SOLVABLE AND SUPERSOLVABLE GROUPS IN WHICH EVERY ELEMENT IS CONJUGATE TO ITS INVERSE

J. LENNART (JOHN) BERGGREN
SOLVABLE AND SUPERSOLVABLE
GROUPS IN WHICH EVERY ELEMENT IS
CONJUGATE TO ITS INVERSE

J. L. BERGGREN

Let $\mathcal{E}$ be the class of finite groups in which every element is conjugate to its inverse. In the first section of this paper we investigate solvable groups in $\mathcal{E}$; in particular we show that if $G \in \mathcal{E}$ and $G$ is solvable then the Carter subgroup of $G$ is a Sylow 2-subgroup and we show that any finite solvable group may be embedded in a solvable group in $\mathcal{E}$. In the second section the main theorem reduces the study of supersolvable groups in $\mathcal{E}$ to the study of groups in $\mathcal{E}$ whose orders have the form $2^\alpha p^\beta$, $p$ an odd prime.

NOTATION. The notation here will be as in [1] with the addition of the notation $G = X\, Y$ to mean $G$ is a split extension of $Y$ by $X$. Also, $F(G)$ will denote the Fitting subgroup of $G$ and $\Phi(G)$ the Frattini subgroup of $G$. We will denote the maximal normal subgroup of $G$ of odd order by $O^o(G)$. Further, $\text{Hol}(G)$ will denote the split extension of $G$ by its automorphism group.

If $K$ and $T$ are subgroups of $G$ we will call $K$ a $T$-group if $T \subseteq N_o(K)$ and we say $K$ is a $T$-indecomposable $T$-group if $K = K_1 \times K_2$, where $K_1$ and $K_2$ are $T$-groups, implies $K_1 = \langle 1 \rangle$ or $K_2 = \langle 1 \rangle$.

1. Burnside [2] proved that if $P$ is a Sylow $p$-subgroup of the finite group $G$ and if $X$ and $Y$ are $P$-invariant subsets of $P$ which are not conjugate in $N_o(P)$ then they are not conjugate in $G$. Using Burnside's method one may prove a similar fact about the Carter subgroups. The proof is easy and we omit it.

LEMMA 1.1. Let $C$ be a Carter subgroup of the solvable group $G$ and let $A$ and $B$ be subsets of $C$, both normal in $C$. If $A \neq B$ then $A$ and $B$ are not conjugate in $G$.

THEOREM 1.1. If $G$ is a solvable group in $\mathcal{E}$ then a Carter subgroup of $G$ is a Sylow 2-subgroup of $G$.

Proof. Let $C$ be a Carter subgroup of $G$. If $C$ has a nonidentity element of odd order then $C$ has a nonidentity central element $g$ of odd order, since $C$ is nilpotent. Then with $A = \{g\}$ and $B = \{g^{-1}\}$ the hypotheses of Lemma 1.1 are satisfied and, since $A \neq B$, $g$ and $g^{-1}$ are not conjugate in $G$, contradicting our supposition that $G \in \mathcal{E}$. 
Hence $C$ is a 2-group. As $C$ is self-normalizing in $G$, $C$ must be a Sylow 2-subgroup of $G$.

NOTE. This proof implies, also, that $Z(C)$ is an elementary abelian 2-group. However, the theorem of Burnside we mentioned can be used to show that if $T$ is a Sylow 2-subgroup of any group $G \in \mathcal{S}$ (whether solvable or not) then $Z(T)$ is elementary abelian. Thus, if $G \in \mathcal{S}$ and $T$ is a Sylow 2-subgroup of $G$ the ascending central series of $T$ has elementary abelian factors.

**Corollary 1.1.** If $T$ is a Sylow 2-subgroup of a solvable group $G \in \mathcal{S}$ then $N_G(T) = T$.

**Proof.** By Theorem 1.1 $T$ is a Carter subgroup of $G$. Carter subgroups are self-normalizing.

**Corollary 1.2.** If $G$ and $T$ are as in Corollary 1.1, and if $T$ is abelian, then $G$ has a normal 2-complement.

**Proof.** By Corollary 1.1 and the assumption $T$ is abelian, $T$ is in the center of its normalizer. The result follows from a well-known theorem of Burnside.

