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**SOLVABLE AND SUPERSOLVABLE GROUPS IN WHICH  
EVERY ELEMENT IS CONJUGATE TO ITS INVERSE**

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## SOLVABLE AND SUPERSOLVABLE GROUPS IN WHICH EVERY ELEMENT IS CONJUGATE TO ITS INVERSE

J. L. BERGGREN

Let  $\mathfrak{S}$  be the class of finite groups in which every element is conjugate to its inverse. In the first section of this paper we investigate solvable groups in  $\mathfrak{S}$ ; in particular we show that if  $G \in \mathfrak{S}$  and  $G$  is solvable then the Carter subgroup of  $G$  is a Sylow 2-subgroup and we show that any finite solvable group may be embedded in a solvable group in  $\mathfrak{S}$ . In the second section the main theorem reduces the study of supersolvable groups in  $\mathfrak{S}$  to the study of groups in  $\mathfrak{S}$  whose orders have the form  $2^a p^b$ ,  $p$  an odd prime.

NOTATION. The notation here will be as in [1] with the addition of the notation  $G = XY$  to mean  $G$  is a split extension of  $Y$  by  $X$ . Also,  $F(G)$  will denote the Fitting subgroup of  $G$  and  $\Phi(G)$  the Frattini subgroup of  $G$ . We will denote the maximal normal subgroup of  $G$  of odd order by  $O_2(G)$ . Further,  $\text{Hol}(G)$  will denote the split extension of  $G$  by its automorphism group.

If  $K$  and  $T$  are subgroups of  $G$  we will call  $K$  a  $T$ -group if  $T \leq N_G(K)$  and we say  $K$  is a  $T$ -indecomposable  $T$ -group if  $K = K_1 \times K_2$ , where  $K_1$  and  $K_2$  are  $T$ -groups, implies  $K_1 = \langle 1 \rangle$  or  $K_2 = \langle 1 \rangle$ .

1. Burnside [2] proved that if  $P$  is a Sylow  $p$ -subgroup of the finite group  $G$  and if  $X$  and  $Y$  are  $P$ -invariant subsets of  $P$  which are not conjugate in  $N_G(P)$  then they are not conjugate in  $G$ . Using Burnside's method one may prove a similar fact about the Carter subgroups. The proof is easy and we omit it.

LEMMA 1.1. *Let  $C$  be a Carter subgroup of the solvable group  $G$  and let  $A$  and  $B$  be subsets of  $C$ , both normal in  $C$ . If  $A \neq B$  then  $A$  and  $B$  are not conjugate in  $G$ .*

THEOREM 1.1. *If  $G$  is a solvable group in  $\mathfrak{S}$  then a Carter subgroup of  $G$  is a Sylow 2-subgroup of  $G$ .*

*Proof.* Let  $C$  be a Carter subgroup of  $G$ . If  $C$  has a nonidentity element of odd order then  $C$  has a nonidentity central element  $g$  of odd order, since  $C$  is nilpotent. Then with  $A = \{g\}$  and  $B = \{g^{-1}\}$  the hypotheses of Lemma 1.1 are satisfied and, since  $A \neq B$ ,  $g$  and  $g^{-1}$  are not conjugate in  $G$ , contradicting our supposition that  $G \in \mathfrak{S}$ .

Hence  $C$  is a 2-group. As  $C$  is self-normalizing in  $G$ ,  $C$  must be a Sylow 2-subgroup of  $G$ .

NOTE. This proof implies, also, that  $Z(C)$  is an elementary abelian 2-group. However, the theorem of Burnside we mentioned can be used to show that if  $T$  is a Sylow 2-subgroup of any group  $G \in \mathfrak{S}$  (whether solvable or not) then  $Z(T)$  is elementary abelian. Thus, if  $G \in \mathfrak{S}$  and  $T$  is a Sylow 2-subgroup of  $G$  the ascending central series of  $T$  has elementary abelian factors.

COROLLARY 1.1. *If  $T$  is a Sylow 2-subgroup of a solvable group  $G \in \mathfrak{S}$  then  $N_G(T) = T$ .*

*Proof.* By Theorem 1.1  $T$  is a Carter subgroup of  $G$ . Carter subgroups are self-normalizing.

