let \( M(A) \) denote the number of lines in a minimal covering of \( A \). A basis of \( A \) is a positive subdiagonal of \( A \) of maximal length. \( m(A) \) denotes the length of a basis of \( A \). The \( j \)th column of \( A \) is essential to \( A \) if \( M(A(\{j\})) < M(A) \).

We now give the two versions of König’s Theorem and their proofs:

K. T. 1. If \( A \) is an \( m \times n \) matrix, then \( m(A) = M(A) \).

K. T. 2. If \( A \) is an \( n \)-square matrix, then \( A \) has \( k \) zeros on every diagonal \((k > 0)\) if and only if \( A \) contains an \( s \times t \) zero-submatrix with \( s + t = n + k \).

This is a generalized version of a theorem of Frobenius. The following theorem appears in [8] (we reproduce it here in a hypothetical form).
E. T.: If $A$ is an $m \times n$ matrix and K.T.I. holds for $A$, then there exists a minimal covering of $A$ (called essential covering) containing precisely the essential columns of $A$ (and may be some rows).

**Proof of K. T. 1.** $m(A) \leq M(A)$ holds trivially. The theorem is clearly true for $1 \times n$ matrices for all $n$. Assume that the theorem is true for all $\mu \times n$ matrices, $\mu < m$ and all $n$. Let $A$ be an $m \times n$ matrix. Consider $A' = A({m}\setminus N]$. $A'$ is an $(m - 1) \times n$ matrix so that K.T.1, holds for $A'$ and hence E.T. holds for $A'$. Let $Q$ be the essential covering of $A'$.

**Case 1.** $Q$ is a covering of $A$. Then $m(A) \geq m(A') = M(A') \geq M(A)$.

**Case 2.** $Q$ is not a covering of $A$. Then there exists $j_0 \in N$ for which $a_{m,j_0} > 0$ which is not covered by $Q$ and hence the $j_0$th column is not essential to $A'$. Then clearly there exists a basis $b'$ of $A'$ without elements in the $j_0$th column. Then $b = b' \cup \{a_{m,j_0}\}$ is a sub-diagonal of $A$ and hence $M(A) \leq M(A') + 1 = m(A') + 1 \leq m(A)$. This proves K. T. 1.

**Proof of K. T. 2.** Necessity. If $A$ has $k$ zeros on every diagonal then $m(A) \leq n - k$. By K.T.1, $M(A) \leq n - k$. Apply a minimal covering to $A$. Then there remains an $s \times t$ zero-matrix of $A$ which is not covered, with $s + t \geq 2n - M(A) \geq n + k$.

Sufficiency. Let $A$ contain an $s \times t$ zero-submatrix with $s + t = n + k$. Then there are positive elements on at most $2n - (n + k) = n - k$ lines, meaning that there are at least $k$ zero-rows, which proves the sufficiency.

**References**


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