We now investigate two families of solvable groups in $\mathcal{S}$.

**Theorem 1.2.** If $G \in \mathcal{S}$ and a Sylow 2-subgroup of $G$ is cyclic then $G = TK$ where $K$ is an abelian normal subgroup of odd order and $T = \langle \alpha \rangle$ with $\alpha^2 = 1$ and $g^\alpha = g^{-1}$ for all $g \in K$.

**Proof.** As $G$ has a cyclic Sylow 2-subgroup, $G$ is solvable. By Corollary 1.2 $G = TK$, $T = \langle \alpha \rangle$ is a Sylow 2-subgroup of $G$ and $K$ is a normal subgroup of odd order. By the Note after Theorem 1.1, $\alpha^2 = 1$. If $\alpha$ did not induce a fixed-point-free automorphism of $K$ then $C_G(T) \cap K \not\supseteq \langle 1 \rangle$, so $N_G(T) \not\supseteq T$, contradicting Corollary 1.1. Thus $g \rightarrow g^\alpha$ is a fixed-point-free automorphism of $K$. It is known that if $K$ has a fixed-point-free automorphism $\alpha$ of order 2 then $\alpha(k) = k^{-1}$ for all $k \in K$ and hence $K$ is abelian.

**Theorem 1.3.** Let $G$ be a finite solvable group in $\mathcal{S}$ and suppose a Sylow 2-subgroup $T$ of $G$ has order 4. Then $T$ is elementary abelian, $G$ has a normal 2-complement $K$, and $K^{(1)}$ is nilpotent.

**Proof.** As $G$ is solvable, Corollary 1.1 and 1.2 imply that $G =$
TK where $|T| = 4$ and $K$ is a normal subgroup of odd order. The Note after Theorem 1.1 implies $T$ is elementary, say $T = \langle \alpha \rangle \times \langle \beta \rangle$. Let $K_\alpha$ and $K_\beta$ denote the set of fixed points of the automorphisms of $K$ induced by $\alpha$ and $\beta$ respectively. Then $\langle 1 \rangle = C_K(T) \supseteq K_\alpha \cap K_\beta$. Hence, as $T$ is abelian, $K_\alpha$ is $\beta$-invariant and $\beta$ induces a fixed-point free automorphism of $K_\alpha$. Thus $K_\alpha$ is abelian. Then, by [4], $K^{(1)}$ is nilpotent.

Finally, we show that any finite solvable group can be embedded in a solvable group in $\mathfrak{S}$. We shall need the following lemma.

**Lemma 1.2.** Let $G \in \mathfrak{S}$ and let $\langle x \rangle$ be a cyclic group of order $p$, where $p$ is an odd prime. Let $\alpha$ be an involution and define $H = \langle Gw\langle x \rangle, \alpha \rangle$, where $x^\alpha = x^{-1}$ and $b^\alpha = b$ for all $b \in G$. Then $H \in \mathfrak{S}$.

**Proof.** Let $K = G \times G^x \times \cdots \times G^{x^{p-1}}$ be the base subgroup of $Gw\langle x \rangle$. Then $K \in \mathfrak{S}$ since $G \in \mathfrak{S}$. Suppose $h_i \in H$ and