COROLLARY 1.2. *If  $G$  and  $T$  are as in Corollary 1.1, and if  $T$  is abelian, then  $G$  has a normal 2-complement.*

*Proof.* By Corollary 1.1 and the assumption  $T$  is abelian,  $T$  is in the center of its normalizer. The result follows from a well-known theorem of Burnside.

We now investigate two families of solvable groups in  $\mathfrak{S}$ .

THEOREM 1.2. *If  $G \in \mathfrak{S}$  and a Sylow 2-subgroup of  $G$  is cyclic then  $G = TK$  where  $K$  is an abelian normal subgroup of odd order and  $T = \langle \alpha \rangle$  with  $\alpha^2 = 1$  and  $g^\alpha = g^{-1}$  for all  $g \in K$ .*

*Proof.* As  $G$  has a cyclic Sylow 2-subgroup,  $G$  is solvable. By Corollary 1.2  $G = TK$ ,  $T = \langle \alpha \rangle$  is a Sylow 2-subgroup of  $G$  and  $K$  is a normal subgroup of odd order. By the Note after Theorem 1.1,  $\alpha^2 = 1$ . If  $\alpha$  did not induce a fixed-point-free automorphism of  $K$  then  $C_G(T) \cap K \cong \langle 1 \rangle$ , so  $N_G(T) \cong T$ , contradicting Corollary 1.1. Thus  $g \rightarrow g^\alpha$  is a fixed-point-free automorphism of  $K$ . It is known that if  $K$  has a fixed-point-free automorphism  $\alpha$  of order 2 then  $\alpha(k) = k^{-1}$  for all  $k \in K$  and hence  $K$  is abelian.

THEOREM 1.3. *Let  $G$  be a finite solvable group in  $\mathfrak{S}$  and suppose a Sylow 2-subgroup  $T$  of  $G$  has order 4. Then  $T$  is elementary abelian,  $G$  has a normal 2-complement  $K$ , and  $K^{(1)}$  is nilpotent.*

*Proof.* As  $G$  is solvable, Corollary 1.1 and 1.2 imply that  $G =$

$TK$  where  $|T| = 4$  and  $K$  is a normal subgroup of odd order. The Note after Theorem 1.1 implies  $T$  is elementary, say  $T = \langle \alpha \rangle \times \langle \beta \rangle$ . Let  $K_\alpha$  and  $K_\beta$  denote the set of fixed points of the automorphisms of  $K$  induced by  $\alpha$  and  $\beta$  respectively. Then  $\langle 1 \rangle = C_K(T) \cong K_\alpha \cap K_\beta$ . Hence, as  $T$  is abelian,  $K_\alpha$  is  $\beta$ -invariant and  $\beta$  induces a fixed-point free automorphism of  $K_\alpha$ . Thus  $K_\alpha$  is abelian. Then, by [4],  $K^{(1)}$  is nilpotent.

Finally, we show that any finite solvable group can be embedded in a solvable group in  $\mathfrak{S}$ . We shall need the following lemma.

**LEMMA 1.2.** *Let  $G \in \mathfrak{S}$  and let  $\langle x \rangle$  be a cyclic group of order  $p$ , where  $p$  is an odd prime. Let  $\alpha$  be an involution and define  $H = \langle Gw\langle x \rangle, \alpha \rangle$ , where  $x^\alpha = x^{-1}$  and  $b^\alpha = b$  for all  $b \in G$ . Then  $H \in \mathfrak{S}$ .*

*Proof.* Let  $K = G \times G^x \times \dots \times G^{x^{p-1}}$  be the base subgroup of  $Gw\langle x \rangle$ . Then  $K \in \mathfrak{S}$  since  $G \in \mathfrak{S}$ . Suppose  $h_1 \in H$  and

