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A NON-COMPACT KREIN-MILMAN THEOREM

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This paper describes a class of closed bounded convex sets which are the closed convex hulls of their extreme points. It includes all compact ones and those with the positive binary intersection property.

Let K be a closed bounded convex subset of a Hausdorff locally convex linear topological space F . Denote by EK the extreme points of K , by $\text{co } EK$ their convex hull and let $\overline{\text{co } EK}$ be its closure. We are interested in showing when

$$K = \overline{\text{co } EK}.$$

The principal known results are the following:

THEOREM 1.1. *If either*

(a) K is compact;

or (b) K has the positive binary intersection

property;

then

$$K = \overline{\text{co } EK}.$$

Case (a) is the Krein-Milman Theorem [3, p. 131]. Case (b) was proved by Nachbin in [6], and he poses in [5, p. 346] the problem of obtaining a theorem of which both (a) and (b) are specializations. This is answered by Theorem 4.2. For the whole of this paper, S is a Stonean (extremally disconnected compact Hausdorff) space.¹

A simplified version of Theorem 4.2 reads as follows:

THEOREM 1.2. *Let X be a normed linear space. Then any norm-closed ball in the space $\mathfrak{B}(X, C(S))$ of continuous linear operators from X to $C(S)$ is the closure of the convex hull of its extreme points in the strong neighborhood topology.*

The result concerning the unit ball of a dual Banach space in its weak*-topology and that concerning the unit ball in $C(S)$ in its norm topology are special cases of Theorem 1.2.

A sublinear function P from a vector space X to a partially ordered space V satisfies

$$P(x + y) \leq P(x) + P(y)$$

and

¹ Theorem 2.3 and its proof are valid when S is zero-dimensional.

$$P(tx) = tP(x)$$

for all x, y in X and $t \geq 0$.

A linear operator T from X to V is *dominated by P* if $Tx \leq Px$ for all x in X . The set of all linear operators from X to V dominated by P will be written $L(P)$.

2. Let P be a sublinear function into $C(S)$, where S is Stonean. We obtain a compact approximation to $L(P)$ by considering a finite partition $\mathcal{U} = \{U_1, \dots, U_M\}$ of S into disjoint open-and-closed sets. Let $C(S_{\mathcal{U}})$ denote the set of all function in $C(S)$ whose restrictions $f|U_k$ are constant. The constant values will be written as $f(U_k)$.

LEMMA 2.1. *Let P be a sublinear function from a vector space X to $C(S_{\mathcal{U}})$ and let $L(P_{\mathcal{U}})$ be the set of all linear operators from X to $C(S_{\mathcal{U}})$ dominated by P . Then*

$$EL(P_{\mathcal{U}}) \subseteq EL(P) .$$

Proof. Suppose $T \in EL(P_{\mathcal{U}})$. For $k = 1, \dots, M$ let t_k be chosen arbitrarily in U_k . If $G, H \in L(P)$ and $T = 1/2(G+H)$ define $G', H' \in L(P_{\mathcal{U}})$ by

$$G'x = \sum_{k=1}^M (Gx)(t_k)\chi_k \qquad H'x = \sum_{k=1}^M Hx(t_k)\chi_k$$

where χ_k is the characteristic function of U_k . Since $1/2(G' + H') = T$ and $T \in EL(P_{\mathcal{U}})$, we have $G' = H' = T$. Hence, for each $x \in X$ and $k = 1, \dots, M$,

$$G'x(U_k) = H'x(U_k) = Tx(U_k)$$

so that

$$Gx(t_k) = Hx(t_k) = Tx(t_k) .$$

Since t_k was chosen arbitrarily in U_k , $G = H = T$. Hence $T \in EL(P)$.

DEFINITION 2.2. Let X and E be linear topological spaces and let $\mathfrak{B}(X, E)$ be the space of all continuous linear operators from X to E . The *strong neighborhood topology* for $\mathfrak{B}(X, E)$ is the topology with a base given by sets of the form

$$N(T; x_1, \dots, x_n; U) = \{S \in \mathfrak{B}(X, E) : (T-S)x_i \in U, i = 1, \dots, n\}$$

where $x_1, \dots, x_n \in X$ and U is a neighborhood of 0 in E .

If E is normed, then we write

$N(T; x_1, \dots, x_n; \varepsilon)$ for $N(T; x_1, \dots, x_n; U)$ when U is the open ε -ball about 0.

