Pacific Journal of Mathematics

ON SPECTRAL THEORY AND SCATTERING FOR ELLIPTIC OPERATORS WITH SINGULAR POTENTIALS

RICHARD WILLIAM BEALS

Vol. 37, No. 1 January 1971

PACIFIC JOURNAL OF MATHEMATICS
Vol. 37, No. 1, 1971

ON SPECTRAL THEORY AND SCATTERING FOR ELLIPTIC OPERATORS WITH SINGULAR POTENTIALS

RICHARD BEALS

General conditions have been found which imply that the perturbation $A + q$ of an elliptic differential operator A by a singular potential term $q(x)$ has a closed extension B in $L^2(R^n)$ having the same essential spectrum as A. The purpose of this paper is to sharpen the known results slightly and to estimate the characteristic numbers of the operator $(A + \lambda)^p - (B + \lambda)^p$. Under an appropriate assumption on $q(x)$, this operator is shown to be of trace class for large p. In the self-adjoint case it follows then from results of Kato that wave operators for the pair (A, B) exist and that the absolutely continuous parts of these operators are unitarily equivalent.

Let r be a positive integer and let

$$A(x, D) = \sum_{|\alpha| \leqq r, |\beta| \leqq r} D^\alpha(a_{\alpha\beta}(x)D^\beta)$$

be a differential operator of order $m = 2r$. Here

$$x \in R^n,\ \alpha = (\alpha_1, \cdots, \alpha_n),\ |\alpha| = \sum \alpha_j\ ,$$

and $D^\alpha = (-i)^{|\alpha|} \prod (\partial/\partial x_j)^{\alpha_j}$. We assume throughout that $a_{\alpha\beta}$ has continuous derivatives of order $\leqq \max\{|\alpha|, |\beta|\}$, and the derivatives are uniformly bounded. For $|\alpha| = |\beta| = r$ we assume $a_{\alpha\beta}$ uniformly continuous. Finally, $A(x, D)$ is uniformly strongly elliptic: there is a constant $a_0 > 0$ such that

$$Re\Big(\sum_{|\alpha|=|\beta|=r} a_{\alpha\beta}(x)\xi^\alpha\xi^\beta\Big) \geqq a_0 |\xi|^m$$

for all $x \in R^n$, $\xi \in R^n$.

Let A_0 be the restriction of $A(x, D)$ to $\mathscr{D}$, the smooth functions with compact support. Let A be the closure of A_0 in $L^2 = L^2(R^n)$. Various conditions have been given on a potential term $q(x)$ such that $A + q$ have a closed extension B with the same essential spectrum as A, either generally or in the particular case $A = -\Delta$; [1], [2], [4], [6]. The most general result of this sort seems to be that of Schechter [8], [9]. We shall sharpen Schechter's result and then investigate the characteristic numbers of $(A + \lambda)^{-p} - (B + \lambda)^{-p}$.

For $\mu > -n$ and $x \neq 0$, set $\omega_\mu(x) = |x|^\mu$ if $\mu < 0$,

$$\omega_0(x) = (-\log|x|)^+,\ \omega_\mu(x) = 1$$

if $\mu > 0$. Suppose q is a measurable function defined on R^n and suppose $\mu > -n$, $p > 0$, $\delta > 0$, $x \in R^n$. Let

$$\begin{aligned} M_{\mu,p,\delta,x}(q) &= \int_{|y|<\delta} |q(x-y)|^p\, \omega_\mu(y)dy\ , \\ M_{\mu,p,x}(q) &= M_{\mu,p,1,x}(q)\ , \\ M_{\mu,p,\delta}(q) &= \sup_x M_{\mu,p,\delta,x}(q)\ , \\ M_{\mu,p}(q) &= M_{\mu,p,1}(q)\ . \end{aligned}$$

We shall assume throughout that

$$\text{(I)} \qquad \begin{aligned} &M_{2m-n,1}(q) < \infty,\ M_{m-n,1}(\operatorname{Im} q) < \infty, \text{ and} \\ &M_{m-n,1,\delta}((\operatorname{Re}\ q)^-) \to 0 \text{ as } \delta \to 0, \text{ if } m \leqq n\ . \end{aligned}$$

Note that these conditions are implied by

$$\text{(I)}'. \qquad \begin{aligned} &M_{2m-n,1}((\operatorname{Re}\ q)^+) < \infty,\ M_{m-n,1}(\operatorname{Im}\ q) < \infty\ , \\ &M_{\mu,1}((\operatorname{Re}\ q)^-) < \infty \text{ for some } \mu < m-n\ . \end{aligned}$$

In fact $M_{\mu,p}(q) \leqq c(\mu, \nu, \delta)M_{\nu,p,\delta}(q)$ if $\mu \leqq \nu$. Thus the conditions (I)$'$ imply the first two conditions in (I). If $m \leqq n$, take $\mu < \nu < m - n$. The Schwarz inequality gives

$$\begin{aligned} [M_{\nu,1,\delta,x}(q)]^2 &\leqq M_{\mu,1,\delta,x}(q)\int_{|y|<\delta} |q(x-y)|\,|y|^{2\nu-\mu}\, dy \\ &\leqq [M_{\mu,1,\delta}(q)]^2\, \delta^{2\nu-2\mu}\ , \end{aligned}$$

which implies the third condition in (I). Note that if $m > n$ then (I) and (I)$'$ coincide; in fact $M_{\nu,p}\ (f)$ is constant for $\nu > 0$.