$$h_1 = x^r g_0 \cdot g_1^x \cdots g_{p-1}^{x^{p-1}},$$

where  $r \not\equiv 0(p)$ . Writing  $[j]$  for  $x^j$  we may write

$$h_1 = x^r \cdot g_0 \cdot g_r^{[r]} \cdots g_{(p-1)r}^{[(p-1)r]}.$$

Now, if  $g \in G$  then  $(g^{[i]})^{x^r} = g^{[i+r]}$  implies that

$$(g^{[i]})^{-1} x^{-r} g^{[i]} x^r = (g^{[i]})^{-1} g^{[i+r]},$$

and hence  $(g^{[i]})^{-1} x^r g^{[i]} = x^r (g^{[i+r]})^{-1} g^{[i]}$ . Thus if  $\beta = g_{i_r}^{[(e-1)r]}$  then  $(x^r)^\beta = x^r (g_{e_r}^{-1})^{[er]} (g_{e_r})^{[(e-1)r]}$ . Writing  $h_1^\beta = x^r \cdot f_0 \cdot f_r^{[r]} \cdots f_{(p-1)r}^{[(p-1)r]}$ , where  $f_i \in G$  for all  $i$ , we see that  $f_{i_r} = g_{i_r}$  if  $i \neq e$ ,  $e-1$  while  $f_{e_r} = 1$ . Thus first changing the rightmost  $g_{i_r}^{[i_r]}$  in  $h_1$  to 1 by conjugation and proceeding to the left we may conjugate  $h_1$  to an element  $h = x^r g$ , where  $g \in G = G^{[0]}$ .

Pick  $a \in G$  such that  $g^a = g^{-1}$  and let  $u = aa^x \cdots a^{x^{p-1}}$ . Then with  $\gamma = \alpha u x^{-r}$  we have  $h^\gamma = h^{-1}$ . It remains to consider elements of  $H$  of the form  $h = \alpha \cdot x^r \cdot g_0 \cdot g_1^{[1]} \cdots g_{p-1}^{[p-1]}$ , where  $[j]$  denotes  $x^j$ . If  $r \not\equiv 0(p)$  then let  $e$  be an integer such that  $2e \equiv -r(p)$ . Then  $h$  conjugated by  $x^e$  has the form  $\alpha y_0 y_1^{[1]} \cdots y_{p-1}^{[p-1]}$  where the  $y_i \in G$ .

We now exploit the fact that, since  $x^\alpha = x^{-1}$  and  $g^\alpha = g$  for all  $g \in G = G^{[0]}$ ,  $g_{p-1}^{[p-1]} = (g_{p-1}^{[1]})^\alpha$ ,  $g_{p-2}^{[p-2]} = (g_{p-2}^{[2]})^\alpha$ , etc. Thus

$$\alpha^{r(p-1)} = \alpha (g_{p-1}^{-1})^{[p-1]} (g_{p-1})^{[1]},$$

where  $\gamma(p-1) = g_{p-1}^{[1]}$ . Performing this computation for

$$\gamma(p-1), \gamma(p-2), \dots, \gamma((p+1)/2),$$

where  $\gamma(e) = g_e^{[p-e]}$  and observing that  $u = \gamma(p-1) \cdots \gamma((p+1)/2)$

has the identity in  $G^{[i]}$  as its  $i$ -th component for  $i > ((p+1)/2)$  we see that  $h^u$  has the form  $h_1 = \alpha \cdot f_0 \cdot f_1^{[1]} \cdots f_r^{[r]}$  where  $r = (p-1)/2$  and  $f_i \in G$  for all  $i$ . Then  $h_1^{-1} = \alpha \cdot f_0^{-1} \cdot ((f_1^{-1})^{[1]} \cdots (f_r^{-1})^{[r]})^\alpha$ . Now for all  $i = 0, \dots, r$  pick  $a_i \in G$  such that  $f_i^{\alpha_i} = f_i^{-1}$  and let  $u = a_0 \cdot v \cdot v^\alpha$  where  $v = a_1^{[1]} \cdots a_r^{[r]}$ . Taking  $x = u\alpha$  it is easy to see that  $h_1^x = h_1^{-1}$ , using the fact that  $(vv^\alpha, \alpha) = (g_0, vv^\alpha) = 1$ . This disposes of all cases.

**Theorem 1.4.** *If  $G$  is a finite solvable group then there exists a solvable group  $L \in \mathfrak{S}$  and a monomorphism  $\tau: G \rightarrow L$ .*

