INTERPOLATION IN $C(\Omega)$

Benjamin Baxter Wells, Jr.
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It is known from the work of Bade and Curtis that if $\mathfrak{A}$ is a Banach subalgebra of $C(\Omega)$, $\Omega$ a compact Hausdorff space, and if $\Omega$ is an $F$-space in the sense of Gillman and Hendriksen then $\mathfrak{A} = C(\Omega)$. This paper is concerned with the extension of this and similar results to the setting of Grothendieck spaces ($G$-spaces for short). An important feature of the extension is that emphasis is shifted from the underlying topological structure of $\Omega$ to the linear topological character of $C(\Omega)$.

As a corollary we show that if $\Omega_1$ and $\Omega_2$ are infinite compact Hausdorff spaces, then $\Omega_1 \times \Omega_2$ is not a $G$-space. Consequently if $\Omega$ is a $G$-space then $C(\Omega)$ is not linearly isomorphic to $C(\Omega \times \Omega)$.

If $A$ is a commutative Banach algebra whose spectrum is a totally disconnected $G$-space, a second corollary of our extension is that the Gelfand homomorphism is onto. This establishes for $G$-spaces a result due to Seever for $N$-spaces.

Two definitions of $G$-space are to be found in the literature.
(A) A Banach space $X$ is a $G$-space if every weak-* convergent sequence in $X^*$, the dual of $X$, is weakly convergent.
(B) A compact Hausdorff space $\Omega$ is a $G$-space if $C(\Omega)$ is a $G$-space in the sense of (A).

Unless otherwise noted we shall accept (B) as our definition.

It is known from the work of Seever [7] that if $\Omega$ is an $F$-space, i.e., if disjoint open $F_\sigma$ subsets of $\Omega$ have disjoint closures, then $\Omega$ is a $G$-space. A result due to Rudin [3] states that if $\Omega_1$ and $\Omega_2$ are infinite compact Hausdorff spaces then $\Omega_1 \times \Omega_2$ is not an $F$-space. Corollary 2.6 is an extension of this to $G$-spaces. Although an example of a $G$-space which is not an $F$-space is given in [7], no necessary and sufficient topological characterization of the $G$ property is known.

1. Preliminaries. Let $M(\Omega)$ be the space of regular Borel measures on $\Omega$ equipped with the total variation norm. A sequence $\{\mu_n\}$ in $M(\Omega)$ converges for the weak-* topology if for each $f$ in $C(\Omega)$, the space of continuous complex valued functions on $\Omega$, the sequence $\{\mu_n(f)\}$ is convergent. Weak convergence of $\{\mu_n\}$ means convergence of $\{\gamma(\mu_n)\}$ for every $\gamma$ in $M^*(\Omega)$, the dual of $M(\Omega)$. If $\Omega$ is any set $l_1(\Omega)$ will denote the Banach space of point mass measures on $\Omega$ with the total variation norm.

A Banach subalgebra (subspace) $\mathfrak{A}$ of $C(\Omega)$ is a subalgebra (subspace)
of \( C(\Omega) \) under the pointwise operations and is a Banach algebra (space) such that the embedding \( \mathcal{A} \to C(\Omega) \) is continuous. \( \mathcal{A} \) is said to be normal if for each pair \( F_1, F_2 \) of disjoint compact subsets of \( \Omega \) there is an \( f \in \mathcal{A} \) such that \( f = 1 \) on \( F_1 \) and \( f = 0 \) on \( F_2 \). Following [2] we call \( \mathcal{A} \varepsilon \)-normal if for each pair \( F_1, F_2 \) of disjoint compact subsets of \( \Omega \) there exists an \( f \in \mathcal{A} \) satisfying

\[
(i) \quad |f(\omega) - 1| < \varepsilon, \quad \omega \in F_1,
(ii) \quad |f(\omega)| < \varepsilon, \quad \omega \in F_2.
\]

If \( \Omega_1 \) and \( \Omega_2 \) are compact Hausdorff spaces the projective tensor product \( V = C(\Omega_1) \widehat{\otimes} C(\Omega_2) \) is the set of all functions of the form

\[
\sum_{i=1}^{\infty} f_i(x)g_i(y), \quad f_i(x) \in C(\Omega_1)
\]

and \( g_i(y) \in C(\Omega_2) \) such that \( \sum_{i=1}^{\infty} ||f_i||_\infty ||g_i||_\infty < \infty \). If \( h \in V \) then

\[
||h||_V = \inf \left\{ \sum_{i=1}^{\infty} ||f_i||_\infty ||g_i||_\infty : h = \sum_{i=1}^{\infty} f_i g_i \right\}.
\]

Two Banach spaces \( X_1 \) and \( X_2 \) are isomorphic if there is a one-to-one continuous linear map from \( X_1 \) onto \( X_2 \). If \( X_2 \) is a closed subspace of \( X_1 \), it is said to be complemented in \( X_1 \) if there exists a closed subspace \( Y \) of \( X_1 \) such that \( X_2 + Y = X_1 \) and \( X_2 \cap Y = \{0\} \). We write \( X_1 = X_2 \oplus Y \).