Take measurable functions ρ, σ, τ such that

$$(1) \qquad \rho^2 = (\operatorname{Re}\ q)^+,\ \sigma^2 = (\operatorname{Re}\ q)^-,\ \tau^2 = \operatorname{Im}\ q\ .$$

For real l let

$$(2) \qquad ||u||_l^2 = \int (1 + |\xi|^2)^l\, |\hat{u}(\xi)|^2 d\xi,\ u \in \mathscr{D}\ ,$$

where u denotes the Fourier transform of u. Let H^l be the completion of $\mathscr{D}$ with respect to this norm. Then $H^0 = L^2$. When l is a nonnegative integer, an equivalent norm is

$$(2)' \qquad \sum_{|\alpha| \leqq l} ||D^\alpha u||_0^2\ .$$

Let W^r be the completion of $\mathscr{D}$ with respect to the norm

(3) $$|u|^2 = \|u\|_r^2 + \|\rho u\|_0^2 .$$

Let

$$b(u, v) = a(u, v) + (qu, v) = \sum (a_{\alpha\beta} D^\beta u, D^\alpha v) + (qu, v)$$

for $u, v \in \mathscr{D}$, where (u, v) is the L^2-inner product. As in [9], this form extends to W^r. Define B_0 as an operator in H^0 by $B_0 u = Au + qu$ for $u \in \mathscr{D}$ such that $qu \in H^0$. Define B by $Bu = f$ in H^0 if and only if $u \in W^r$ and $b(u, v) = (f, v)$ for all $v \in W^r$.

THEOREM 1. *B is a closed extension of B_0. There is a constant λ_0 such that $A + \lambda$ and $B + \lambda$ are $1 - 1$ and onto for* Re $\lambda \geqq \lambda_0$. *If $a_{\alpha\beta} = \bar{a}_{\alpha\beta}$ and q is real-valued, then A and B are self-adjoint.*

The *essential spectrum* of a closed operator T in the sense of Schechter [7], $\sigma_{em}(T)$, is the complement of the set of complex λ such that $T - \lambda$ has finite-dimensional null space N and closed range of codimension equal to the dimension of N.

THEOREM 2. *Suppose*

(II) $$M_{2m-n,1,x}(q) \longrightarrow 0 \quad as \quad |x| \longrightarrow \infty .$$

Then for Re $\lambda \geqq \lambda_0$, *$(A + \lambda)^{-1} - (B + \lambda)^{-1}$ is compact. Consequently A and B have the same essential spectrum.*

The *characteristic numbers* of a bounded operator S in a Hilbert space are defined by

(4) $$\mu_j(S) = \inf_{\mathrm{codim}(H_j) < j} \ \sup_{u \in H_j, \|u\| = 1} \|Su\|, \quad j = 1, 2, \cdots .$$

For compact S, these are the eigenvalues of $(S^*S)^{1/2}$ arranged in non-increasing order.

THEOREM 3. *Suppose $a > 0$ and*

$(\mathrm{III})_a$ $$M_{2m-n,1,x}((\mathrm{Re}\ q)^+) = 0(|x|^{-2a}) \quad as \quad |x| \longrightarrow \infty ,$$
$$M_{m-n,1,x}(q - (\mathrm{Re}\ q)^+) = 0(|x|^{-a}) \quad as \quad |x| \longrightarrow \infty ,$$

or

$(\mathrm{III})_a'$ $$M_{m-n,1,x}(q) = 0(|x|^{-a}) \quad as \quad |x| \longrightarrow \infty .$$

Then for any $\varepsilon > 0$ there is an integer $p(\varepsilon)$ such that

$$\mu_j((A + \lambda)^{-p} - (B + \lambda)^{-p}) = 0(j^{\varepsilon - a/n}) \quad as \quad j \longrightarrow \infty$$

for $p \geqq p(\varepsilon)$, Re $\lambda \geqq \lambda_0$.

THEOREM 4. *Suppose* $(\mathrm{III})_a$ or $(\mathrm{III})'_a$ *holds for some* $a > n$, *and suppose* A *and* B *are self-adjoint. Let* P_0 *and* P_1 *be the projections on the absolutely continuous subspaces of* A *and* B *respectively. Then the wave operators*

$$W_\pm(B, A) = s - \lim_{t\to\pm\infty} e^{itB}\, e^{-itA} P_0 \ ,$$
$$W_\pm(A, b) = s - \lim_{t\to\pm\infty} e^{itA}\, e^{-itB} P_1$$

exist. The operators AP_0 *and* BP_1 *are unitarily equivalent.*

REMARKS 1. Theorem 1 is proved in [9] under the stronger assumption (I)′.