$$h_i = x^r g_0 \cdot g_1^r \cdots g_{p-1}^r,$$

where $r \neq 0(p)$. Writing $[j]$ for $x^j$ we may write

$$h_i = x^r g_0 \cdot [g_i^r] \cdots [g_{(p-1)r}]^r.$$

Now, if $g \in G$ then $(g^{[i]}x^r = g^{[i+r]}$ implies that

$$(g^{[i]})^{-1}x^{-r}g^{[i]}x^r = (g^{[i]})^{-1}g^{[i+r]}x^r,$$

and hence $(g^{[i]})^{-1}x^r g^{[i]} = x^r (g^{[i+r]})^{-1} g^{[i]}$. Thus if $\beta = g^{[i]}_{[r]}$ then $(x^r)^\beta = x^r (g^{[i]}_{[r]})^{[r]} (g^{[i]}_{[r]})^{-1}$. Writing $h_i = x^r f_0 f_1^r \cdots f_{(p-1)r}^r$, where $f_i \in G$ for all $i$, we see that $f_{ir} = g_{ir}$ if $i \neq e, e-1$ while $f_{er} = 1$. Thus first changing the rightmost $g_{ir}^r$ in $h_i$ to 1 by conjugation and proceeding to the left we may conjugate $h_i$ to an element $h = x^r g$, where $g \in G = G^{[0]}$.

Pick $a \in G$ such that $g^a = g^{-1}$ and let $u = a^x \cdots a^{x^{p-1}}$. Then with $\gamma = axax^{-r}$ we have $h_i \gamma = h_i$. It remains to consider elements of $H$ of the form $h = \alpha \cdot x^r g_0 \cdot g_1^{[1]} \cdots g_{p-1}^{[p-1]}$, where $[j]$ denotes $x^j$. If $r \neq 0\ (p)$ then let $e$ be an integer such that $2e = -r(p)$. Then $h$ conjugated by $x^e$ has the form $\alpha y_0 y_1^{[1]} \cdots y_{p-1}^{[p-1]}$ where the $y_i \in G$.

We now exploit the fact that, since $x^a = x^{-1}$ and $g^a = g$ for all $g \in G = G^{[0]}$, $g_0^{[p-1]} = (g_0^{[1]})^a, g_1^{[p-2]} = (g_1^{[2]})^a$, etc. Thus

$$\alpha g_i^{[p-1]} = \alpha (g_{i+1}^{[p-1]} (g_{i+1})^{[1]}$$

where $\gamma(p-1) = g_{i+1}^{[1]}$. Performing this computation for

$$\gamma(p-1), \gamma(p-2), \cdots, \gamma((p+1)/2),$$

where $\gamma(e) = g_{i+1}^{[p-1]}$ and observing that $u = \gamma(p-1) \cdots \gamma((p+1)/2)$
has the identity in $G^{[1]}$ as its $i$-th component for $i > ((p + 1)/2)$ we see that $h_u$ has the form $h_i = \alpha \cdot f_i \cdot f_i^{[1]} \cdots f_i^{[r]}$ where $r = (p - 1)/2$ and $f_i \in G$ for all $i$. Then $h_i^{-1} = \alpha \cdot f_i^{-1} \cdot ((f_i^{-1})^{[1]} \cdots (f_i^{-1})^{[r]})^\sigma$. Now for all $i = 0, \cdots, r$ pick $a_i \in G$ such that $f_i^{a_i} = f_i^{-1}$ and let $u = a_o \cdot v \cdot v^\sigma$ where $v = a_1^{\sigma} \cdots a_r^{\sigma}$. Taking $x = u\alpha$ it is easy to see that $h_i^{-1} = h_i^{-1}$, using the fact that $(vv^\sigma, \alpha) = (g_o, vv^\sigma) = 1$. This disposes of all cases.

Theorem 1.4. If $G$ is a finite solvable group then there exists a solvable group $L \in \mathcal{S}$ and a monomorphism $\tau: G \to L$.