If \( D \) is a discrete space, \( C(D) \) will denote the bounded continuous functions on \( D \). It is well known that \( C(D) \) is isometrically isomorphic to \( C(\beta D) \) where \( \beta D \) is the Stone-Cech compactification of \( D \). A compact Hausdorff space is totally disconnected if there is a basis for the topology consisting of open and closed neighborhoods.

2. We shall need to recall here a criterion due to Grothendieck [5] for relative weak compactness in \( M(\Omega) \). Namely, a bounded sequence \( \{\mu_n\} \) in \( M(\Omega) \) is relatively weakly compact if and only if for every sequence \( \{0_i\} \) of pairwise disjoint Borel sets \( \lim_{i \to \infty} \mu_n(0_i) = 0 \) uniformly in \( n \). By the Eberlein Smulian theorem this is equivalent to every subsequence of \( \{\mu_n\} \) having a weakly convergent subsequence.

**Lemma 2.1.** If \( \Omega \) is a \( G \)-space and \( K \) is a closed subspace of \( \Omega \), then \( K \) is a \( G \)-space.

**Proof.** Suppose \( \{\mu_n\} \) in \( M(K) \) is weak-* convergent. One may regard \( \{\mu_n\} \) as a weak-* convergent sequence in \( M(\Omega) \). It is therefore weakly convergent as a sequence in \( M(\Omega) \), and so by the Hahn-Banach Theorem it is a weakly convergent sequence in \( M(K) \).
LEMMA 2.2. Let $\Omega$ be a $G$-space and $X$ a dense Banach subspace such that $X \not= C(\Omega)$. Then for every $M > 0$ there is a measure $\mu$ with no atomic part such that $||\mu|| \geq M$ and $\sup\{|\mu(f)| : f \in X, ||f||_X \leq 1\} \leq 1$.

Proof. We shall write $\mu_a$ for the atomic part of $\mu$ and $\mu_c$ for the continuous part. By a well known theorem of Banach there is a sequence $\{\mu_n\}$ of measures such that $\mu_n \overset{\text{weak-}*}{\to} \mu$ and $\sup\{|\mu_n(f)| : f \in X, ||f||_X \leq 1\} \leq 1$ for each $n$. Since $X$ is dense in $C(\Omega)$ setting $\nu_n = \mu_n\mu_n$ we have $\lim_n \nu_n = 0$ weak-* and hence $\lim_n \nu_n = 0$ weakly since $\Omega$ is a $G$-space.

The natural projection $p: M(\Omega) \to l_1(\Omega)$ given by $p\mu = \mu_a$ is continuous and hence weakly continuous. Hence $\lim_n \nu_n = 0$ weakly. Since in $l_1(\Omega)$ weakly convergent sequences are norm convergent, it follows that $\lim_n \nu_n = 0$. Thus for an appropriate sequence of scalars $\{c_n\}$ we have $\lim_n |\nu_n(f)| = 0$ and $\sup\{|c_n\nu_n(f)| : f \in X, ||f||_X \leq 1\} \leq 1$ for every $n$.

THEOREM 2.3. Let $\Omega$ be a $G$-space and let $X$ be a dense Banach subspace of $C(\Omega)$. Then there exists a finite open covering $U_1, \ldots, U_n$ of $\Omega$ such that $X \setminus U_i = C(U_i)$, $1 \leq i \leq n$.

Proof. From the compactness of $\Omega$ it suffices to show that each point $p$ of $\Omega$ has a neighborhood $U_p$ such that $X \setminus U_p = C(U_p)$. Suppose this fails for some $p$, and choose $U$ a neighborhood of $p$. Let $X$ denote the quotient space of $X$ by all functions in $X$ vanishing on $U$.

Applying Lemmas 2.1 and 2.2 it follows that there is a regular Borel measure $\mu$ with no atomic part such that $||\mu|| \geq 1$, $\sup \mu \subseteq \overline{U}$, and such that $|\mu(f)| \leq ||f||_X \leq ||f||_x$ for every $f \in X$.