2. Theorem 2 is proved in [9] under the assumptions (I)′ and (II)′, where

$$(\mathrm{II})' \qquad \begin{aligned} &M_{\nu,1}((\mathrm{Re}\ q)^+) < \infty \ \text{ for some } \nu < 2m - n\ , \\ &\int_{|y|\leq 1} |q(x - y)|\, dy \to 0 \ \text{ as } |x| \to \infty\ . \end{aligned}$$

Again these assumptions imply ours. If $2m > n$, (II)′ is the same as (II). Otherwise take $\nu < \eta < 2m - n$. Then $M_{2m-n,1,x}(q) \leqq cM_{\eta,1,x}(q)$ and Holder's inequality gives

$$M_{\eta,1,x}(q) \leqq [M_{\eta p,1,x}(q)]^{1/p}\Big[\int_{|y|\leq 1} (q(x - y)\Big]^{1/p'}$$

for $1 < p < \infty$, $1/p + 1/p' = 1$. Taking $p = \nu/\eta$ and using (II)′, we get (II).

3. Theorems 3 and 4 are not difficult when r is large relative to n or when q is smooth. However in the Schrodinger case $r = 1$, and in general one wants to allow singular q. Our methods for handing the general case are cumbersome, but effective.

4. There is a large body of literature on the existence of wave operators, but none of the previous work seems to cover explicitly general potentials as locally singular as those considered here.

2. Proof of Theorem 1.

The proof is a modification of the proofs of similar theorems in [8] and [9], to which we refer for details.

For $s > 0$, let g_s be the tempered distribution with Fourier transform $(2\pi)^{-n/2}(1 + |\xi|^2)^{-s/2}$. Then g_s is in L^1 and

$$(5) \qquad \begin{aligned} |g_s(x)| &\leqq c_s\omega_{s-n}(x) \ \text{ for } |x| \leqq 1\ , \\ &\leqq c_s\, exp(-d_s|x|) \ \text{ for } |x| > 1, \text{ where } d_s > 0\ . \end{aligned}$$

From (2) we get

$$(6) \qquad ||u||_{t+s} = ||G_s u||_t, \text{ all real } t ,$$

where $G_s u$ is the convolution $g_s * u$. In fact

$$[G_s u]^\wedge = (1 + |\xi|^2)^{s/2}\hat{u} .$$

We can use this equation to define G_s for $s \leqq 0$; then (6) holds and $G_s G_t = G_{s+t}$ for all s, t.

LEMMA 1. *If $s > 0$ and $M_{2s-n,2}(f) < \infty$, then multiplication by f maps H^s into H^0 and H^0 into H^{-s}. Moreover*

$$(7) \qquad ||fu||_0^2 \leqq cM_{2s-n,2}(f)\, ||u||_s^2 ,$$

$$(8) \qquad ||fu||_{-s}^2 \leqq cM_{2s-n,2}(f)\, ||u||_0^2 ,$$

c independent of u and f.

This is proved in [8]. We also need a sharper form.

LEMMA 2. *If $s > 0$ and $M_{2s-n,2}(f) < \infty$, then for all $\delta > 0$,*

$$(9) \qquad ||fu||_0^2 \leqq cM_{2s-n,2,\delta}(f)\, ||u||_s^2 + c(\delta)M_{2s-n,2}(f)\, ||u||_0^2 ,$$

with c independent of f, u, and δ.

Proof. Choose $\varphi \in \mathscr{D}$ such that $\varphi(x) = 1$, $|x| < 1/4$, $\varphi(x) = 0$, $|x| > 1/2$. Let $\varphi_\delta(x) = \varphi(\delta^{-1}x)$. Let $g_s = \varphi_\delta g_s + (1 - \varphi_\delta)g_s = k_\delta + l_\delta$. Let $K_\delta u = k_\delta^* u$, $L_\delta u = l_\delta^* u$. Then

$$(10) \qquad ||fu||_0 = ||fG_sG_{-s}u||_0 \leqq ||fK_\delta G_s u||_0 + ||fL_\delta G_s u||_0 .$$

We want to compute $||fK_\delta||$, the norm as operator in H^0. But $||fK_\delta||^2 = ||fK_\delta(K_\delta f)^*|| = ||fK_\delta K_\delta^* f^*||$, where f^* is the complex conjugate function. Now $K_\delta K_\delta^*$ is convolution with $k = k_\delta^* k_\delta^*$ and this is easily shown to have $|k(x)| \leqq c_0\omega_{2s-n}(x)$ for $|x| < \delta$, $k(x) = 0$, $|x| > \delta$, with c_0 independent of δ. Then the Schwarz inequality gives

$$\begin{aligned} |(fk_\delta K_\delta^* f^* u, v)|^2 &\leqq \int |f(x)^2 k(x-y) u(y)^2|\, dydx \int |f(y)^2 k(x-y) v(x)^2|\, dxdy \\ &\leqq c_1^2 [M_{2s-n,2,\delta}(f)]^2\, ||u||_0^2\, ||v||_0^2 . \end{aligned}$$

This shows that $||fK_\delta||^2 \leqq c_1 M_{2s-n,2,\delta}(f)$. Finally, l_δ is a smooth function of rapid decrease, so L_δ is continuous from H^0 to H^s. Combining these facts with (6), (7), and (10) we get (9).