Proof. If $G$ is abelian let $L = \langle G, \alpha \rangle$ where $\alpha^\sigma = 1$ and $g^\sigma = g^{-1}$ for all $g \in G$. Then in $L$ every element of $G$ is conjugate to its inverse and all other elements lie in the coset $G\alpha$ which consists of involutions, so $L \in \mathcal{S}$ and $L$ is solvable. Hence the theorem is true for all abelian groups $G$. Induct on $|G|$ and assume it is true for all solvable groups of order less than the order of $G$. Now let $H \lhd G$ such that $[G: H] = p$, $p$ a prime. Our induction hypothesis says there is a solvable $K \in \mathcal{S}$ and a monomorphism of $HwC_p$ into $KwC_p$, where $C_p$ is cyclic of order $p$. By Satz 15.9 [3] (Chapter I) there is a monomorphism of $G$ into $HwC_p$, so $G$ may be imbedded in $KwC_p$. If $p = 2$ then by Theorem 1.1 of [1] $KwC_p \in \mathcal{S}$, and it is solvable since $K$ is. If $p > 2$ then by Lemma 1.2 $KwC_p$ has a solvable extension $\langle KwC_p, \alpha \rangle \in \mathcal{S}$.

Thus, in this case as well, $G$ may be imbedded in a solvable group in $\mathcal{S}$.

This concludes our investigation of solvable groups in $\mathcal{S}$.

2. In § 1 we showed that if $G \in \mathcal{S}$ is a solvable group with an abelian Sylow 2-subgroup $T$ then $T$ has a normal complement in $G$. Of course, if $G$ is supersolvable then (by the Sylow Tower Theorem) $T$ has a normal complement $K$, regardless whether $T$ is abelian or $G \in \mathcal{S}$. If we assume that $G \in \mathcal{S}$, where $G$ is supersolvable, then with the above notation we assert.

Theorem 2.1. The Sylow 2-subgroup $T$ is in $\mathcal{S}$, and $K$ and $\Phi(T)$ are contained in $F(G)$.

Proof. That $T \in \mathcal{S}$ was remarked in [1]. Since $G$ is supersolvable $G^{(1)} \leq F(G)$. Now $G \in \mathcal{S}$ implies $G/G^{(1)} \in \mathcal{S}$ and since $G/G^{(1)}$ is abelian $G/G^{(1)}$ is an elementary abelian 2-group. Thus $\Phi(T) \leq G^{(1)}$, and since $(2, |K|) = 1, K \leq G^{(1)}$.

Remark. If $G \in \mathcal{S}$ is supersolvable Theorem 2.1 implies $G$ is a
split extension of a nilpotent group $K$ by a two-group $T$ in $\mathfrak{S}$. If $S$ is a Sylow 2-subgroup of $F(G)$ then $S \triangleleft G$, so $G/S \in \mathfrak{S}$. But by Theorem 2.1 $G/S$ is isomorphic to a split extension $EK$ of the nilpotent group $K$ by an elementary abelian two-group $E$. Thus given a supersolvable $G$ in $\mathfrak{S}$ there exists a supersolvable $G^* \in \mathfrak{S}$ such that $O_2(G^*) \cong O_2(G)$ but $G^*$ has an elementary abelian Sylow 2-subgroup.

Now let $G = TK \in \mathfrak{S}$ be given, where $G$ is supersolvable and $T$ and $K$ are as above. Let $P_1, \cdots, P_r$ be the Sylow subgroups of $K$, so $K = P_1 \times \cdots \times P_r$. If $\pi_i$ is the projection of $K$ onto $P_i$ let $H_i = \text{ker}(\pi_i)$. Then $H_i \lhd G$ and $G/H_i \cong TP_i$, a split extension of $P_i$ by $T$ which is supersolvable and in $\mathfrak{S}$. We have now reduced the study of supersolvable groups in $\mathfrak{S}$ to two questions:

(1) Given a 2-group $T \in \mathfrak{S}$ and a $p$-group $P$ ($p$ an odd prime) find the split extensions $TP$ of $P$ by $T$ which are supersolvable and in $\mathfrak{S}$.

(2) Given split extensions $TP_1, \cdots, TP_n$ of $P_i$-groups by $T$ (where the $p_i$ are distinct odd primes) which are supersolvable and in $\mathfrak{S}$, when is $TP_1 \wedge TP_2 \wedge \cdots \wedge TP_n \in \mathfrak{S}$? (For a definition of the symbol $\wedge$ see [3], Satz 9.11.)

