RAMSEY BOUNDS FOR GRAPH PRODUCTS

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Here we show that Ramsey numbers $M(k_1, \ldots, k_n)$ give sharp upper bounds for the independence numbers of product graphs, in terms of the independence numbers of the factors.

The Ramsey number $M(k_1, \ldots, k_n)$ is the smallest integer $m$ with the property that no matter how the $m \choose 2$ edges of the complete graph on $m$ nodes are partitioned into $n$ colors, there will be at least one index $i$ for which a complete subgraph on $k_i$ nodes has all of its edges in the $i$th color. Ramsey's Theorem tells that these numbers exist but only a few exact values are known.

The complement graph $\bar{G}$ has the same nodes as $G$ and the complementary set of edges.

The independence number $\alpha(G)$ of a graph $G$, is the largest number of nodes in any complete subgraph of $G$.

The product $G_1 \times \cdots \times G_n$ of graphs $G_1, \ldots, G_n$ is the graph whose nodes are all the ordered $n$-tuples $(a_1, \ldots, a_n)$ in which $a_i$ is a node of $G_i$ for each $i$ from 1 to $n$, and whose edges are as follows. A set of two nodes $\{(a_1, \ldots, a_n), (b_1, \ldots, b_n)\}$ will be an edge of $G_1 \times \cdots \times G_n$ if and only if the nodes are distinct and for each $i$ from 1 to $n$, $a_i = b_i$ or $\{a_i, b_i\}$ is an edge of $G_i$.

**Theorem 1.** For arbitrary graphs $G_1, \ldots, G_n$

$$\alpha(G_1 \times \cdots \times G_n) < M(\alpha(G_1) + 1, \ldots, \alpha(G_n) + 1).$$

**Proof.** We have a complete subgraph of $\bar{G}_1 \times \cdots \times \bar{G}_n$ on $\alpha(G_1) \times \cdots \times \alpha(G_n)$ nodes. Its edges can be $n$ colored by the following rule: give $\{(a_1, \ldots, a_n), (x_1, \ldots, x_n)\}$ color $i$ if $i$ is the first index for which $\{a_i, x_i\}$ is an edge of $\bar{G}_i$.

With this coloration any case where all the edges on $k$ nodes have color $i$ requires a complete $k$ subgraph of $\bar{G}_i$ and so requires $k < \alpha(G_i) + 1$. With the definition of the Ramsey number this ensures that

$$\alpha(G_1 \times \cdots \times G_n) < M(\alpha(G_1) + 1, \ldots, \alpha(G_n) + 1).$$

**Theorem 2.** If $k_1, \ldots, k_n$ are given, there exist graphs $G_1, \ldots, G_n$ such that for each index $i$ from 1 to $n$, $\alpha(G_i) = k_i$ and

$$\alpha(G_1 \times \cdots \times G_n) = M(k_1 + 1, \ldots, k_n + 1) - 1.$$
Proof. From the definition of the Ramsey number there must exist an $n$ color partition of the edges of the complete graph on $M(k_1, 1, \ldots, k_n, 1) - 1 = m$ modes such that for every $i$ from 1 to $n$ the largest complete subgraph in the $i$th color is on $k_i$ nodes. For each $i$ let $G_i$ be the graph on the same $m$ nodes having all the edges not of color $i$. Thus for each $i$, $\alpha(G_i) = k_i$. These $G_i$ make the diagonal a complete $m$ subgraph of $G_1 \times \cdots \times G_n$, and so

$$\alpha(G_1 \times \cdots \times G_n) \geq m.$$ 

Applying Theorem 1 we have

$$\alpha(G_1 \times \cdots \times G_n) = M(k_1, 1, \ldots, k_n, 1) - 1.$$

**Theorem 3.** If $n$ and $k$ are given, there exists a graph $G$ such that $\alpha(G) = k$ and putting $k_i = k$ for every $i$,

$$\alpha(G^n) = M(k_1, 1, \ldots, k_n, 1) - 1.$$

**Proof.** With $m = M(k_1, 1, \ldots, k_n, 1) - 1$ and every $k_i = k$, refer to the graphs $G_1, \cdots, G_n$ as specified for Theorem 2. Now construct $G$ as follows. Let the nodes of $G$ be all the ordered pairs $(a, i)$ such that $1 \leq i \leq n$ and $a$ is a node of $G_i$. Let $\{(a, i), (b, j)\}$ be an edge of $G$ if and only if $i \neq j$ or $\{a, b\}$ is an edge of $G_i$.

Thus constructed $\alpha(G) = k$ because each $\alpha(G_i) = k$. $G^n$ will have a subgraph isomorphic to $G_1 \times \cdots \times G_n$ and consequently

$$\alpha(G^n) \geq \alpha(G_1 \times \cdots \times G_n) = m.$$

So again with Theorem 1 we have

$$\alpha(G^n) = m = M(k_1, 1, \ldots, k_n, 1) - 1.$$

A question remains whether for every $k, n$ with

$$k^2 \leq n < M(k + 1, k + 1)$$

there exists $G$ such that $\alpha(G) = k$ and $\alpha(G^n) = n$. It is known that $M(4, 4) = 18$, and for each $n$ between 9 and 17 we have found a graph $G$ such that $\alpha(G) = 3$ and $\alpha(G^n) = n$. However it is only known that $37 < M(5, 5) < 58$ and for example we have no proof that there exists a graph $G$ such that $\alpha(G) = 4$ and $\alpha(G^n) = M(5, 5) - 2$.

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