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**MULTIPLIERS AND UNCONDITIONAL CONVERGENCE OF
BIORTHOGONAL EXPANSIONS**

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MULTIPLIERS AND UNCONDITIONAL CONVERGENCE OF BIORTHOGONAL EXPANSIONS

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We solve in the affirmative a problem raised by B. S. Mityagin in 1961, namely, we prove that if (x_n, f_n) is a biorthogonal system for a Banach space E with (f_n) total over E , such that the set of multipliers $M(E, (x_n, f_n))$ contains all sequences (ε_i) with $\varepsilon_i = \pm 1$ for each i , then (x_n) is an unconditional basis for E .

Let E be a Banach space, and let (x_n, f_n) be a biorthogonal system for E (i.e., $(x_n) \subset E$, $(f_n) \subset E^*$ and $f_n(x_m) = \delta_{nm}$) which has (f_n) total over E (i.e., $f_n(x) = 0$ for all n implies $x = 0$). A scalar sequence (γ_n) is called a *multiplier* of an element x in E with respect to (x_n, f_n) (write $(\gamma_n) \in M(x, (x_n, f_n))$) if there is an element y of E such that $f_n(y) = \gamma_n f_n(x)$ for all n (call this element $x_{(\gamma_n)}$). The set of multipliers for E with respect to (x_n, f_n) is

$$M(E, (x_n, f_n)) = \cap \{M(x, (x_n, f_n)) \mid x \in E\}.$$

Here we consider the following two problems:

P 1: (Mityagin [6], Kadec-Pelczynski [4], Pelczynski [7]). Let E be separable and suppose that $M(E, (x_n, f_n))$ contains all sequences (ε_i) with $\varepsilon_i = \pm 1$ for each i . Is (x_n) an unconditional basis for E ?

P 2: (Kadec-Pelczynski [4]). Let E be separable and suppose $M(x, (x_n, f_n))$ contains all sequences (ε_i) with $\varepsilon_i = \pm 1$ for each i . Does the formal expansion $\sum_n f_n(x)x_n$ converge unconditionally to x ?

Problem 2 (and hence also problem 1) is known to have an affirmative answer in the following cases [4]:

- 1°. $M(x, (x_n, f_n)) \supset m$ (the space of bounded sequences).
- 2°. E contains no subspace isomorphic to c_0 (the space of sequences converging to 0) and $M(x, (x_n, f_n)) \supset c_0$.
- 3°. $sp(f_n)$ (= linear span of (f_n)) is norming (i.e.,

$$\|x\| = \sup \{ \|f(x)\| \mid f \in sp(f_n), \|f\| \leq 1 \}$$

defines a norm on E equivalent to the original norm on E).

Problem 1 is known to have an affirmative answer in the case when $[x_n] = E$, where $[x_n]$ denotes the closed linear span of $\{x_n\}$ ([5]; see also [1], Theorem 3.4, implication (4) \Rightarrow (3)).

In the present paper we give an affirmative solution for problem

1. Our method also provides a more elementary proof of 3° than that given in [4].

THEOREM 1. *Let E be a separable Banach space and let (x_n, f_n) be a biorthogonal system for E with (f_n) total over E . If $M(E, (x_n, f_n))$ contains all sequences (ε_i) with $\varepsilon_i = \pm 1$ for each i , then (x_n) is an unconditional basis for E .*

If the hypothesis $[x_n] = E$ is added then a much simpler proof of the theorem is obtained (see the Remark following Lemma 3 below).

LEMMA 1. $M(E, (x_n, f_n)) \supset \{(\varepsilon_i) \mid \varepsilon_i = \pm 1 \text{ for all } i\}$ if and only if $M(E, (x_n, f_n)) \supset \{(\varepsilon_i) \mid \varepsilon_i = 0 \text{ or } 1 \text{ for all } i\}$.

Proof. Obvious.

LEMMA 2. *Suppose $(\varepsilon_i) \in M(E, (x_n, f_n))$, where $\varepsilon_i = 0$ or 1 for all i and define $S_{(\varepsilon_i)} = E \rightarrow E$ by*

$$(1) \quad S_{(\varepsilon_i)}(x) = x_{(\varepsilon_i)} \quad (x \in E).$$

Then $S_{(\varepsilon_i)}$ is a continuous linear mapping.

Lemma 2 is well known (see e.g. [8]).

In the particular case when $\varepsilon_i = 1$ for $i = 1, \dots, n$ and $\varepsilon_i = 0$ $i = n + 1, n + 2, \dots$ we shall use for $S_{(\varepsilon_i)}$ the notation S_n . Obviously,

$$(2) \quad S_n(x) = \sum_{i=1}^n f_i(x)x_i \quad (x \in E, n = 1, 2, \dots).$$

If σ is a subset of the positive integers, we define the mapping $S_\sigma: E \rightarrow E$ by

$$(3) \quad S_\sigma = S_{(\varepsilon_i)},$$

where $\varepsilon_i = 1$ for $i \in \sigma$ and $\varepsilon_i = 0$ for $i \notin \sigma$.