The answer to (2) is not "Always." For example let

$$TP_1 = \langle x, y, a, b \rangle$$

where $\langle x, y \rangle$ is the non-abelian group of order 27 and exponent 3, $\langle a, b \rangle$ is the four-group, and $(x, a) = x$, $(x, b) = 1$, $(y, a) = 1$, $(y, b) = y$. Let $TP_2 = \langle u, v, a, b \rangle$ where $\langle u, v \rangle$ is the nonabelian group of order 125 and exponent 5 with $(u, a) = u$, $(u, b) = 1$, $(v, a) = 1$, $(v, b) = v$. Then $TP_1$ and $TP_2$ are supersolvable and in $\mathfrak{S}$, but $TP_1 \wedge TP_2 \in \mathfrak{S}$.

The next theorem answers (1) when $T$ and $P$ are abelian. It may be used to show that for certain $P$ no $T$ exists such that $TP \in \mathfrak{S}$. 

THEOREM 2.2. If $G = TK$ is a group in $\mathfrak{S}$ such that $K$ is abelian of odd order ($K \triangleleft G$) and $T$ is an abelian two-group then $T$ is elementary and we may pick a basis $x_1, \cdots, x_n$ for $K$ and a basis $\alpha, \beta_1, \cdots, \beta_m$ for $T$ such that $x_i^\alpha = x_i^{-1}$ for all $i = 1, \cdots, n$ and $x_i^{\beta_j} = x_i^{\beta_1}$ for all $i, j$. Conversely any such group is in $\mathfrak{S}$.

Proof. Since $G/K \cong T$, $T \in \mathfrak{S}$. Being abelian $T$ must be elementary. Since $K$ is a finite $T$-group we may write $K = K_i \times \cdots \times K_n$ where each $K_i$ is a $T$-indecomposable $T$-group. Now pick any $\gamma \in T$. Since $|\gamma| \leq 2$ and $K_i$ is abelian of odd order, $K_i = I_\gamma \times F_\gamma$ where

$$I_\gamma = \{x \in K_i | x^\gamma = x^{-1}\} \quad \text{and} \quad F_\gamma = \{x \in K_i | x^\gamma = x\}.$$
(For clearly $K_i \geq I_r \times F_r$. For any $x \in K_i$ let $z = xx^r$ and $w = x(x^{-1})^r$. Observe that $z \in F_r$, $w \in I_r$, and $x^2 = zw$. Since $x^2 \in I_r \times F_r$ and $K_i$ has odd order, $x \in I_r \times F_r$. Thus $K_i = I_r \times F_r$.) Since $T$ is abelian and $K_i$ is a $T$-group, $I_r$ and $F_r$ are also $T$-groups. But $K_i$ is $T$-indecomposable so $I_r = \langle 1 \rangle$ or $F_r = \langle 1 \rangle$. This means that each $\gamma \in T$ either inverts every element of $K_i$ or fixes every element of $K_i$. Hence in any decomposition of $K_i$ as a direct product of cyclic groups each direct factor is a $T$-group. As $K_i$ is $T$-indecomposable we conclude $K_i$ is cyclic. Let $K_i = \langle x_i \rangle$. Because $G \in \mathfrak{S}$ there exists $\alpha \in T$ such that $(x_1 \cdots x_n)^{\alpha} = x_1^{-\alpha} \cdots x_n^{-\alpha}$. Hence $x_i^\alpha = x_i^{-\alpha}$ for all $i$ and therefore $x^\alpha = x^{-\alpha}$ for all $x \in K$. Now let $\alpha, \beta_1, \cdots, \beta_m$ be a basis of $T$, where $\alpha$ is as above. We found that for an arbitrary $\gamma \in T$ and an arbitrary $x \in K_i$, $x^r = x$ or $x^r = x^{-1}$. Hence for each $j$ and $i$, $x_{ij} = x_i$, where $\varepsilon = \pm 1$.

