

Pacific Journal of Mathematics

**SINGULAR PERTURBATIONS OF DIFFERENTIAL EQUATIONS
IN ABSTRACT SPACES**

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In a recent paper, Kisynski studied the solutions of the abstract Cauchy problem $\varepsilon x''(t) + x'(t) + Ax(t) = 0$, $x(0) = x_0$ and $x'(0) = x_1$ where $0 \leq t \leq T$, $\varepsilon > 0$ is small parameter and A is a nonnegative self-adjoint operator in a Hilbert space H . With the aid of the functional calculus of the operator A , he has showed that as $\varepsilon \rightarrow 0$ the solution of this problem converges to the solution of the unperturbed Cauchy problem $x'(t) + Ax(t) = 0$, $x(0) = x_0$. Smoller has proved the same result for equation of higher order.

The purpose of this paper is to study the solution of a similar problem and allowing the operator A to depend on t .

To be precise, we shall show that if the initial data is taken from a suitable dense subset of H , then the solution of the Cauchy problem:

$$(1.1) \quad \varepsilon x''(t) + x'(t) + A(t)x(t) = 0, \quad x(0) = x_0, \quad x'(0) = x_1$$

converges to the solution of the unperturbed Cauchy problem

$$(1.2) \quad x'(t) + A(t)x(t) = 0, \quad x(0) = x_0$$

as $\varepsilon \rightarrow 0$ where $0 \leq t \leq T$, $\varepsilon > 0$ is a small parameter, $A(t)$ is a continuous semi-group of nonnegative self-adjoint operators in H with infinitesimal generator A .

2. The problem (1.1) when $H = R_1$. Before considering (1.1) in the general case, it is necessary to consider (1.1) in the case when $H = R_1$ (i.e., the real line). Thus we consider the Cauchy problem:

$$(2.1) \quad \varepsilon u''(t) + u'(t) + e^{\mu t}u(t) = 0. \quad u(0) = x_0, \quad u'(0) = x_1$$

when $t \geq 0$, $\mu \geq 0$, $\varepsilon > 0$.

According to theorem (1) in [2], equation (2.1) has two linearly independent solutions:

$$u_1 = \sum_0^{m-1} u_{1j}(t)\varepsilon^j + \varepsilon^m E_0, \quad u_i = \sum_0^{m-1} u_{ij}(t)\varepsilon^j + \varepsilon^{m-1} E_1$$

$$u_2 = \sum_0^{m-1} u_{2j}(t)\varepsilon^j e^{-t/\varepsilon} + \varepsilon^m E_0, \quad u_2 = \sum_0^{m-1} (d/dt)[u_{2j}(t)e^{-t/\varepsilon}]\varepsilon^j + \varepsilon^{m-1} E_1$$

where $u_{ij}(t)$ ($i = 1, 2$) are C^∞ functions on $[0, T]$ and $u_{i0}(t)$ ($i = 1, 2$) does not vanish at any point of $[0, T]$ and E_0, E_1 are functions of ε and others, but bounded for small $\varepsilon \geq 0$.

Hence the general solution of equation (2.1) is $u = c_1 u_1 + c_2 u_2$. Solving for c_1 and c_2 by using the initial condition we obtain $u = x_0 s_{00} + x_1 s_{01}$ and $u^* = x_0 s_{10} + x_1 s_{11}$ where

$$(2.3) \quad \begin{aligned} s_{00} &= H^{-1}(\varepsilon)[u_2(0)u_1(t) - u_1(0)u_2(t)] \\ s_{01} &= H^{-1}(\varepsilon)[u_1(0)u_2(t) - u_2(0)u_1(t)] \\ s_{10} &= s_{00} = \frac{d}{dt}s_{00} \\ s_{11} &= s_{01} = \frac{d}{dt}s_{01} \end{aligned}$$

and

$$H(\varepsilon) = u_1(0)u_2(0) - u_2(0)u_1(0)$$

How taking the limit as $\varepsilon \rightarrow 0$, we find that

$$(2.4) \quad \begin{aligned} s_{00}(t, \varepsilon, \mu) &\longrightarrow x_0 u_{10}(t) \\ s_{01}(t, \varepsilon, \mu) &\longrightarrow 0. \end{aligned}$$

Consequently, $u(t, \varepsilon) \rightarrow x_0 u_{10}(t)$. From equation 15 in [2] we find that $u_{10}(t)$ is the solution of the equation

$$(2.5) \quad u^* + e^{\mu t} u = 0$$

and this is what we wished to show.

