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RAMSEY BOUNDS FOR GRAPH PRODUCTS

PAUL ERDŐS, ROBERT JAMES MCELIECE AND HERBERT TAYLOR

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Here we show that Ramsey numbers $M(k_1, \dots, k_n)$ give sharp upper bounds for the independence numbers of product graphs, in terms of the independence numbers of the factors.

The Ramsey number $M(k_1, \dots, k_n)$ is the smallest integer m with the property that no matter how the $\binom{m}{2}$ edges of the complete graph on m nodes are partitioned into n colors, there will be at least one index i for which a complete subgraph on k_i nodes has all of its edges in the i th color. Ramsey's Theorem tells that these numbers exist but only a few exact values are known.

The complement graph \bar{G} has the same nodes as G and the complementary set of edges.

The independence number $\alpha(G)$ of a graph G , is the largest number of nodes in any complete subgraph of \bar{G} .

The product $G_1 \times \dots \times G_n$ of graphs G_1, \dots, G_n is the graph whose nodes are all the ordered n -tuples (a_1, \dots, a_n) in which a_i is a node of G_i for each i from 1 to n , and whose edges are as follows. A set of two nodes $\{(a_1, \dots, a_n), (b_1, \dots, b_n)\}$ will be an edge of $G_1 \times \dots \times G_n$ if and only if the nodes are distinct and for each i from 1 to n , $a_i = b_i$ or $\{a_i, b_i\}$ is an edge of G_i .

THEOREM 1. *For arbitrary graphs G_1, \dots, G_n*

$$\alpha(G_1 \times \dots \times G_n) < M(\alpha(G_1) + 1, \dots, \alpha(G_n) + 1) .$$

Proof. We have a complete subgraph of $\overline{G_1 \times \dots \times G_n}$ on $\alpha(G_1 \times \dots \times G_n)$ nodes. Its edges can be n colored by the following rule: give $\{(a_1, \dots, a_n), (x_1, \dots, x_n)\}$ color i if i is the first index for which $\{a_i, x_i\}$ is an edge of \bar{G}_i .

With this coloration any case where all the edges on k nodes have color i requires a complete k subgraph of \bar{G}_i and so requires $k < \alpha(G_i) + 1$. With the definition of the Ramsey number this ensures that

$$\alpha(G_1 \times \dots \times G_n) < M(\alpha(G_1) + 1, \dots, \alpha(G_n) + 1) .$$

THEOREM 2. *If k_1, \dots, k_n are given, there exist graphs G_1, \dots, G_n such that for each index i from 1 to n , $\alpha(G_i) = k_i$ and*

$$\alpha(G_1 \times \dots \times G_n) = M(k_1 + 1, \dots, k_n + 1) - 1 .$$

Proof. From the definition of the Ramsey number there must exist an n color partition of the edges of the complete graph on $M(k_1 + 1, \dots, k_n + 1) - 1 = m$ nodes such that for every i from 1 to n the largest complete subgraph in the i th color is on k_i nodes. For each i let G_i be the graph on the same m nodes having all the edges not of color i . Thus for each i , $\alpha(G_i) = k_i$. These G_i make the diagonal a complete m subgraph of $\overline{G_1 \times \dots \times G_n}$, and so

$$\alpha(G_1 \times \dots \times G_n) \geq m.$$

Applying Theorem 1 we have

$$\alpha(G_1 \times \dots \times G_n) = M(k_1 + 1, \dots, k_n + 1) - 1$$

THEOREM 3. *If n and k are given, there exists a graph G such that $\alpha(G) = k$ and putting $k_i = k$ for every i ,*

$$\alpha(G^n) = M(k_1 + 1, \dots, k_n + 1) - 1.$$

Proof. With $m = M(k_1 + 1, \dots, k_n + 1) - 1$ and every $k_i = k$, refer to the graphs G_1, \dots, G_n as specified for Theorem 2. Now construct G as follows. Let the nodes of G be all the ordered pairs (a, i) such that $1 \leq i \leq n$ and a is a node of G_i . Let $\{(a, i), (b, j)\}$ be an edge of G if and only if $i \neq j$ or $\{a, b\}$ is an edge of G_i .

Thus constructed $\alpha(G) = k$ because each $\alpha(G_i) = k$. $\overline{G^n}$ will have a subgraph isomorphic to $\overline{G_1 \times \dots \times G_n}$ and consequently

$$\alpha(G^n) \geq \alpha(G_1 \times \dots \times G_n) = m.$$

So again with Theorem 1 we have

$$\alpha(G^n) = m = M(k_1 + 1, \dots, k_n + 1) - 1.$$

A question remains whether for every k, n with

$$k^2 \leq n < M(k + 1, k + 1)$$

there exists G such that $\alpha(G) = k$ and $\alpha(G^2) = n$. It is known that $M(4, 4) = 18$, and for each n between 9 and 17 we have found a graph G such that $\alpha(G) = 3$ and $\alpha(G^2) = n$. However it is only known that $37 < M(5, 5) < 58$ and for example we have no proof that there exists a graph G such that $\alpha(G) = 4$ and $\alpha(G^2) = M(5, 5) - 2$.

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