*Proof.* If  $G$  is abelian let  $L = \langle G, \alpha \rangle$  where  $\alpha^2 = 1$  and  $g^\alpha = g^{-1}$  for all  $g \in G$ . Then in  $L$  every element of  $G$  is conjugate to its inverse and all other elements lie in the coset  $G\alpha$  which consists of involutions, so  $L \in \mathfrak{S}$  and  $L$  is solvable. Hence the theorem is true for all abelian groups  $G$ . Induct on  $|G|$  and assume it is true for all solvable groups of order less than the order of  $G$ . Now let  $H \triangleleft G$  such that  $[G:H] = p$ ,  $p$  a prime. Our induction hypothesis says there is a solvable  $K \in \mathfrak{S}$  and a monomorphism of  $HwC_p$  into  $KwC_p$ , where  $C_p$  is cyclic of order  $p$ . By Satz 15.9 [3] (Chapter I) there is a monomorphism of  $G$  into  $HwC_p$ , so  $G$  may be imbedded in  $KwC_p$ . If  $p = 2$  then by Theorem 1.1 of [1]  $KwC_p \in \mathfrak{S}$ , and it is solvable since  $K$  is. If  $p > 2$  then by Lemma 1.2  $KwC_p$  has a solvable extension  $\langle KwC_p, \alpha \rangle \in \mathfrak{S}$ .

Thus, in this case as well,  $G$  may be imbedded in a solvable group in  $\mathfrak{S}$ .

This concludes our investigation of solvable groups in  $\mathfrak{S}$ .

2. In §1 we showed that if  $G \in \mathfrak{S}$  is a solvable group with an abelian Sylow 2-subgroup  $T$  then  $T$  has a normal complement in  $G$ . Of course, if  $G$  is supersolvable then (by the Sylow Tower Theorem)  $T$  has a normal complement  $K$ , regardless whether  $T$  is abelian or  $G \in \mathfrak{S}$ . If we assume that  $G \in \mathfrak{S}$ , where  $G$  is supersolvable, then with the above notation we assert.

**THEOREM 2.1.** *The Sylow 2-subgroup  $T$  is in  $\mathfrak{S}$ , and  $K$  and  $\Phi(T)$  are contained in  $F(G)$ .*

*Proof.* That  $T \in \mathfrak{S}$  was remarked in [1]. Since  $G$  is supersolvable  $G^{(1)} \leq F(G)$ . Now  $G \in \mathfrak{S}$  implies  $G/G^{(1)} \in \mathfrak{S}$  and since  $G/G^{(1)}$  is abelian  $G/G^{(1)}$  is an elementary abelian 2-group. Thus  $\Phi(T) \leq G^{(1)}$ , and since  $(2, |K|) = 1$ ,  $K \leq G^{(1)}$ .

**REMARK.** If  $G \in \mathfrak{S}$  is supersolvable Theorem 2.1 implies  $G$  is a

split extension of a nilpotent group  $K$  by a two-group  $T$  in  $\mathfrak{S}$ . If  $S$  is a Sylow 2-subgroup of  $F(G)$  then  $S \triangleleft G$ , so  $G/S \in \mathfrak{S}$ . But by Theorem 2.1  $G/S$  is isomorphic to a split extension  $EK$  of the nilpotent group  $K$  by an elementary abelian two-group  $E$ . Thus given a supersolvable  $G$  in  $\mathfrak{S}$  there exists a supersolvable  $G^* \in \mathfrak{S}$  such that  $O_2, (G^*) \cong O_2, (G)$  but  $G^*$  has an elementary abelian Sylow 2-subgroup.

Now let  $G = TK \in \mathfrak{S}$  be given, where  $G$  is supersolvable and  $T$  and  $K$  are as above. Let  $P_1, \dots, P_r$  be the Sylow subgroups of  $K$ , so  $K = P_1 \times \dots \times P_r$ . If  $\pi_i$  is the projection of  $K$  onto  $P_i$  let  $H_i = \ker(\pi_i)$ . Then  $H_i \triangleright G$  and  $G/H_i \cong TP_i$ , a split extension of  $P_i$  by  $T$  which is supersolvable and in  $\mathfrak{S}$ . We have now reduced the study of supersolvable groups in  $\mathfrak{S}$  to two questions:

(1) Given a 2-group  $T \in \mathfrak{S}$  and a  $p$ -group  $P$  ( $p$  an odd prime) find the split extensions  $TP$  of  $P$  by  $T$  which are supersolvable and in  $\mathfrak{S}$ .

(2) Given split extensions  $TP_1, \dots, TP_n$  of  $P_i$ -groups by  $T$  (where the  $p_i$  are distinct odd primes) which are supersolvable and in  $\mathfrak{S}$ , when is  $TP_1 \wedge TP_2 \wedge \dots \wedge TP_n \in \mathfrak{S}$ ? (For a definition of the symbol  $\wedge$  see [3], Satz 9.11.)

