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ON COVERING SPACES AND GALOIS EXTENSIONS

LINDSAY NATHAN CHILDS

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L. N. CHILDS

Let X be a connected compact Hausdorff space, and G a finite abelian group. In this note we obtain a short exact sequence (Theorem 1) which describes the group of isomorphism classes of regular covering spaces of X with group G . The sequence is derived as an immediate translation of a similar sequence involving the group of commutative Galois extensions with group G of $C(X)$, the ring of complex-valued continuous functions on X .

The translation is obtained in part by showing (Theorem 2) that there is an equivalence between the category of finite covering spaces of X and the dual of the category of separable $C(X)$ -algebras which are finitely generated projective $C(X)$ -modules. This equivalence may be known to students of [8], but I am unaware of any reference for it, so we sketch a proof here.

1. **The sequence.** Let $X, C(X)$ be as above, and let G be a finite group. A not necessarily connected covering space Y over X is called regular with group G if G acts as a fixed point free group of homeomorphisms of Y which preserve the covering map. Denote by $\text{Cov}(X, G)$ the set of isomorphism classes of regular covering spaces of X with group G (where Y, Z , two covering spaces with group G , are isomorphic if there is a homeomorphism from Y to Z which commutes with the covering maps and the action of G). Denote by $\text{Pic}(X)$ the group (under tensor product of fibers) of isomorphism classes of line bundles on X . Denote by $H_s^2(G, U(C(X)))$ the subgroup of $H^2(G, U(C(X)))$ (group cohomology, with G acting trivially on $U(C(X))$) which is the image of the symmetric 2-cocycles—those cocycles f from $G \times G$ into the units of $C(X)$ which satisfy $f(s, t) = f(t, s)$ for all s, t in G .

THEOREM 1. *Let G be a finite abelian group. Then $\text{Cov}(X, G)$ has an abelian group structure so that the following sequence of abelian groups is exact:*

$$0 \rightarrow H_s^2(G, U(C(X))) \rightarrow \text{Cov}(X, G) \rightarrow \text{Hom}(G, \text{Pic}(X)) \rightarrow 0.$$

Proof. If G is a group of order n and R is a commutative ring with unity, a commutative R -algebra S is a Galois extension of R with group G if G acts as a group of R -algebra automorphisms of S , R is the fixed ring under the action of G , and ([3, 1.3f]) for each

maximal ideal m of S and $\sigma \neq 1$ in G , there is an s in S so that $\sigma(s) - s \in m$. Two Galois extensions S', S'' with group G are isomorphic if there is an R -algebra isomorphism $S' \rightarrow S''$ which preserves the action of G . If G is an abelian group, Harrison [9] has shown that the set of isomorphism classes of commutative Galois extensions with group G forms an abelian group, $\text{Comm}(R, G)$. Viewing a Galois extension of R with group G as a rank one projective $R[G]$ -module defines a homomorphism from $\text{Comm}(R, G)$ to $\text{Pic}(R[G])$ (the group under tensor product of isomorphism classes of rank one projective $R[G]$ -modules), whose kernel consists of the set $NB(R, G)$ of isomorphism classes of commutative Galois extensions with normal basis. If R has no idempotents but 0 and 1 and contains $1/n$ and a primitive n th root of unity, then the image is isomorphic to $\text{Hom}(G, \text{Pic}(R))$ [6, Theorem 9], so that we have the short exact sequence

$$(*) \quad 0 \rightarrow NB(R, G) \rightarrow \text{Comm}(R, G) \rightarrow \text{Hom}(G, \text{Pic}(R)) \rightarrow 0.$$

We set $R = C(X)$ in (*) and translate. $\text{Pic}(C(X)) \cong \text{Pic}(X)$ by Swan [11]; $NB(C(X), G) \cong H^2(G, U(X))$ by Theorems 2.2 and 4.4 of [5]. It suffices to show that $\text{Comm}(C(X), G) \cong \text{Cov}(X, G)$. This will be a corollary of Theorem 2.