Our assumptions on $A(x, D)$ imply [8]

$$(11) \qquad \text{Re } a(u, u) = \text{Re}(A_0 u, u) \geqq c_1 ||u||_r^2 - c_2 ||u||_0^2,\ u \in \mathscr{D} ,$$

where $c_1 > 0$. Conversely it is clear that

$$|a(u, v)| \leqq c_3 \|u\|_r \|v\|_r, \; u, v \in \mathscr{D} . \tag{12}$$

From (12), (7), (3) and assumption (I) we get

$$|b(u, v)| \leqq c_4 |u| \, |v|, \; u, v \in \mathscr{D} . \tag{13}$$

Therefore $b(u, v)$ extends uniquely to a continuous form on W^r. When $m < n$, assumption (I) and (9), (11) give

$$\begin{aligned} \operatorname{Re} b(u, u) &= \operatorname{Re} a(u, u) + \|\rho u\|_0^2 - \|\sigma u\|_0^2 \\ &\geqq c_5 |u|^2 - c_6 \|u\|_0^2 , \end{aligned} \tag{14}$$

with $c_5 > 0$. This also holds when $m \geqq n$; here multiplication by σ is continuous from H^s to H^0 for any $s > 0$. Taking $0 < s < r$ and using the well-known fact that for $\varepsilon > 0$,

$$\|u\|_s^2 \leqq \varepsilon \|u\|_r^2 + c(\varepsilon) \|u\|_0^2 ,$$

we again can get (14).

It is now easy to show that B as defined in § 1 is a closed extension of B_0, cf. [9]. The fact that $B + \lambda$ is $1 - 1$ and onto for $\operatorname{Re} \lambda \geqq \lambda_0 > c_6$ follows from (14) and the Lax-Milgram lemma as in [9]. Finally, if $a_{\alpha\beta} = \bar{a}_{\alpha\beta}$ and q is real-valued, then for u, v in the domain of B,

$$(Bu, v) = b(u, v) = b(v, u)^* = (Bv, u)^* = (u, Bv) .$$

Thus B is symmetric. For $\lambda \geqq \lambda_0$, $(B + \lambda)^{-1}$ is symmetric, hence self-adjoint. Then $B + \lambda$ and B are self-adjoint. It can be shown that A as defined above is the same as A defined by $Au = f \in H^0$ if and only if $u \in H^r$ and $a(u, v) = (f, v)$, all $v \in H^r$. Then the same results hold for A.

3. Proof of Theorem 2. Suppose $\operatorname{Re} \lambda \geqq \lambda_0$. For convenience we may replace A, B by $A + \lambda$, $B + \lambda$, and assume that A and B are invertible. We want to show that $A^{-1} - B^{-1}$ is compact.

The space H^{-r} is dual to H^r via the inner product in H^0. Specifically, we can consider H^s as contained in H^t for $s \geqq t$ in the obvious way. Then H^{-r} is the completion of H^0 with respect to the norm (identical to the previously defined norm)

$$\|u\|_{-r} = \sup_{v \in H^r, \|v\|_r = 1} |(u, v)| .$$

Similarly, define W^{-r} as the completion of H^0 with respect to the norm

$$|u|_- = \sup_{v \in W^r, \|v\|=1} |(u, v)| .$$

Because of the definitions of the norms we have natural inclusions

$$W^r \subset H^r \subset H^0 \subset H^{-r} \subset W^{-r} . \tag{15}$$

LEMMA 3. *A and B have unique bounded extensions mapping H^r onto H^{-r} and W^r onto W^{-r} respectively.*

Proof. For u in the domain of B, (13) gives

$$|Bu|_- = \sup_{|v|=1} |(Bu, v)| = \sup_{|v|=1} |b(u, v)| \leqq c_4 |u| .$$

Thus B has a unique continuous extension B_1 mapping W^r into W^{-r}. With our replacement of B by $B + \lambda$ we have $c_6 < 0$ in (14), so B_1 has closed range in W^{-r}. The range includes H^0, hence is all of W^{-r}. Similarly A has an extension A_1.

From now on we shall drop the subscripts on A_1, B_1 and consider A, B as being defined either on their original domains or on H^r, W^r. Since A^* maps H^m onto H^0 [8], we can also, by duality, consider A as mapping H^0 onto H^{-m}. With these conventions,

$$\begin{aligned} A^{-1} - B^{-1} &= A^{-1}(B - A)B^{-1} = A^{-1}qB^{-1} \\ &= (A^{-1}\rho)(\rho B^{-1}) - (A^{-1}\sigma)(\sigma B^{-1}) + i(A^{-1}\tau)(\tau B^{-1}) \end{aligned} \tag{16}$$

on W^{-r}. In fact (ρB^{-1}) is bounded from W^{-r} to H^0 via W^r, while A^{-1} is bounded from H^0 to itself via H^{-m}, by Lemma 1. Similarly for the other terms. We want to show that each term on the right in (16) is compact, but shall consider in detail only the first term.

Take $\varphi \in \mathscr{D}$ with $\varphi(x) = 1$, $|x| < 1/2$, and $\varphi(x) = 0$, $|x| > 1$. Let $\varphi_t(x) = \varphi(t^{-1}x)$, $t > 0$. We consider multiplication by φ as an operator, also denoted by φ. Write

$$A^{-1}\rho^2 B^{-1} = A^{-1}\rho[\varphi_t \rho B^{-1}] + [A^{-1}\rho(1 - \varphi_t)]\rho B^{-1} .$$

Lemma 1 and assumption (II) imply that the norm of the second term on the right as operator in H^0 goes to 0 as $t \to \infty$. Therefore in suffices to show that the first term on the right is compact. Assumption (I)′ implies that $A^{-1}\rho\varphi_t$ is compact [9]. We shall show that in any case, $\varphi_t \rho B^{-1}$ is compact. In general, if S is bounded from W^{-r} or H^{-r} to H^0, φS will not be compact in H^0. We must use particular properties of B^{-1}.