3. Estimates for the Functions $s_{ij}(t, \varepsilon, \mu)$. In this section we would like to find estimates for the functions $s_{ij}(t, \varepsilon, \mu)$ ($i, j = 0, 1$). We may do so by solving for $u_{ij}(t)$ ($i = 1, 2; j = 0, 1, \dots, m-1$) from equation 15 in [2]. Since this would be rather tedious we will take the simpler approach of estimating $u_i(t, \varepsilon, \mu)$ and $u_i(t, \varepsilon, \mu)$ ($i = 1, 2$). Multiplying (2.1) by u^* and integrating between 0 and t we obtain:

$$\frac{\varepsilon u^{*2}}{2} + \int_0^t u^{*2} + \frac{u^2}{2} e^{\mu t} - \frac{1}{2} \mu \int_0^t u^2 e^{\mu t} = c.$$

Consequently

$$u^2 \leq 2|c| + \mu \int_0^t u^2 e^{\mu t} dt.$$

Now using Bellman's lemma, we obtain

$$(3.1) \quad w^2 \leq 2/c/e^{\mu t}.$$

For estimating $u^*(t)$, we multiply equation (2.1) by $e^{-\mu t}u^*$, integrating between 0 and t and using Bellman's lemma we obtain:

$$(3.2) \quad u^{*2}(t) \leq 2\varepsilon^{-1}/c/e^{2\mu t}.$$

In [2] page 323 we proved that for all small $\varepsilon \geq 0$ $H(\varepsilon) \neq 0$, therefore we see that (2.3), (3.1), and (3.2) yield,

$$(3.3) \quad |s_{00}| \leq K(\varepsilon) \exp\left(\frac{e^{\mu t}}{2}\right)$$

$K(\varepsilon)$ is a bounded function in ε , and

$$(3.4) \quad |s_{01}| \leq \bar{K}(\varepsilon) \exp(e^{\mu t/2})$$

$\bar{K}(\varepsilon)$ is a bounded function in ε .

To obtain an estimate for s_{ij} ($i, j = 1, 2$) we write equation (2.1) in amatrix form as:

$$U \cdot = AU$$

when

$$A = \begin{pmatrix} 0 & 1 \\ -\bar{\varepsilon}^{-1} \exp(\mu t) & -\bar{\varepsilon}^{-1} \end{pmatrix}.$$

Hence

$$U = \exp\left[\int A(s)ds\right] = \begin{pmatrix} s_{00} & s_{01} \\ s_{10} & s_{11} \end{pmatrix}$$

and from the equation

$$(3.5) \quad \begin{aligned} (d/dt) \begin{pmatrix} s_{00} & s_{01} \\ s_{10} & s_{11} \end{pmatrix} &= \begin{pmatrix} s_{00} & s_{01} \\ s_{10} & s_{11} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\bar{\varepsilon}^{-1} \exp(\mu t) & -\bar{\varepsilon}^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -\bar{\varepsilon}^{-1} \exp(\mu t) & -\bar{\varepsilon}^{-1} \end{pmatrix} \begin{pmatrix} s_{00} & s_{01} \\ s_{10} & s_{11} \end{pmatrix} \end{aligned}$$

we obtain

$$(3.6) \quad s_{10} = -s_{01}\bar{\varepsilon}^{-1} \exp(\mu t)$$

$$(3.7) \quad s_{11} = s_{00} - \bar{\varepsilon}^{-1}s_{01}.$$

4. The problem (1.1) in abstract Hilbert space. We shall now consider the problem (1.1) in any Hilbert space H with norm $\|\cdot\|$.

Since $\{A(t)\}$ is a semi-group of a nonnegative selfadjoint operator in H , with infinitesimal generator A , there is a resolution of the identity E_μ such that $A(t)$ has the spectral representation:

$$A(t) = \int_0^\infty e^{\mu t} dE_\mu.$$

We shall next use the functional calculus of the operator $A(t)$. For fixed $\varepsilon > 0$, $t \geq 0$, we define the operator S_{ij} on H by

$$(4.1) \quad S_{ij}(t, \varepsilon) = \int_0^\infty s_{ij}(t, \varepsilon, \mu) dE_\mu \quad (i, j = 0, 1)$$

where the $s_{ij}(t, \varepsilon, \mu)$ are defined by (2.3). If we let D denote the dense domain of the operator $e^{A^2(t)}$ for all t , then our estimates (3.2) through (3.7) imply that D is contained in the domain of $S_{ij}(t, \varepsilon)$ for every $i, j = 0, 1$.

For x_0 and x_1 in D , we write

$$(4.2) \quad x_\varepsilon(t) = S_{00}(t, \varepsilon)x_0 + S_{01}(t, \varepsilon)x_1$$

and we see that $x_\varepsilon(t)$ is in the domain of $A(t)$ for every $\varepsilon > 0$. We now state the main theorem.