From the regularity of $\mu$ we may choose open $U \subseteq U_p$, $p \in U_\ast$ such that $||\mu|| \geq 1/2 |||\mu|||$. Since $X \setminus U_\ast \neq C(U_\ast)$ we may choose in the same way a $\mu$ with no atomic part such that $sup \mu \subseteq U$, $||\mu|| \geq 2$ and $|\mu(f)| \leq ||f||_X$ for all $f \in X$.

Continuing in this fashion, define inductively a sequence of measures $\{\mu_n\}$ with no atomic parts such that $||\mu_n|| \geq n$, $|\mu_n(f)| \leq ||f||_x$ for every $f \in X$, $sup \mu_n \subseteq \overline{U}$, and $|\mu_n| (\overline{U}_n \setminus \overline{U}_{n+1}) > 1/2 ||\mu_n||$.

Setting $\nu_n = \mu_n||\mu_n||$ we see $\lim_n \nu_n = 0$ weak-* from the density of $X$. However, since $|\nu_n| (\overline{U}_n \setminus \overline{U}_{n+1}) > 1/2$ for each $n$, $\{\nu_n\}$ is not weakly convergent by the Grothendieck criterion. This contradiction establishes the theorem.

REMARK. Theorem 2.3 is the sharpest result in the sense that
for every compact Hausdorff space $\Omega$ there is a dense Banach subspace $X$ of $C(\Omega)$ such that $X \neq C(\Omega)$. By a result of [8] (corollary 3.2 page 201) there are closed subspaces $Y, W$ of $C(\Omega)$ such that $Y + W$ is dense in $C(\Omega)$ but $Y + W \neq C(\Omega)$; in the terminology of that paper every $C(\Omega)$ contains a quasi-complemented uncomplemented subspace. Setting $X = Y \oplus W$ we have the result.

Our next theorem is an extension to $G$-spaces of a result of [2]. The work is all done by the following:

**Lemma 2.4.** [2] Let $\Omega$ be a compact Hausdorff space, and let $\mathfrak{A}$ be a Banach subalgebra of $C(\Omega)$ such that

(i) $\mathfrak{A}$ is $\varepsilon$-normal for some $\varepsilon < 1/2$,

(ii) There is an open covering $U_1, \ldots, U_n$ of $\Omega$ such that $\mathfrak{A} \vert U_i = C(U_i)$, $1 \leq i \leq n$.

Then $\mathfrak{A} = C(\Omega)$.

Combining this with Theorem 2.3 and the remark that density implies $\varepsilon$-normality we obtain:

**Theorem 2.5.** Let $\Omega$ be a $G$-space, and let $\mathfrak{A}$ be a dense Banach subalgebra of $C(\Omega)$. Then $\mathfrak{A} = C(\Omega)$.

**Remark.** As demonstrated in [2] $\varepsilon$-normality for some $\varepsilon < 1/4$ and density of a Banach subspace of $C(\Omega)$ are equivalent in case $\Omega$ is an $F$-space. We do not know if “dense” may be replaced by “$\varepsilon$-normal” in Theorem 2.5.

**Corollary 2.6.** If $\Omega_1$ and $\Omega_2$ are infinite compact Hausdorff spaces then $\Omega_1 \times \Omega_2$ is not a $G$-space.

**Proof.** We need only take $\mathfrak{A} = C(\Omega_1) \otimes C(\Omega_2)$ and note that $\mathfrak{A}$ is a dense Banach subalgebra of $C(\Omega_1 \times \Omega_2)$. ($\mathfrak{A}$ happens to be normal as well.) But it is well known that $\mathfrak{A} \neq C(\Omega_1 \times \Omega_2)$.

Let $X_1$ and $X_2$ be Banach spaces such that $X_2$ is a continuous linear image of $X_1$. It is an easy consequence of the Hahn Banach theorem that if $X_1$ is a $G$-space in the sense of definition A, then so is $X_2$. Consequently if $\Omega$ is a $G$-space then $C(\Omega \times \Omega)$ is not even a continuous linear image of $C(\Omega)$. This is contrasted with a result of Milutin [6, p. 42] which states that if $\Omega_1$ and $\Omega_2$ are uncountable compact metric spaces then $C(\Omega_1)$ is isomorphic to $C(\Omega_2)$. In particular for such $\Omega$, $C(\Omega)$ is isomorphic to $C(\Omega \times \Omega)$.