LEMMA 4. *Let S be a bounded operator in H^0 such that one of the following holds:*

(a) *S is bounded from L^2 to H^s, some $s > 0$;*

(b) *S is bounded from* H^{-s} *to* L^2, *some* $s > 0$.

Suppose $\varphi \in \mathscr{D}$. *Then in case* (a), φS *is compact in* H^0, *while in case* (b), $S\varphi$ *is compact in* H^0.

Proof. This follows immediately from the well-known fact that φ is compact as operator from H^s to H^0 or from H^0 to H^{-s}.

We shall say that a bounded operator S in H^0 is *acceptable* if it is of the form

$$S = (\varphi_1 S_1 \psi_1)(\varphi_2 S_2 \psi_2) \cdots (\varphi_k S_k \psi_k) \tag{17}$$

with $\varphi_j, \psi_j \in \mathscr{D}$ and each S_j bounded in H^0. We say that S in the form (17) is of *weight*:-N if there are integers $N_j \geqq 0$ with $\Sigma N_j \geqq N$ such that for each j, S_j is either bounded from H^0 to H^{N_j} or from H^{-N_j} to H^0.

If $\varphi \in \mathscr{D}$, φ clearly maps W^r and W^{-r} into themselves. On W^{-r} the commutator $[B^{-1}, \varphi] = B^{-1}\varphi - \varphi B^{-1}$ can be written

$$[B^{-1}, \varphi] = B^{-1}[\varphi, B]B^{-1} = B^{-1}[\varphi, A]B^{-1} = B^{-1}C(\varphi)B^{-1} ,$$

where $C(\varphi) = [\varphi, A]$ is a differential operator of order $< m$. It is easily seen that for $u, v \in \mathscr{D}$,

$$(C(\varphi)u, v) = \Sigma(c_{\alpha\beta}D^\beta u, D^\alpha v)$$

with the summation over $|\alpha| \leqq r, |\beta| \leqq r, \ |\alpha| + |\beta| < m$. The coefficients $c_{\alpha\beta}$ have bounded derivatives of order $\leqq \max(|\alpha|, |\beta|)$, with the bounds depending only on the bounds of derivatives of the $a_{\alpha\beta}$ and of φ. It follows readily from this, duality, and the equivalent form (2)′ of the norm that

$$\|C(\varphi)u\|_{k-m-1} \leqq c(\varphi)\, \|u\|_k, \ k = -1, 0, 1, \cdots, m , \tag{18}$$

with

$$c(\varphi) = c \sup_{|\alpha| \leqq 2m, x \in R^n} |D^\alpha \varphi(x)| , \tag{19}$$

c independent of φ. Note also that if $\psi\varphi = \varphi$, then

$$C(\varphi) = \psi C(\varphi) = C(\varphi)\psi .$$

Let $s = r$ if r is even, $s = r - 1$ if r is odd. Define $\Lambda^\pm = G_{\mp s}$, where G_t is as in § 2. Then $\Lambda^+ = (1 - \Delta)^{s/2}$ is a differential operator of order s.

LEMMA 5. *Suppose* $\varphi, \psi \in \mathscr{D}$ and N *is an integer* $\geqq 0$. *Then* $\varphi\Lambda^- C(\psi)B^{-1}$ *can be written as a sum of acceptable terms of weight*

$-s-1$ *plus a remainder term of the form* ST *with* S *acceptable of weight*-N *and* T *bounded in* H^0.

Proof. Induce on N. For $N=0$, take $S=\varphi$, $T=\Lambda^- C(\psi)B^{-1}$. Otherwise take $\psi=\psi_0, \psi_1, \psi_2 \in \mathscr{D}$ with $\psi_{j+1}\psi_j=\psi_j$. Set $C=C(\psi)$, $C_1=C(\psi_1)$, $R=B^{-1}$. Then

$$\begin{aligned}\varphi\Lambda^- CR &= \varphi\Lambda^- C\psi_1 R \\ &= \varphi\Lambda^- CR\psi_1 + \varphi\Lambda^- CRC_1R ,\end{aligned} \tag{20}$$

$$R\psi_1 = R\psi_1\Lambda^+\Lambda^-\psi_2 = (R\psi_1\Lambda^+\psi_2)(\psi_2\Lambda^-\psi_1) . \tag{21}$$

Now $\Lambda^- CR\psi_1\Lambda^+$ is bounded from H^{-1} to H^0 if $s=r-1$ and from H^0 to H^1 if $s=r$, so the first term on the right in (20) can be expressed in the desired form (with no remainder). As for the second term,

$$\begin{aligned}\varphi\Lambda^- CRC_1R &= \varphi\Lambda^- CR\psi_1\Lambda^+\Lambda^- C_1R \\ &= [\varphi\Lambda^- CR\psi_1\Lambda^+\psi_2][\psi_2\Lambda^- C_1R] .\end{aligned}$$

The first term in brackets is again acceptable of weight -1, and we may apply the induction assumption to the second term.