The answer to (2) is *not* "Always." For example let

$$TP_1 = \langle x, y, a, b \rangle$$

where  $\langle x, y \rangle$  is the non-abelian group of order 27 and exponent 3,  $\langle a, b \rangle$  is the four-group, and  $(x, a) = x, (x, b) = 1, (y, a) = 1, (y, b) = y$ . Let  $TP_2 = \langle u, v, a, b \rangle$  where  $\langle u, v \rangle$  is the nonabelian group of order 125 and exponent 5 with  $(u, a) = u, (u, b) = 1, (v, a) = 1, (v, b) = v$ . Then  $TP_1$  and  $TP_2$  are supersolvable and in  $\mathfrak{S}$ , but  $TP_1 \wedge TP_2 \notin \mathfrak{S}$ .

The next theorem answers (1) when  $T$  and  $P$  are abelian. It may be used to show that for certain  $P$  no  $T$  exists such that  $TP \in \mathfrak{S}$ .

**THEOREM 2.2.** *If  $G = TK$  is a group in  $\mathfrak{S}$  such that  $K$  is abelian of odd order ( $K \triangleleft G$ ) and  $T$  is an abelian two-group then  $T$  is elementary and we may pick a basis  $x_1, \dots, x_n$  for  $K$  and a basis  $\alpha, \beta_1, \dots, \beta_m$  for  $T$  such that  $x_i^\alpha = x_i^{-1}$  for all  $i = 1, \dots, n$  and  $x_i^{\beta_j} = x_i^{\pm 1}$  for all  $i, j$ . Conversely any such group is in  $\mathfrak{S}$ .*

*Proof.* Since  $G/K \cong T, T \in \mathfrak{S}$ . Being abelian  $T$  must be elementary. Since  $K$  is a finite  $T$ -group we may write  $K = K_1 \times \dots \times K_n$  where each  $K_i$  is a  $T$ -indecomposable  $T$ -group. Now pick any  $\gamma \in T$ . Since  $|\gamma| \leq 2$  and  $K_i$  is abelian of odd order,  $K_i = I_\gamma \times F_\gamma$  where

$$I_\gamma = \{x \in K_i \mid x^\gamma = x^{-1}\} \quad \text{and} \quad F_\gamma = \{x \in K_i \mid x^\gamma = x\}.$$

(For clearly  $K_i \cong I_\gamma \times F_\gamma$ . For any  $x \in K_i$  let  $z = xx^\gamma$  and  $w = x(x^{-1})^\gamma$ . Observe that  $z \in F_\gamma$ ,  $w \in I_\gamma$ , and  $x^2 = zw$ . Since  $x^2 \in I \times F_\gamma$  and  $K_i$  has odd order,  $x \in I_\gamma \times F_\gamma$ . Thus  $K_i = I_\gamma \times F_\gamma$ .) Since  $T$  is abelian and  $K_i$  is a  $T$ -group,  $I_\gamma$  and  $F_\gamma$  are also  $T$ -groups. But  $K_i$  is  $T$ -indecomposable so  $I_\gamma = \langle 1 \rangle$  or  $F_\gamma = \langle 1 \rangle$ . This means that each  $\gamma \in T$  either inverts every element of  $K_i$  or fixes every element of  $K_i$ . Hence in any decomposition of  $K_i$  as a direct product of cyclic groups each direct factor is a  $T$ -group. As  $K_i$  is  $T$ -indecomposable we conclude  $K_i$  is cyclic. Let  $K_i = \langle x_i \rangle$ . Because  $G \in \mathfrak{S}$  there exists  $\alpha \in T$  such that  $(x_1 \cdots x_n)^\alpha = x_1^{-1} \cdots x_n^{-1}$ . Hence  $x_i^\alpha = x_i^{-1}$  for all  $i$  and therefore  $x^\alpha = x^{-1}$  for all  $x \in K$ . Now let  $\alpha, \beta_1, \dots, \beta_m$  be a basis of  $T$ , where  $\alpha$  is as above. We found that for an arbitrary  $\gamma \in T$  and an arbitrary  $x \in K_i$ ,  $x^\gamma = x$  or  $x^\gamma = x^{-1}$ . Hence for each  $j$  and  $i$ ,  $x_i^{\beta_j} = x_i^{\varepsilon_j}$ , where  $\varepsilon_j = \pm 1$ .