LEMMA 3. *Let (x_n, f_n) be a biorthogonal system for E (not necessarily separable), with (f_n) total over E . If $M(E, (x_n, f_n))$ contains all sequences (ε_i) with $\varepsilon_i = \pm 1$ for all i , then $(\|S_n\|)$ is bounded.*

Consequently, (x_n) is an unconditional basic sequence (i.e., an unconditional basis of its closed linear span $[x_n]$) and hence, if $[x_n] = E$, then (x_n) is an unconditional basis of E .

Proof. Assume that $(\|S_n\|)$ is unbounded. Let (n_k) be an increasing sequence of integers such that $\|S_{n_k}\| \geq 2^k + \|S_{n_{k-1}}\|$, whence

$\|S_{n_k} - S_{n_{k-1}}\| \rightarrow \infty$. Let $(M_p; p = 1, 2, \dots)$ be a countable collection of pairwise disjoint, infinite subsets of the positive integers, $I_k = \{n_{k-1} + 1, \dots, n_k\}$, and $\sigma_p = \bigcup_{k \in M_p} I_k$. The projection S_{σ_p} is continuous by Lemma 2. Moreover, if k is in M_p and x is in E , we have

$$\begin{aligned} \|(S_{n_k} - S_{n_{k-1}})x\| &= \left\| \sum_{i=n_{k-1}+1}^{n_k} f_i(x)x_i \right\| = \left\| \sum_{i=n_{k-1}+1}^{n_k} f_i(S_{\sigma_p}x)x_i \right\| \\ &= \|S_{n_k} - S_{n_{k-1}}\|_{S_{\sigma_p}x} \leq \|S_{n_k} - S_{n_{k-1}}\|_{X_p} \|S_{\sigma_p}\| \|x\| \end{aligned}$$

where

$$X_p = \{x \in E \mid f_j(x) = 0 \text{ if } j \notin \sigma_p\}.$$

It follows that $\|S_{n_k} - S_{n_{k-1}}\|_{X_p}$ is unbounded as k runs through M_p . Choose $u_p \in X_p$, $k_p \in M_p$ such that $\|u_p\| \leq 2^{-p}$ and $\|(S_{n_{k_p}} - S_{n_{k_p-1}})u_p\| \geq 1$. Let $\sigma = \bigcup_{p=1}^{\infty} I_{k_p}$. Now $\sigma \cap \sigma_p = I_{k_p}$ so that if $y_p \in X_p$ then $f_i(S_{\sigma}y_p) = f_i[(S_{n_{k_p}} - S_{n_{k_p-1}})y_p]$ for all i , whence $S_{\sigma}y_p = (S_{n_{k_p}} - S_{n_{k_p-1}})y_p$. Thus $\sum_p u_p$ converges while $S_{\sigma}(\sum_p u_p) = \sum_p S_{\sigma}(u_p) = \sum_p (S_{n_{k_p}} - S_{n_{k_p-1}})u_p$ doesn't converge, contradicting Lemma 2, that S_{σ} is continuous. Thus (x_n) is [2] a basic sequence. Since the same argument remains valid for every permutation $(x_{\rho(n)})$ of (x_n) , it follows that (x_n) is an unconditional basic sequence, which completes the proof.

REMARK. One can give a much simpler proof of the fact that under the hypotheses of Lemma 3 we have

$$(4) \quad \sup_n \|S_n|_{[x_j]}\| < \infty,$$

whence (x_n) is an unconditional basic sequence (and, if $[x_n] = E$, then (x_n) is an unconditional basis of E). Indeed, if (4) does not hold, then there exist increasing sequences of positive integers $(p_n), (q_n)$ with $p_{n-1} + 1 \leq q_{n-1} + 1 \leq p_n$ ($n = 1, 2, \dots; p_0 = q_0 = 0$) and a sequence (u_n) with $u_n \in [x_{q_{n-1}+1}, \dots, x_{q_n}]$ ($n = 1, 2, \dots$) such that $\|S_{p_n}u_n\| = 1$, $\|u_n\| \leq 1/2^n$ ($n = 1, 2, \dots$), whence $(\sum_{j=1}^n u_j)$ is convergent, but for $\sigma = \{1, \dots, p_1, q_1 + 1, \dots, p_2, \dots\}$ the sequence $(S_{\sigma}(\sum_{j=1}^n u_j)) = (\sum_{j=1}^n S_{p_j}u_j)$ is not convergent. Thus, S_{σ} is not continuous, which contradicts Lemma 2, completing the proof.