LEMMA 6. *Suppose* $\varphi \in \mathscr{D}$ *and* N *is an integer* $\geqq 0$. *Then* φB^{-1} *can be expressed as a sum of terms of the form* $\varphi B^{-1}\psi\Lambda^+ S$ *with* S *acceptable of weight* $\leqq -s$ *and* $\psi \in \mathscr{D}$, *plus a remainder term of the form* $\varphi B^{-1}\psi\Lambda^+ ST$ *with* S *acceptable of weight* $-N$ *and* T *bounded in* H^0.

Proof. Take $\psi=\psi_0, \psi_1, \psi_2 \in \mathscr{D}$ *with* $\psi\varphi=\varphi$, $\psi_{j+1}\psi_j=\psi_j$. Let $C=C(\psi)$, $R=B^{-1}$. Then

$$\varphi R = \varphi\psi R = \varphi R\psi + \varphi RCR .$$

Now $\varphi R\psi$ can be expressed as in (21), while

$$\varphi RCR = \varphi R\psi_1 CR = [\varphi R\psi_1\Lambda^+\psi_2][\psi_2\Lambda^- CR] .$$

Applying Lemma 5 to the second term in brackets, we get what we want.

COROLLARY. *If* $\varphi \in \mathscr{D}$, $\varphi\rho B^{-1}$ *is a compact operator in* H^0.

Proof. Lemma 6 shows that $\varphi\rho B^{-1}$ is a sum of terms of the form $[\varphi\rho B^{-1}\psi\Lambda^+]S$, ψ, $\eta \in \mathscr{D}$ and S bounded. Moreover, if $s>0$ then by Lemma 4 S is compact; if $s=0$ then $\Lambda^+=I$ and the term in brackets is compact, mapping H^{-r} into H^0.

The conclusion of the Corollary applies with ρ replaced by σ or τ, so $A^{-1} - B^{-1}$ is compact. This proves Theorem 2.

The following will be used in the proof of Theorem 3.

LEMMA 7. *Suppose $\varphi \in \mathscr{D}$ and N is an integer $\geqq 0$. Then φB^{-1} can be expressed as a sum of acceptable operators of weight $-m$, plus remainder terms of the form ST, S acceptable of weight $-N$ and T bounded in H^0.*

Proof. When $s = r$, $B^{-1}\psi\Lambda^+$ is bounded from H^0 to H^r and this conclusion follows immediately from Lemma 6. When $s = r - 1$,

$$\varphi B^{-1}\psi\Lambda^+\psi_1 = \varphi B^{-1}\psi\Lambda^+(1 - \Delta)\psi_1(1 - \Delta)^{-1}\psi_2 .$$

Breaking up $\Lambda^+(1 - \Delta)$ into monomial differential operators, we get a sum of terms $\varphi B^{-1}E_1\psi_1\psi_2E_2(1 - \Delta)^{-1}\psi_2$, with E_1, E_2 differential operators of orders $\leqq r$ and $\leqq 1$ respectively. These terms are then acceptable of weight $(-r) + (-1)$. Thus again the conclusion follows from Lemma 6.

LEMMA 8. *Suppose $\varphi \in \mathscr{D}$ and N an integer $\geqq 0$. Then $A^{-1}\varphi$ can be expressed as a sum of terms of the form $SA^{-1}\varphi$ with S acceptable of weight 0, plus a remainder term of the form $TSA^{-1}\varphi$ with S acceptable of weight $-N$ and T bounded in H^0.*

Proof. Take $\psi_0, \psi_1, \psi_2, \cdots$ in $\mathscr{D}$ with $\psi_2\varphi = \varphi$, $\psi_{j+1}\psi_j = \psi_j$. Let $R = A^{-1}$, $C_j = [\varphi_j, A]$. Then

$$\begin{aligned} R\varphi &= \psi_0 R - RC_0R\varphi \\ &= \psi_0R\varphi - \psi_1RC_0R\varphi + RC_1RC_0R\varphi . \\ &= \cdots . \end{aligned}$$

Now RC_j maps H^{-1} to H^0, so $\psi_{j+2}RC_j\psi_{j+1}$ is acceptable of weight -1. Continuing as above, we get acceptable terms of the form we want, plus a remainder term

$$\pm RC_N[RC_{N-1}R \cdots RC_0]R\varphi .$$

Take $\pm RC_N$ as T and write the term in brackets as a product of terms $\psi_{j+2}RC_j\psi_{j+1}$.

4. Proof of Theorems 3 and 4. Once again, given $\psi \in C_c^\infty$, let $\psi_t(x) = \psi(t^{-1}x)$. It follows from the equivalent form (2)$'$ of the norm that for integer $l \geqq 0$,

(22) $$\|\varphi u\|_l \leqq c_l \sup_{|\alpha|\leqq l, x\in R^n} |D^\alpha\varphi(x)|\,\|u\|_l .$$

Thus ψ_t is bounded uniformly with respect to t, $t \geqq 1$, as a map from H^l to itself. By duality the same is true for integers $l < 0$.