2. The equivalence. $X, C(X)$ are as above.

THEOREM 2. *There are category equivalences between the category of finite covering spaces of X , the dual of the category of separable $C(X)$ -algebras which are finitely generated projective $C(X)$ -modules, and the category of nonramified affine coverings of $\text{Spec}(C(X))$ [8]. The first equivalence is induced by: if Y is a covering space, $Y \rightarrow C(Y)$; if S is a separable R -algebra, $S \rightarrow \text{Max}(S)$. The functor from the first to the third sends S to $\text{Spec}(S)$.*

Here $\text{Max}(S)$ is the space of maximal ideals of S with the Stone topology (= the topology induced on the geometric points from the Zariski topology on $\text{Spec}(S)$).

COROLLARY. *$\text{Cov}(X, G)$ is an abelian group isomorphic to $\text{Comm}(C(X), G)$.*

Proof of corollary. If $R = C(X)$, $S = C(Y)$, it follows easily from the definition of Galois extension given above that S is a Galois extension of R with group G if and only if Y is a regular covering space of X with group G . Hence there is a bijection between $\text{Cov}(X, G)$ and $\text{Comm}(R, G)$. The group structure on $\text{Cov}(X, G)$ is

the one induced from $\text{Comm}(R, G)$.

Concerning Theorem 2, we have included the category of non-ramified affine coverings of $\text{Spec}(C(X))$ only to make more explicit the relationship with [8]. We shall show only that the correspondences $Y \rightarrow C(Y)$, $S \rightarrow \text{Max}(S)$ give inverse bijections between the objects of the first two categories; the proof of the rest of the theorem is straightforward and will be omitted.

In what follows, the phrase “ S is a finitely generated projective R -algebra” will mean that S is an R -algebra which is finitely generated and projective as an R -module.

Proof of Theorem 2. We recall some facts about a compact Hausdorff space X (see [7]): The topology of X has a basis consisting of the complements of zero sets of continuous functions on X (“cozero sets”). [7, 3.2, p. 38]. If f is a continuous function let $Z(f) =$ the zeros of f and $V(f) = X - Z(f)$. For any closed set F , if $C(X)|_F$ denotes the restriction to F of the continuous functions on X , then $C(X)|_F = C(F)$ [7, 3.11(c), p. 43]. For any open set $V = V(f)$, $C(X)|_V = C(X)_f$, the localization of $C(X)$ with respect to the multiplicative set consisting of the powers of f . $\text{Max}(C(X))$, the set of maximal ideals of $C(X)$, is in one-to-one correspondence with the points of X , since any maximal ideal of $C(X)$ is of the form $\{f \text{ in } C(X) \mid f(p) = 0\}$ for some point p of X . If $\text{Max}(C(X))$ is given the Stone topology: basic closed sets are of the form $\{x \mid f \text{ is in } x\} = Z(f)$ for f in $C(X)$, then $\text{Max}(C(X))$ is homeomorphic to X [7, 4.9, p. 58].

Assume now that X is a compact Hausdorff space and Y a finite covering space, that is, there is a continuous map $p: Y \rightarrow X$ and for each x in X a neighborhood U of x (a canonical neighborhood) such that $p^{-1}(U)$ is the disjoint union of a finite number n of open sets of Y each homeomorphic to U .