THEOREM 2.3. *Let \mathscr{W} be a finite partition of S into nonempty open-and-closed subsets. Let P be a sublinear function from a linear space X into $C(S_{\mathscr{W}})$. Then $L(P) = \overline{\text{co}} EL(P)$, with the closure in the strong neighborhood topology of $\mathfrak{B}(X, C(S))$.*

Proof. Let \mathscr{U} be any finite partition of S into nonempty open-and-closed sets. From Lemma 2.1, $\overline{\text{co}} EL(P) \cong \overline{\text{co}} EL(P_{\mathscr{U}})$. Now $L(P_{\mathscr{U}})$ can be linearly identified with a certain compact convex subset of a finite product $X^* \times \dots \times X^*$, where X^* is the algebraic dual of X with the topology $w(X^*, X)$. Hence, from the Krein-Milman Theorem, $\overline{\text{co}} EL(P_{\mathscr{U}}) = L(P_{\mathscr{U}})$.

Let $T \in L(P)$ and let $N(T; x_1, \dots, x_n; \varepsilon)$ be a strong neighborhood of T . The functions $\{Tx_i: i = 1, \dots, n\}$ are continuous so for each fixed i there is a finite covering

$$\mathscr{V}^{(i)} = \{V_1^i, \dots, V_{N_i}^i\}$$

of S by open sets such that

$$\text{Var}(Tx_i, V_k^i) < \varepsilon$$

for all k .

Since S is zero-dimensional, there is a finite partition

$$\mathscr{U} = \{U_1, \dots, U_M\}$$

of S into nonempty open-and-closed sets that simultaneously refines $\mathscr{V}^{(1)}, \dots, \mathscr{V}^{(n)}$. By taking a further refinement if necessary, \mathscr{U} may also be assumed to be a refinement of \mathscr{W} and then the functions $P(x)$ are constant on each of the sets U_k .

For each $k = 1, \dots, M$ define a sublinear functional q_k on X by $q_k(x) = \sup \{Tx(t): t \in U_k\}$. From the Hahn-Banach Theorem, there exists a linear functional ϕ_k on X dominated by q_k . Define $T_1: X \rightarrow C(S_{\mathscr{U}})$ by

$$T_1x = \sum_{k=1}^M \phi_k(x) \chi_{U_k}.$$

Then $T_1 \in L(P_{\mathscr{U}})$ and, for $i = 1, \dots, n$,

$$\|(T_1 - T)x_i\| \leq \sup_R \text{Var}(Tx_i, U_k) < \varepsilon.$$

DEDUCTION of THEOREM 1.2. With X and S as in the statement of the theorem, let \mathfrak{B}_1 be the closed unit ball in $\mathfrak{B}(X, C(S))$.

The set \mathfrak{B}_1 is $L(P)$, where P is the sublinear function $P(x) = \|x\| e$, e being the unit function in $C(S)$. By Theorem 2.3 $\mathfrak{B}_1 = \text{co } E\mathfrak{B}_1$ and the result for any closed ball then follows by a scalar multiplication and translation.

3. Nachbin's problem. Let K be a closed bounded convex subset of a linear topological space E . Recall that K has the *positive binary intersection property* if every pairwise-intersecting subfamily of

$$\{x + \lambda K : x \in E, \lambda \geq 0\}$$

has nonempty intersection.

If K is bounded and has the above property, it may be shown to be centrally symmetric with a unique centre c , and to have the *binary intersection property* where the restriction $\lambda \geq 0$ is removed. This is proved in [6].

Results in [4] and [2] then show that the set $K_0 = K - c$ generates a subspace of E which is a hyperconvex normed space and isomorphic to $C(S)$, with S Stonean.

THEOREM 3.1. *Let E be a locally convex Hausdorff linear topological space containing a closed bounded convex subset K with the positive binary intersection property. Let p be a continuous sublinear functional on a locally convex Hausdorff linear topological space X .*

If L is the set of linear maps $T: X \rightarrow E$ such that for all x in X

$$Tx \in \frac{1}{2} [p(x) - p(-x)] e + \frac{1}{2} [p(x) + p(-x)] K_0$$

where e is any extreme point of K_0 , then $L = \overline{\text{co}} L$, with the closure taken in $\mathfrak{B}(X, E)$ with the strong neighborhood topology.

Proof. Because p is continuous the set $L(P)$ is closed in the space $\mathfrak{B}(X, E)$ in the strong neighborhood topology. Since K is centrally symmetric, K_0 has the binary intersection property and is linearly isomorphic to the unit ball in a space $C(S)$ with S Stonean. The isomorphism may be chosen as in [4] so that e is mapped onto the unit function of $C(S)$. Using e to denote also this unit function, we may define a sublinear function $P(x) = p(x) e$ from X to $C(S)$, which is the situation of Theorem 3.1. with $\mathscr{W} = \{S\}$.