LEMMA 9. *Suppose $\varphi \in \mathscr{D}$ and $\varphi_t(x) = \varphi(t^{-1}x)$. For each integer $l \geqq 0$, each $t \geqq 1$, and each integer $j \geqq 1$ there is a subspace H_j of H^0 having codimension $j-1$, such that*

(23) $$\|\varphi_t u\|_{-l} \leqq cj^{-l/n}\cdot t^l\,\|u\|_0,\ u \in H_j .$$

The constant $c = c(l)$ can be chosen independent of j and $t, t \geqq 1$.

Proof. Let Ω_t be the cube $\{x\,|\,|x_k| < t, k = 1, \cdots, n\}$. We may assume that $\varphi(x) = 0$ outside Ω_1. Choose $\psi \in \mathscr{D}$ with $\int\psi(\xi)d\xi = (2\pi)^{n/2}$, and let η be the inverse Fourier transform of ψ. Then $\eta(0) = 1$. Replacing ψ by $\varepsilon^{-n}\psi(\varepsilon^{-1}x)$ for small enough $\varepsilon > 0$, we may assume both that $|\eta(x)| \geqq \frac{1}{2}$ on Ω_1 and that ψ vanishes outside Ω_1. Let $\eta_t(x) = \eta(t^{-1}x)$. For a given $t \geqq 1$ and α an n-tuple of integers, let $\alpha\cdot x = \Sigma\alpha_k x_k$, $|\alpha| = \Sigma|\alpha_k|$, and set

$$\begin{aligned} e_\alpha(x) &= (2t)^{-n/2}\exp(t^{-1}\pi i\alpha\cdot x),\ x\in\Omega_t ,\\ &= 0,\ \notin \Omega_t ;\\ f_\alpha(x) &= (2t)^{-n/2}\,\eta_t(x)\exp(t^{-1}\pi i\alpha\cdot x),\ \text{all } x. \end{aligned}$$

Then

(24) $$\hat f_\alpha(\xi) = (t/2)^{n/2}\,\psi(t\xi + \pi i\alpha) .$$

It follows that $\hat f_\alpha \hat f_\beta \equiv 0$ for $\alpha \neq \beta$, so the f_α are orthogonal as elements of H^{-l}. Let $\alpha(1), \alpha(2), \cdots,$ be an enumeration of the n-tuples α with $|\alpha(j)| \leqq |\alpha(j+1)|$. Let $e_k = e_{\alpha(k)}, f_k = f_{\alpha(k)}$. It follows from (24) and (2) that

(25) $$\|f_k\|_{-l} \leqq c_0 t^l\,|\alpha(k)|^{-l} \leqq c_1 t_k^{l-l/n}$$

for $|\alpha| > 0$, with c_1 independent of k and $t, t \geqq 1$.

Let H_j be the orthogonal complement of $\{e_1, e_2, \cdots, e_{j-1}\}$ in H^0. The e_k are an orthonormal basis for $L^2(\Omega_t)$, so for any $u \in H^0$,

(26) $$u = \Sigma a_k e_k \text{ on } \Omega_t,\ \Sigma|a_k|^2 \leqq \|u\|_0^2 ,$$

with $a_k = (u, e_k)$. If $u \in H_j$,

(27) $$\varphi_t u = \varphi_t \sum_{k\geqq j} a_k e_k = (\varphi_t\eta_t^{-1})\sum_{k\geqq j} a_k f_k .$$

Since the f_k are orthogonal in H^{-l}, (25), (26), (27) and the remarks preceding the lemma give (23), with c independent of j and t, $t \geqq 1$.

LEMMA 10. *Suppose S is bounded in H^0 and either*
(a) $\|Su\|_0 \leqq M\|u\|_{-l}$, *all* $u \in H^0$
or
(b) $\|Su\|_l \leqq M\|u\|_0$, *all* $u \in H^0$,
where l is a positive integer. Suppose $\varphi \in \mathscr{D}$ and $\varphi_t(x) = \varphi(t^{-1}x)$. Set $T_t = S\varphi_t$ in case (a), $T_t = \varphi_t S$ *in case* (b). *Then the characteristic numbers satisfy*

$$\mu_j(T_t) \leqq cM_j^{-l/n} \cdot t^l , \tag{28}$$

where c is independent of j and t, $t \geqq 1$.

Proof. Suppose (a). Given $t \geqq 1$, let H_j be as in Lemma 9 and apply (4), (23), and (a). Case (b) follows from (a) and the fact that $\mu_j(T) = \mu_j(T^*)$.