Fix an x in X , let U be a canonical neighborhood, and let F be a closed subneighborhood. Then $p^{-1}(F)$ is a disjoint union of closed sets of Y each homeomorphic to F , so $C(p^{-1}(F)) \cong C(F)^n$ (the product as rings of n copies of $C(F)$). Let $V = V(f)$ be a cozero set containing x and contained in F . Then $V_Y(f) = V_Y(f \circ p) = p^{-1}(V_X(f)) \subseteq p^{-1}(F)$ is a disjoint union of open sets of Y each homeomorphic to $V_X(f)$. So $C(Y)_f = C(Y)|_{V_Y(f)} = C(p^{-1}(F))|_{p^{-1}(V_X(f))}$ is the ring of continuous functions on a finite disjoint union of open sets each homeomorphic to $V_X(f)$, whence $C(Y)_f \cong (C(X)_f)^n$. Thus for each maximal ideal x of $C(X)$ there is a f not in x so that $C(Y)_f$ is a finitely generated projective separable $C(X)_f$ -algebra. By [4, § 5] and [1] $C(Y)$ is therefore a finitely generated projective separable $C(X)$ -algebra.

For the other direction we need a

LEMMA. Let S be a finitely generated projective separable R -algebra, $R = C(X)$. Then for each x in X there is a neighborhood $V(h)$ of x so that $S_h \cong (R_h)^n$, a product as rings of n copies of R_h .

Proof of lemma. R_x , the localization of R with respect to the maximal ideal x , is a local ring, so by Theorems 3.1, 2.2 and 2.8 of [10], $S_x = R_x[\theta]$, where θ satisfies a polynomial $f(t) = t^n + a_1 t^{n-1} + \dots + a_n$ with coefficients in R_x and with n distinct roots s_1, \dots, s_n in $R/x \cong C$. Let $\theta = a/b$ with a in S, b in $R - x$, let k be the product of the denominators of the coefficients of $f(t)$, and let d be the discriminant [10] of $f(t)$. Then on $V(g)$ with $g = bdk, R_g[\theta]$ is a separable R_g -algebra contained in S_g , and is a finitely generated projective R_g -module of the same rank as S_g . So S_g is a projective $R_g[\theta]$ -module of rank one. But since S_g is a finitely generated projective $R_g[\theta]$ -algebra, $R_g[\theta]$ is a $R_g[\theta]$ -direct summand of S_g . Thus $S_g = R_g[\theta]$. Since $f(t)$ has coefficients in $R_g, S_g \cong R_g[t]/(f(t))$.

Claim: There exists a subneighborhood $V = V(h)$ of $V(g)$ containing x , and continuous functions r_1, \dots, r_n in $C(X)$ so that on $V, f(t) = \prod_{i=1}^n (t - r_i)$. This follows from the implicit function theorem applied to the function $F(\bar{a}, t) = t^n + a_1 t^{n-1} + \dots + a_n$ at $\bar{a} = \bar{a}(x) = (a_1(x), \dots, a_n(x)) \in C^n, t = s_i$. For since $F(\bar{a}(x), t)$ has n distinct roots, the partial derivative $F_i(\bar{a}(x), s_i) \neq 0$, so there exists a neighborhood U of $\bar{a}(x)$ in C^n and continuous functions $t_i: U \rightarrow C$ such that $F(\bar{a}, t_i(\bar{a})) = 0$ for all \bar{a} in U . Since $t_i(\bar{a}(x)) = s_i \neq s_j = t_j(\bar{a}(x))$ for all $i \neq j$ we can pick U so small that $t_i(U) \cap t_j(U) = \emptyset$ for all $i \neq j$. If we set $\chi: V(g) \rightarrow C^n$ by $\chi(y) = \bar{a}(y)$, then χ is a continuous function, so $V(g) \cap \chi^{-1}(U)$ is a neighborhood of x on which there exist continuous functions $\tilde{r}_i = t_i \circ \chi, i = 1, \dots, n$, with disjoint images, which are roots of f . Hence there is a basic open subneighborhood $V(h)$ (containing x) of a closed subneighborhood of $V(g) \cap \chi^{-1}(U)$ and n elements r_1, \dots, r_n of $C(X)$ so that on $V(h), f(t) = \prod_{i=1}^n (t - r_i)$. The lemma follows easily.

Suppose now that S is a finitely generated projective separable R -algebra, and set $Y = \text{Max}(S)$. Put the Stone topology on Y . Let $p: Y \rightarrow X$ by $p(y) = y \cap R$, a maximal ideal since S is integral over R ([2]).