Conversely, if $G = TK$ is as in the conclusion of the theorem then $g \in G$ either has the form $x_i^{\alpha_i} \cdots x_n^{\alpha_n}$ (which is conjugated to its inverse by $\alpha$) or the form $\gamma x_i^{\alpha_i} \cdots x_n^{\alpha_n}$, with $\gamma \in T$. In this case it is easy to see that $g^\beta = g^{-1}$, where $\beta = \gamma \alpha$.

As an example of how this theorem might be applied we shall show that if $P = \langle x, y \mid x^{p^{n-1}} = y^p = 1, x^r = x^{p^n-2} \rangle$, where $p$ is an odd prime and $n \geq 3$, then there is no two-group $T$ and supersolvable extension $TP$ such that $TP \in \mathfrak{S}$. For suppose there were such a $T$, with $TP \in \mathfrak{S}$. We may assume, by previous remarks, that $T$ is elementary abelian. Then $TP/\Phi(P) \in \mathfrak{S}$ and by the foregoing theorem there exists $\alpha \in T$ such that $x^\alpha = x^{-1}x^{p^k}$ and $y^\alpha = y^{-1}x^{p^r}$. Then

$$
(x^\alpha)^{\alpha} = (x^{1+p^{n-2}})^{\alpha} = x^{-1-p^{n-2}}x^{p^k}
$$

while $(x^\alpha)^{y^\alpha} = (x^{-1}x^{p^k})^{y^{-1}} = (x^{-1})^{y^{-1}}x^{p^k} = x^{-1+p^{n-2}}x^{p^k}$. Since $(x^\alpha)^{\alpha} = (x^\alpha)^{\gamma}$ we conclude that $x^{-p^{n-2}} = x^{p^{n-2}}$. Therefore $x^{2^{p^{n-2}}} = 1$, contradicting the supposition that $p$ is odd. Hence no such $G$ exists.

3. We now give an example of a solvable group satisfying the hypotheses of Theorem 1.3 which does not have a nilpotent normal 2-complement. Thus the second assertion of Theorem 2.1 does not generalize to solvable groups with a normal 2-complement. Let

$$
H = \langle x, y, z \mid x^r = y^r = z^2 = 1, x^r = x^2, x^r = x^{-1}, y^r = y \rangle,
$$

so $H = \text{Hol}(C_7)$, where $C_7$ is a cyclic group of order 7. Let

$$
C_2 = \langle u \mid u^2 = 1 \rangle
$$

and define $K = HwC_2$. In $K$ let $a = x$, $b = x^e$, $c = y(y^e)^a$, $d = z^e$, $e = u$, and consider the subgroup $G = \langle a, b, c, d, e \rangle$. Then $G$ has defining
relations $a^7 = b^7 = c^8 = d^2 = e^2 = 1$, $(a, b) = (c, d) = (d, e) = 1$, $a^4 = a^{-1}$, $b^4 = b^{-1}$, $a^6 = a^2$, $b^e = b^4$, $c^e = c^{-1}$, and $a^e = b$.

Consider the subgroup $\langle a, b, d, e \rangle$. Elements of the form $ea^ib^j$, $a^ib^j$, $da^ib^j$, and $ed^aib^j$ are conjugated to their inverses by, respectively, $a^jda^-1$, $d$, $1$ and $e$. We may now consider elements $c'e^jd^a^k b^m$, $\varepsilon = \pm 1$. Such an element is always conjugate to an element of the form $ce^id^j a^{k} b^{m}$. Now $ceda^k b^m$ and $cea^k b^m$ are conjugated to their inverses by $ce$ and $ced$ respectively. Finally $ca^k b^m$ and $cda^k b^m$ are conjugated to their inverses by $a^k b^m e a^{-k} b^{-5m}$ and $a^2 k^4 m e a^{-2k} b^{-4m}$ respectively.