Now consider Theorem 3. Again we may assume $\lambda = 0$ and look at $A^{-1} - B^{-1}$, p a positive integer. On W^{-r},

$$\begin{aligned} A^{-p} - B^{-p} &= \sum_{j=0}^{p-1} A^{j+1-p}[A^{-1} - B^{-1}]B^{-j} \\ &= \sum_{j=0}^{p-1} A^{j-p}\, qB^{-j-1} . \end{aligned} \tag{29}$$

Once again take $\varphi \in \mathscr{D}$ with $\varphi(x) = 1$, $|x| < \frac{1}{2}$, and $\varphi(x) = 0$, $|x| > 1$, and set $\varphi_t(x) = \varphi(t^{-1}x)$. Repeated applications of Lemmas 6, 7 and 8 show that

$$A^{j-1}\varphi_t \rho^2 \varphi_t B^{1-j}$$

can be written as a sum of terms of the form

$$T_1 S_1 (A^{-1}\rho)\varphi_t^2 (B^{-1}) S_2 T_2 . \tag{30}$$

Here $\psi, \eta \in \mathscr{D}$, T_1 and T_2 are bounded in H^0, and S_1 and S_2 are acceptable of weights s_1 and s_2 with

$$s_1 = (p - j - 1)m,\ s_2 = jm . \tag{31}$$

Furthermore the numbers of summands and factors are independent of t, and the operator norms are bounded independent of $t \geqq 1$. Recall also that $(A^{-1}\rho)$ is bounded in H^0 and ρB^{-1} is bounded from H^{-r} to H^0.

In general, characteristic numbers satisfy

$$\mu_{jk+1}(S_1 S_2 \cdots S_k) \leqq \mu_{j+1}(S_1)\mu_{j+1}(S_2) \cdots \mu_{j+1}(S_k) , \tag{32}$$

$$\mu_{jk+1}(S_1 + \cdots + S_k) \leqq \mu_{j+1}(S_1) + \cdots + \mu_{j+1}(S_k) ; \tag{33}$$

[3], Corollary X1.9.3. Also clearly $\mu_j(S) \leqq ||S||$.

Now write each term on the right in (29) as a sum of a term with $q\varphi_t^2$ and a term with $q(1-\varphi_t^2)$. Grouping the former and latter terms together we get

$$A^{-p} - B^{-p} = D_t + E_t . \tag{34}$$

Our assumptions $(\mathrm{III})_a$ or $(\mathrm{III})'_a$ together with Lemma 1 and the fact that $(1-\varphi_t^2) = 0$ for $|x| < t/2$ give

$$||E_t|| = 0(t^{-a}) , \tag{35}$$

where again the norm is the operator norm in H^0. In case $(\mathrm{III})'_a$ this comes from (8) applied to $A^{-1}q(1-\varphi_t^2)^{1/2}$ and (7) applied to $(1-\varphi_t^2)^{1/2}qB^{-1}$, while in case $(\mathrm{III})_a$ we apply (7) to $A^{-1}\rho(1-\varphi_t^2)$ and treat the terms with σ and τ as in case $(\mathrm{III})'_a$.

Finally, (28) – (35) imply

$$\mu_j(A^{-p} - B^{-p}) \leqq c(t^{-a} + t^l \cdot j^{-l/n}) , \tag{36}$$

with c independent of j and t, $t \geqq 1$, and with

$$l = (p - j - 1)m + r + jm = (p - \tfrac{1}{2})m ;$$

the extra r coming from $\varphi_t q B^{-1}$. Now let $t = j^b$ with $b = l/n(a+l)$. Then (36) becomes

$$\mu_j(A^{-1} - B^{-p}) = 0(j^{-\nu}), \ \nu = al/n(a+l) . \tag{37}$$

As $p \to \infty$, $\nu = \nu(p) \to a/n$. This proves Theorem 3.

Finally, under the assumptions of Theorem 4,

$$\mu_j = \mu_j((A+\lambda)^{-p} - (B+\lambda)^{-p}) = 0(j^{-1-\varepsilon})$$

for $\mathrm{Re}\,\lambda \geqq \lambda_0$, large enough p, and small enough $\varepsilon > 0$. Thus $\Sigma\mu_j < \infty$, and $(A+\lambda)^{-p} - (B+\lambda)^{-p}$ is a trace class operator. The conclusions of Theorem 4 now follow from results of Kuroda and Kato; see [5], Theorem 4.12 and Remark 4.13.

REFERENCES

1. E. Balslev, *The essential spectrum of self-adjoint differential operators in* $L^2(R_n)$, Math. Scand., **19** (1966), 193–210.

2. M. S. Birman, *On the spectrum of singular boundary value problems*, Mat. Sb., **97** (1961), 125–174; Amer. Math. Soc. Translations (2), vol. 53 (1966), 23–80.

3. N. Dunford and J. T. Schwartz, *Linear Operators*, Vol. 2, 1963, Interscience.

4. J. S. Howland, *On the essential spectrum of Schrodinger operators with singular potentials*, Pacific J. Math., **25** (1968), 533–542.

5. T. Kato, *Perturbation Theory for Linear Operators*, 1966, Springer-Verlag.

6. P. A. Rejto, *On the essential spectrum of the hydrogen energy and related operators*, Pacific J. Math., **19** (1966), 109–140.

7. M. Schechter, *On the essential spectrum of an arbitrary operator*, J. Math. Anal. Appl., **13** (1966), 205-215.
8. ———, *On the invariance of the essential spectrum of an arbitrary operator, II*, Richerche Mat., **16** (1967), 3-26.
9. ———, *On the invariance of the essential spectrum of an arbitrary operator, III*, Proc. Camb. Phil. Soc., **64** (1968), 975-984.

Received August 4, 1969 and in revised form September 11, 1970. Research supported in part by the Army Research Office, Durham.

UNIVERSITY OF CHICAGO

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics
Vol. 37, No. 1 January, 1971