Let x be a point of X . By the lemma there exists an h in $C(X)$ so that $S_h = (R_h)^n$, a product as rings of copies of R_h . Then $V(h)$ is easily seen to be a canonical neighborhood of x , and p is continuous, so that Y is a covering space of X .

It is easy to verify that the topology defined on Y makes Y into a compact Hausdorff space.

If Y is a covering space and a compact Hausdorff space, then $Y = \text{Max}(C(Y))$ as topological spaces by [7, 3.6, p. 40]. On the other

hand, given S , a finitely generated projective separable R -algebra, S is clearly contained in $C(\text{Max}(S))$. Replacing $\text{Max}(S)$ by each of its connected components as necessary, we may assume that S and $C(\text{Max}(S))$ have no nontrivial idempotents. Embed $C(\text{Max}(S))$ in a finitely generated projective Galois extension T of R containing no idempotents but 0 and 1 (possible by [10, 1.13]). By the fundamental theorem of Galois theory ([3, Theorem 2.3]) S is the fixed ring of some subgroup of $\text{Aut}_R(T)$. But any group which fixes S fixes $C(\text{Max}(S))$, whence $S = C(\text{Max}(S))$. This completes the proof of Theorem 2.

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Gregory Frank Bachelis and Haskell Paul Rosenthal, <i>On unconditionally converging series and biorthogonal systems in a Banach space</i>	1
Richard William Beals, <i>On spectral theory and scattering for elliptic operators with singular potentials</i>	7
J. Lennart (John) Berggren, <i>Solvable and supersolvable groups in which every element is conjugate to its inverse</i>	21
Lindsay Nathan Childs, <i>On covering spaces and Galois extensions</i>	29
William Jay Davis, David William Dean and Ivan Singer, <i>Multipliers and unconditional convergence of biorthogonal expansions</i>	35
Leroy John Derr, <i>Triangular matrices with the isoclinal property</i>	41
Paul Erdős, Robert James McEliece and Herbert Taylor, <i>Ramsey bounds for graph products</i>	45
Edward Graham Evans, Jr., <i>On epimorphisms to finitely generated modules</i>	47
Hector O. Fattorini, <i>The abstract Goursat problem</i>	51
Robert Dutton Fray and David Paul Roselle, <i>Weighted lattice paths</i>	85
Thomas L. Goulding and Augusto H. Ortiz, <i>Structure of semiprime (p, q) radicals</i>	97
E. W. Johnson and J. P. Lediaev, <i>Structure of Noether lattices with join-principal maximal elements</i>	101
David Samuel Kinderlehrer, <i>The regularity of minimal surfaces defined over slit domains</i>	109
Alistair H. Lachlan, <i>The transcendental rank of a theory</i>	119
Frank David Lesley, <i>Differentiability of minimal surfaces at the boundary</i>	123
Wolfgang Liebert, <i>Characterization of the endomorphism rings of divisible torsion modules and reduced complete torsion-free modules over complete discrete valuation rings</i>	141
Lawrence Carlton Moore, <i>Strictly increasing Riesz norms</i>	171
Raymond Moos Redheffer, <i>An inequality for the Hilbert transform</i>	181
James Ted Rogers Jr., <i>Mapping solenoids onto strongly self-entwined, circle-like continua</i>	213
Sherman K. Stein, <i>B-sets and planar maps</i>	217
Darrell R. Turnidge, <i>Torsion theories and rings of quotients of Morita equivalent rings</i>	225
Fred Ustina, <i>The Hausdorff means of double Fourier series and the principle of localization</i>	235
Stanley Joseph Wertheimer, <i>Quasi-compactness and decompositions for arbitrary relations</i>	253
Howard Henry Wicke and John Mays Worrell Jr., <i>On the open continuous images of paracompact Čech complete spaces</i>	265