Given $T \in L(P)$, $x_1, \dots, x_n \in X$ and $\varepsilon > 0$ there exists $A \in \text{co } EL(P)$ with

$$(T - A)x_i \in \varepsilon K_0 \quad (i = 1, \dots, n).$$

Given a neighborhood U of 0 in E , there exists $r > 0$ with $K_0 \subseteq rU$, since K is bounded. So choosing $\varepsilon = r^{-1}$ there exists $A \in \text{co } EL(P)$ with

$$(T - A)x_i \in r^{-1} K_0 \subseteq U \quad (i = 1, \dots, n),$$

which completes the proof.

DEDUCTION OF THEOREM 1.1. (a) Let p_K be the sublinear functional defined on F^* by

$$p_K(f) = \sup \{f(k) : k \in K\}.$$

Then, from the bipolar theorem,

$$L = \{g \in F^{**} : g(f) \leq p_K(f) \text{ for all } f \in F^*\}$$

is identical with the canonical image \hat{K} of K under the evaluation map. Now apply Theorem 3.1 with $E = \mathbf{R}$, $K = [-1, 1]$, $e = 1$ and $X = F^*$, taken with the topology of uniform convergence on compact subsets of F . This shows that \hat{K} is the closure of $\text{co } E\hat{K}$ in the topology $w(F^{**}, F^*)$, which is equivalent to K being the $w(F, F^*)$ and hence the strong closure of $\text{co } EK$ in F .

(b) Apply Theorem 3.1 with $X = \mathbf{R}$ and $E = F$. Then, under the natural isomorphism of $\mathfrak{B}(X, E)$ and E , K_0 corresponds to L , which satisfies $L = \overline{\text{co } EL}$. Since E is a linear topological space we have

$$K = \overline{\text{co } EK}.$$

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E. M. Alfsen and B. Hirsberg, <i>On dominated extensions in linear subspaces of $\mathcal{C}_C(X)$</i>	567
Joby Milo Anthony, <i>Topologies for quotient fields of commutative integral domains</i>	585
V. Balakrishnan, G. Sankaranarayanan and C. Suyambulingom, <i>Ordered cycle lengths in a random permutation</i>	603
Victor Allen Belfi, <i>Nontangential homotopy equivalences</i>	615
Jane Maxwell Day, <i>Compact semigroups with square roots</i>	623
Norman Henry Eggert, Jr., <i>Quasi regular groups of finite commutative nilpotent algebras</i>	631
Paul Erdős and Ernst Gabor Straus, <i>Some number theoretic results</i>	635
George Rudolph Gordh, Jr., <i>Monotone decompositions of irreducible Hausdorff continua</i>	647
Darald Joe Hartfiel, <i>The matrix equation $AXB = X$</i>	659
James Howard Hedlund, <i>Expansive automorphisms of Banach spaces. II</i>	671
I. Martin (Irving) Isaacs, <i>The p-parts of character degrees in p-solvable groups</i>	677
Donald Glen Johnson, <i>Rings of quotients of Φ-algebras</i>	693
Norman Lloyd Johnson, <i>Transition planes constructed from semifield planes</i>	701
Anne Bramble Searle Koehler, <i>Quasi-projective and quasi-injective modules</i>	713
James J. Kuzmanovich, <i>Completions of Dedekind prime rings as second endomorphism rings</i>	721
B. T. Y. Kwee, <i>On generalized translated quasi-Cesàro summability</i>	731
Yves A. Lequain, <i>Differential simplicity and complete integral closure</i>	741
Mordechai Lewin, <i>On nonnegative matrices</i>	753
Kevin Mor McCrimmon, <i>Speciality of quadratic Jordan algebras</i>	761
Hussain Sayid Nur, <i>Singular perturbations of differential equations in abstract spaces</i>	775
D. K. Oates, <i>A non-compact Krein-Milman theorem</i>	781
Lavon Barry Page, <i>Operators that commute with a unilateral shift on an invariant subspace</i>	787
Helga Schirmer, <i>Properties of fixed point sets on dendrites</i>	795
Saharon Shelah, <i>On the number of non-almost isomorphic models of T in a power</i>	811
Robert Moffatt Stephenson Jr., <i>Minimal first countable Hausdorff spaces</i>	819
Masamichi Takesaki, <i>The quotient algebra of a finite von Neumann algebra</i>	827
Benjamin Baxter Wells, Jr., <i>Interpolation in $C(\Omega)$</i>	833