This completes the proof that $G \in \mathfrak{S}$. Notice $G$ satisfies the hypotheses of Theorem 1.3 but the normal 2-complement $K = \langle a, b, c \rangle$ is not nilpotent. In fact $F(K) = K^{[1]}$.

REFERENCES


Received February 10, 1970.

SIMON FRASER UNIVERSITY
<table>
<thead>
<tr>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gregory Frank Bachelis and Haskell Paul Rosenthal, <em>On unconditionally converging series and biorthogonal systems in a Banach space</em></td>
<td>1</td>
</tr>
<tr>
<td>J. Lennart (John) Berggren, <em>Solvable and supersolvable groups in which every element is conjugate to its inverse</em></td>
<td>21</td>
</tr>
<tr>
<td>Lindsay Nathan Childs, <em>On covering spaces and Galois extensions</em></td>
<td>29</td>
</tr>
<tr>
<td>William Jay Davis, David William Dean and Ivan Singer, <em>Multipliers and unconditional convergence of biorthogonal expansions</em></td>
<td>35</td>
</tr>
<tr>
<td>Leroy John Derr, <em>Triangular matrices with the isoclinal property</em></td>
<td>41</td>
</tr>
<tr>
<td>Paul Erdős, Robert James McEliece and Herbert Taylor, <em>Ramsey bounds for graph products</em></td>
<td>45</td>
</tr>
<tr>
<td>Edward Graham Evans, Jr., <em>On epimorphisms to finitely generated modules</em></td>
<td>47</td>
</tr>
<tr>
<td>Hector O. Fattorini, <em>The abstract Goursat problem</em></td>
<td>51</td>
</tr>
<tr>
<td>Robert Dutton Fray and David Paul Roselle, <em>Weighted lattice paths</em></td>
<td>85</td>
</tr>
<tr>
<td>Thomas L. Goulding and Augusto H. Ortiz, <em>Structure of semiprime (p, q) radicals</em></td>
<td>97</td>
</tr>
<tr>
<td>E. W. Johnson and J. P. Lediaev, <em>Structure of Noether lattices with join-principal maximal elements</em></td>
<td>101</td>
</tr>
<tr>
<td>David Samuel Kinderlehrer, <em>The regularity of minimal surfaces defined over slit domains</em></td>
<td>109</td>
</tr>
<tr>
<td>Alistair H. Lachlan, <em>The transcendental rank of a theory</em></td>
<td>119</td>
</tr>
<tr>
<td>Frank David Lesley, <em>Differentiability of minimal surfaces at the boundary</em></td>
<td>119</td>
</tr>
<tr>
<td>Wolfgang Liebert, <em>Characterization of the endomorphism rings of divisible torsion modules and reduced complete torsion-free modules over complete discrete valuation rings</em></td>
<td>123</td>
</tr>
<tr>
<td>Lawrence Carlton Moore, <em>Strictly increasing Riesz norms</em></td>
<td>141</td>
</tr>
<tr>
<td>Raymond Moos Redheffer, <em>An inequality for the Hilbert transform</em></td>
<td>153</td>
</tr>
<tr>
<td>James Ted Rogers Jr., <em>Mapping solenoids onto strongly self-entwined circle-like continua</em></td>
<td>165</td>
</tr>
<tr>
<td>Sherman K. Stein, <em>B-sets and planar maps</em></td>
<td>171</td>
</tr>
<tr>
<td>Darrell R. Turnidge, <em>Torsion theories and rings of quotients of Morita equivalent rings</em></td>
<td>181</td>
</tr>
<tr>
<td>Fred Ustina, <em>The Hausdorff means of double Fourier series and the principle of localization</em></td>
<td>193</td>
</tr>
<tr>
<td>Stanley Joseph Wertheimer, <em>Quasi-compactness and decompositions for arbitrary relations</em></td>
<td>203</td>
</tr>
<tr>
<td>Howard Henry Wicke and John Mays Worrell Jr., <em>On the open continuous images of paracompact Čech complete spaces</em></td>
<td>215</td>
</tr>
</tbody>
</table>