These notions may be of use in solving complementation problems. Suppose that $X_2$ is a complemented subspace of $X_1$. Then if $X_1$ is a $G$-space in the sense of definition A, so is $X_2$. For example, if $D$
denotes an infinite discrete space, \( C(\beta D \times \beta D) \) may be viewed in a natural way as a closed subspace of \( C(D \times D) \). Since \( \beta(D \times D) \) is a \( G \)-space, by the above remarks \( C(\beta D \times \beta D) \) has no complement in \( C(D \times D) \).

**Corollary 2.7.** [cf. [7] corollary 2 p. 278]. Let \( A \) be a commutative Banach algebra whose spectrum \( \Omega \) is a totally disconnected \( G \)-space. Then the Gelfand homomorphism is onto.

**Proof.** By the Šilov idempotent theorem the image of \( A \) in \( C(\Omega) \) contains the characteristic functions of open closed sets. Hence \( A \) is a dense Banach subalgebra of \( C(\Omega) \) and the theorem applies.

**Remark.** An interesting fact suggested by the proof of Theorem 2.3 is that if \( \Omega \) is a \( G \)-space then no normal subalgebra \( A \) of \( C(\Omega) \), closed in the uniform norm, is such that \( C(\Omega)/A \) has countable (infinite) dimension. To see this suppose to the contrary that \( C(\Omega)/A \) has countable dimension. Recall that if \( A \) is a normal subalgebra of \( C(\Omega) \) such that every point \( p \) of \( \Omega \) has a neighborhood \( U_p \) such that \( A \mid U_p = C(U_p) \) then \( A = C(\Omega) \). Thus there is a point \( p \in \Omega \) such that for every neighborhood \( U_p \) of \( p \), \( A \mid U_p \neq C(U_p) \). Since \( A \) contains the constant functions, by a result of Glicksberg [4 p. 421] we may choose \( \mu_1 \in A^1, ||\mu_1|| = 1 \) such that \( |\mu_1| (\bar{U}_p^*) > \delta > 0 \) where \( \bar{U}_p \) is a closed deleted neighborhood of \( p \). By regularity of \( \mu_1 \) we may choose a neighborhood \( U_z \) of \( p \) such that \( \bar{U}_z \subseteq U_1 \) and \( |\mu_1| (\bar{U}_z^*) < \delta/2 \). Again we may choose \( \mu_2 \in A^1, ||\mu_2|| = 1 \), such that \( |\mu_2| (\bar{U}_z^*) > \delta > 0 \). Continuing in this fashion we get a sequence of measures \( \{\mu_n\} \subseteq A^1, ||\mu_n|| = 1 \), and a nested sequence of neighborhoods of \( p, \{U_n\} \subseteq U_1 \) such that \( |\mu_n| (\bar{U}_n - \bar{U}_{n+1}) > \delta/2 \) for each \( n \). By Grothendieck's criterion no subsequence of \( \{\mu_n\} \) is weakly convergent. Since \( C(\Omega)/A \) is separable, the unit ball in \( A_1 \) is weak-* sequentially compact. Thus a subsequence of \( \{\mu_n\} \) may be found which is weak-* convergent and hence weakly convergent. This contradiction completes the proof.

In [7] the following theorem is proved.

**Theorem 2.8.** If \( \Omega \) is an \( F \)-space, and if \( X \) is a normal Banach subspace of \( C(\Omega) \), then \( X = C(\Omega) \).

**Question [7].** In Theorem 2.8 can "\( F \)" be replaced by "\( G \)? In the terminology of that paper is every \( G \)-space an \( N \)-space? The following may be of help in giving an answer.

**Theorem 2.9.** Let \( X \) be a \( G \)-space in the sense of definition \( A \), and let \( Y \) be a closed subspace such that \( X/Y \) is separable. Then \( Y \)
is a $G$-space.

**Proof.** Let $\{y^*_n\}$ denote a sequence in $Y^*$. It suffices to show that if $\lim_{n} y^*_n = 0$ weak-$^*$ then $\{y^*_n\}$ has a subsequence $\{y^*_k\}$ such that $\lim_{k} y^*_k = 0$ weakly. Let $x^*_n$ be any normpreserving extension of $y^*_n$ to all of $X$. Since $X/Y$ is separable, a sequence $\{w_n\}$ in $X$ may be found such that $sp\{w_n\} + Y$ is dense in $X$. By a diagonal argument a subsequence $\{x^*_n\}$ of $\{x^*_n\}$ may be found such that $\{x^*_n\}$ converges on each member of $\{w_n\}$ and hence on $sp\{w_n\} + Y$. Since $\{||x^*_n||\}$ is bounded, $\{x^*_n\}$ is weak-$^*$ convergent in $X$ and hence weakly convergent. Thus $\lim_{k} y^*_k = 0$ weakly.

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**References**


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