Proof of Theorem 1. We prove that $S_n x \rightarrow x$ for each x in E . This will prove the theorem by noting that the same proof works to show that each permutation of (x_n) is a basis for E , so that (x_n) is an unconditional basis for E . Choose x in E such that $(S_n x)$ does not converge (if it converges, its limit must be x by totality of the sequence (f_n)). Let $(n_k), (m_k)$ be sequences of integers such that $m_k + 1 \leq n_k \leq m_{k+1}$ for all k and such that there is $\varepsilon > 0$ with $\varepsilon < \|S_{n_k} - S_{m_k}\|x\|$ for all k . Let $u_k = (S_{n_k} - S_{m_k})x = \sum_{i=m_k+1}^{n_k} f_i(x)x_i$. For each sequence (η_i) such that $\eta_i = 1$ or 0 for each i there is an element of E , denoted

here by $\Sigma\eta_i u_i$, such that $(S_{n_k} - S_{m_k})(\Sigma\eta_i u_i) = \eta_k u_k$ for every k ($\Sigma\eta_i u_i$ is $x_{(\varepsilon_j)}$ where $\varepsilon_j = \eta_k$ for $m_k + 1 \leq j \leq n_k$, $k = 1, 2, \dots$ and 0 for the other j). Since E is separable, and since the set $\{\Sigma\eta_i u_i \mid \eta_i = 1 \text{ or } 0\}$ in E is uncountable, there is a sequence $(y_n)_0^\infty$ with $y_n = \Sigma\eta_i^{(n)} u_i$ such that $y_n \neq y_m$ if $n \neq m$ and $y_n \rightarrow y_0 = \Sigma\eta_i^{(0)} u_i$. Let K be a bound on $\|(S_{n_k} - S_{m_k})\|$ as guaranteed by Lemma 3. Then for p large, and all k , $\|(S_{n_k} - S_{m_k})(y_p - y_0)\| \leq K \|y_p - y_0\| < \varepsilon$, but

$$(S_{n_k} - S_{m_k})(y_p - y_0) = (\eta_k^{(p)} - \eta_k^{(0)})u_k,$$

whence

$$\|(S_{n_k} - S_{m_k})(y_p - y_0)\| = \begin{cases} 0 & \text{if } \eta_k^{(p)} = \eta_k^{(0)} \\ \|\eta_k^{(p)} - \eta_k^{(0)}\| \|u_k\| & \text{otherwise.} \end{cases}$$

Since $y_p \neq y_0$ for all $p \neq 0$, there is a $k = k(p)$ for which

$$\|(S_{n_k} - S_{m_k})(y_p - y_0)\| = \|\eta_k^{(p)} - \eta_k^{(0)}\| \|u_k\| > \varepsilon,$$

which is impossible for large p . Therefore $S_n x \rightarrow x$, which completes the proof of Theorem 1.

REMARK. Using the same method, one can also give a more elementary proof of the result 3° mentioned in the Introduction (actually, of a slightly more general result), than that given in [4]. As above, it is sufficient to show that $(S_n x)$ converges. If not, let $(n_k), (m_k), \varepsilon > 0$ and (u_k) be as in the above proof. Since $sp(f_n)$ is norming, by a technique of [3], or, equivalently, by [4], p. 311, lemma and p. 317, Lemma 5, we may assume (dropping to subsequences of (n_k) and (m_k) if necessary) that the natural projection P_k of $[x_1, \dots, x_{n_k}] \oplus [f_1, \dots, f_{m_{k+1}}]_\perp$ onto $[x_1, \dots, x_{n_k}]$ is of norm $\|P_k\| \leq C$, where $C > 1$ is a constant independent of k (actually, only this projection property is used in the sequel and therefore we obtain a slightly more general result than 3°). As in the above proof of Theorem 1 there is an element of E , denoted by $\Sigma\eta_i u_i$, which is in each of the subspaces $[x_1, \dots, x_{n_k}] \oplus [f_1, \dots, f_{m_{k+1}}]_\perp$, such that $(P_k - P_{k-1})(\Sigma\eta_i u_i) = \eta_k u_k$. The proof is completed in precisely the same manner as before, where now $P_k - P_{k-1}$ take the role of $S_{n_k} - S_{m_k}$.

Note. After this work had been completed, we have learned of the recent paper of G. F. Bachelis and H. P. Rosenthal "On unconditionally converging series and biorthogonal systems in a Banach space" (to appear in Pacific J. Math), where Problem 2 (and hence also Problem 1) is solved, even with the hypothesis "Let E be separable" replaced by the weaker hypothesis "Let E contain no subspace isomorphic to m ". However, our methods are completely different and use more elementary tools.

REFERENCES

1. G. F. Bachelis, *Homomorphisms of annihilator Banach algebras*, Pacific J. Math., **25** (1968), 229-247.
2. M. M. Day, *Normed Linear Spaces*, Springer-Verlag, 1962.
3. ———, *On the basis problem in normed spaces*, Proc. Amer. Math. Soc., **13** (1962), 655-658.
4. M. I. Kadec and A. Pelczynski, *Basic sequences, biorthogonal systems and norming sets in Banach and Frechet spaces*, Studia Math. **25** (1965), 297-323 (Russian).
5. E. R. Lorch, *Bicontinuous linear transformations in certain vector spaces*, Bull. Amer. Math. Soc., **45** (1939), 564-569.
6. B. S. Mityagin, *Approximative dimension and bases in nuclear spaces*, Uspehi Matem. Nauk. **16**, 4(100) (1961), 63-132 (Russian).
7. A. Pelczynski, *Some problems in functional analysis*, Lecture notes, L.S.U. (1966).
8. S. Yamazaki, *Normed rings and unconditional bases in Banach spaces*, Sci. Pap. Coll. Gen. Educ. Univ. Tokyo, **14** (1964), 1-10.

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