Conversely, if  $G = TK$  is as in the conclusion of the theorem then  $g \in G$  either has the form  $x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$  (which is conjugated to its inverse by  $\alpha$ ) or the form  $\gamma x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$ , with  $\gamma \in T$ . In this case it is easy to see that  $g^\beta = g^{-1}$ , where  $\beta = \gamma\alpha$ .

As an example of how this theorem might be applied we shall show that if  $P = \langle x, y \mid x^{2^{n-1}} = y^p = 1, x^y = x^{1+p^{n-2}} \rangle$ , where  $p$  is an odd prime and  $n \geq 3$ , then there is no two-group  $T$  and supersolvable extension  $TP$  such that  $TP \in \mathfrak{S}$ . For suppose there were such a  $T$ , with  $TP \in \mathfrak{S}$ . We may assume, by previous remarks, that  $T$  is elementary abelian. Then  $TP/\Phi(P) \in \mathfrak{S}$  and by the foregoing theorem there exists  $\alpha \in T$  such that  $x^\alpha = x^{-1}x^{p^k}$  and  $y^\alpha = y^{-1}x^{p^e}$ . Then

$$(x^y)^\alpha = (x^{1+p^{n-2}})^\alpha = x^{-1-p^{n-2}}x^{p^k}$$

while  $(x^\alpha)^{y^\alpha} = (x^{-1}x^{p^k})^{y^{-1}x^{p^e}} = (x^{-1})^{y^{-1}x^{p^e}}x^{p^k} = x^{-1+p^{n-2}}x^{p^k}$ . Since  $(x^y)^\alpha = (x^\alpha)^{y^\alpha}$  we conclude that  $x^{-p^{n-2}} = x^{p^{n-2}}$ . Therefore  $x^{2^{p^{n-2}}} = 1$ , contradicting the supposition that  $p$  is odd. Hence no such  $G$  exists.

3. We now give an example of a solvable group satisfying the hypotheses of Theorem 1.3 which does not have a nilpotent normal 2-complement. Thus the second assertion of Theorem 2.1 does not generalize to solvable groups with a normal 2-complement. Let

$$H = \langle x, y, z \mid x^7 = y^3 = z^2 = 1, x^y = x^2, x^z = x^{-1}, y^z = y \rangle,$$

so  $H = \text{Hol}(C_7)$ , where  $C_7$  is a cyclic group of order 7. Let

$$C_2 = \langle u \mid u^2 = 1 \rangle$$

and define  $K = HwC_2$ . In  $K$  let  $a = x, b = x^u, c = y(y^2)^u, d = zz^u, e = u$ , and consider the subgroup  $G = \langle a, b, c, d, e \rangle$ . Then  $G$  has defining

relations  $a^7 = b^7 = c^3 = d^2 = e^2 = 1$ ,  $(a, b) = (c, d) = (d, e) = 1$ ,  $a^d = a^{-1}$ ,  $b^d = b^{-1}$ ,  $a^e = a^2$ ,  $b^e = b^4$ ,  $c^e = c^{-1}$ , and  $a^e = b$ .

Consider the subgroup  $\langle a, b, d, e \rangle$ . Elements of the form  $ea^ib^j$ ,  $a^ib^j$ ,  $da^ib^j$ , and  $eda^ib^j$  are conjugated to their inverses by, respectively,  $a^i d e a^{-j}$ ,  $d$ ,  $1$  and  $e$ . We may now consider elements  $c^\varepsilon e^i d^j a^k b^m$ ,  $\varepsilon = \pm 1$ . Such an element is always conjugate to an element of the form  $ce^i d^j a^k b^m$ . Now  $ceda^k b^m$  and  $cea^k b^m$  are conjugated to their inverses by  $ce$  and  $ced$  respectively. Finally  $ca^k b^m$  and  $cd a^k b^m$  are conjugated to their inverses by  $a^k b^{5m} e a^{-k} b^{-5m}$  and  $a^{2k} b^{4m} e a^{-2k} b^{-4m}$  respectively.

This completes the proof that  $G \in \mathfrak{S}$ . Notice  $G$  satisfies the hypotheses of Theorem 1.3 but the normal 2-complement  $K = \langle a, b, c \rangle$  is not nilpotent. In fact  $F(K) = K^{(1)}$ .

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