THEOREM. *Let $x_\varepsilon(t)$ be defined as in (4.2) when x_0, x_1 are in D . Then $x_\varepsilon(t)$ is the unique solution of the Cauchy problem (1.1) and $x_\varepsilon(t)$ converges to the solution of (1.2) as $\varepsilon \rightarrow 0$.*

To prove this theorem we first prove the following lemmas:

LEMMA 1. *For $x \in D$, $(d/dt)S_{ij}(t, \varepsilon)x$ exists and*

$$(4.3) \quad (d/dt)S_{ij}(t, \varepsilon)x = \int_0^\infty (d/dt)s_{ij}(t, \varepsilon, \mu) dE_\mu x \quad (i, j = 0, 1).$$

Proof. We shall prove the lemma for $i = j = 0$. Since the proofs for the other cases are similar, they will be omitted. For $x \in D$ and $t \geq 0$ fixed, we have:

$$\begin{aligned} & \left\| \frac{S_{00}(t + \Delta t, \varepsilon) - S_{00}(t, \varepsilon)}{\Delta t} \times -S_{10}(t, \varepsilon)x \right\|^2 \\ &= \int_0^\infty \left[\frac{s_{00}(t + \Delta t, \varepsilon, \mu) - s_{00}(t, \varepsilon, \mu)}{\Delta t} - s_{10}(t, \varepsilon, \mu) \right]^2 d \| E_\mu x \|^2 \\ &= \int_0^\infty [s_{10}(t', \varepsilon, \mu) - s_{10}(t, \varepsilon, \mu)]^2 d \| E_\mu x \|^2, \end{aligned}$$

where $t \leq t' \leq t + \Delta t$, using the theorem of the mean and (2.3).

Now there is a T such that $t + \Delta t \leq T$ for all Δt sufficiently small, so that if we use (3.3) through (3.7) we see that

$$\begin{aligned} |s_{10}(t', \varepsilon, \mu) - s_{10}(t, \varepsilon, \mu)| &\leq |s_{10}(t', \varepsilon, \mu)| + |s_{10}(t, \varepsilon, \mu)| \\ &\leq \varepsilon^{-1} e^{\mu T} K(\varepsilon) e^{(1/2)\varepsilon \mu T} \leq N(\varepsilon, T) e^{\varepsilon \mu T} \end{aligned}$$

where $N(\varepsilon, T)$ is a constant depending on T and ε only. Therefore the function $|s_{10}(t', \varepsilon, \mu) - s_{10}(t, \varepsilon, \mu)|^2$ is summable with respect to the measure $d \|E_{\mu} x\|^2$ if Δt is sufficiently small. Furthermore,

$$\lim_{\Delta t \rightarrow 0} [s_{10}(t', \varepsilon, \mu) - s_{11}(t, \varepsilon, \mu)]^2 = 0.$$

So that the Lebesgue dominated convergence theorem yields:

$$\lim_{\Delta t \rightarrow 0} \int_0^{\infty} [s_{10}(t', \varepsilon, \mu) - s_{10}(t, \varepsilon, \mu)]^2 d \|E_{\mu} x\|^2 = 0.$$

This completes the proof of the lemma.

LEMMA 2. For $x \in D$ and $t \geq 0$, we have

$$(4.4) \quad \lim_{\varepsilon \rightarrow 0} \left\| S_{00}(t, \varepsilon)x - \exp\left(-\int A(s)ds\right)x \right\| = 0$$

$$(4.5) \quad \lim_{\varepsilon \rightarrow 0} \|S_{01}(t, \varepsilon)x\| = 0.$$

Proof.

$$\begin{aligned} &\left\| S_{00}(t, \varepsilon)x - \exp\left(-\int A(s)ds\right)x \right\|^2 \\ &= \int_0^{\infty} \left| \left(s_{00}(t, \varepsilon, \mu) - \exp\left(-\int^t e^{\mu s} ds\right) \right) \right|^2 d \|E_{\mu} x\|^2. \end{aligned}$$

From (3.3) we see that $\left[s_{00}(t, \varepsilon, \mu) - \exp\left(-\int^t e^{\mu s} ds\right) \right]^2$ is summable with respect to the measure $d \|E_{\mu} x\|^2$ and, as we have seen in (2.4) and (2.5), the integrand converges pointwise to zero. We apply the Lebesgue dominated convergence theorem to conclude that the integral likewise converges to zero as $\varepsilon \rightarrow 0$. This proves (4.4). Relation (4.5) follows from (2.4) and (2.5) likewise.

LEMMA 3. Let B be a bounded operator in H . If $x'(t) + Bx(t) = 0$, $0 \leq t \leq 0$, and $x(0) = 0$, then $x(t) \equiv 0$.

The proof of the above lemma is in [3] and therefore will be omitted.

The proof of the theorem. That $x_i(t)$ defined by (4.2) is a solu-

tion of (1.1) follows at once from Lemma 1 by direct verification. The uniqueness of $x_\varepsilon(t)$ follows from Lemma 3 just as in [1]. Finally, since $\exp\left(-\int^t A(s)ds\right)x_0$ is the solution of (1.2) Lemma 2 shows that.

$$\lim_{\varepsilon \rightarrow 0} \left\| x_\varepsilon(t) - \exp\left(-\int^t A(s)ds\right)x_0 \right\| = 0 .$$

This completes the proof of the theorem.

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Received December 2, 1969.

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

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