Pacific Journal of Mathematics

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Vol. 37, No. 1 January 1971

PACIFIC JOURNAL OF MATHEMATICS
Vol. 37, No. 1, 1971

STRUCTURE OF SEMIPRIME (p, q) RADICALS

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In this note, the structure of the semiprime (p, q) radicals is investigated. Let $p(x)$ and $q(x)$ be polynomials over the integers. An element a of an arbitrary associative ring R is called (p, q)-*regular* if $a \in p(a) \cdot R \cdot q(a)$. A ring R is (p, q)-*regular* if every element of R is (p, q)-regular. It is easy to prove that (p, q)-regularity is a radical property and also that it is a semiprime radical property (meaning that the radical of a ring is a semiprime ideal of the ring) if and only if the constant coefficients of $p(x)$ and $q(x)$ are ± 1. It is shown that every (p, q)-semisimple ring is isomorphic to a subdirect sum of rings which are either right primitive or left primitive.

Our results follow the ideas in [1]. However, a direct application of the results of [1] is not possible here because condition P_1 [1, p. 302] is not always satisfied in the present case.

Let R be an arbitrary associative ring. Let $p(x) = 1 + n_1 x + \cdots + n_k x^k$ be a polynomial over the integers. For each element $a \in R$, let $F_R(a) = p(a) \cdot R$. In what follows we take $q(x) = 1$. Thus an element a of R is called $(p, 1)$-*regular* if $a \in F_R(a)$. A ring R is called $(p, 1)$-*regular* if every element in R is $(p, 1)$-regular. We shall denote the $(p, 1)$ radical property by F.

A right ideal I of R will be called $(p, 1)$-*modular* if there exists an element $e \notin I$ such that $F_R(e) + eI \subset I$. In order to specify the element e we shall sometimes say that I is $(p, 1)_e$-*modular*. An ideal P of R will be called $(p, 1)$-*primitive* if P is the largest two sided ideal contained in some maximal $(p, 1)_e$-modular right ideal for some e. For a right ideal M of R, let $(M: R) = \{a \in R \mid Ra \subset M\}$ and let $p_0(x) = p(x) - 1$ throughout this paper.

LEMMA 1. *An ideal P of R is $(p, 1)$-primitive if and only if there exists $e \in R$ and a maximal $(p, 1)_e$-modular right ideal M such that $P = (M: R)$.*

Proof. It is clear that $(M: R)$ is a two sided ideal of R. Moreover if $a \in (M: R)$, then $a = p(e) \cdot a - p_0(e) \cdot a \in F_R(e) + Ra \subset M$. Finally if K is an ideal contained in M, then $RK \subset K \subset M$. Hence $K \subset (M: R)$. Thus $(M: R)$ is the largest two sided ideal contained in M.

LEMMA 2. *If I is a $(p, 1)_e$-modular right ideal of R and if $b \in I$, then*

$$F_R(e + b) \subset I .$$

Proof. $p(e+b)\cdot r = p(e)\cdot r + br_1 + ebr_2 + \cdots + e^{k-1}br_k \in F_R(e) + I + eI + \cdots + e^{k-1}I \subset I$.

THEOREM 3. *If P is a $(p, 1)$-primitive ideal of R, then R/P is F-semisimple.*

Proof. Let W/P be a nonzero $(p, 1)$-regular ideal of R/P, where P is $(p, 1)_e$-primitive, say $P = (M\colon R)$. Since P is the largest ideal in M, $W + M$ contains M properly. But $e(W + M) \subset W + M$. Hence $e \in W + M$, since otherwise $W + M$ would be $(p, 1)_e$-modular, violating the maximality of M. Thus, say, $e = w + m$. Since W/P is $(p, 1)$-regular,

$$w + P \in F_{R/P}(w + P) = [F_R(w) + P]/P\ .$$

Now $F_R(w) = F_R(e - m) \subset M$, using Lemma 2. Thus $w \in M + P \subset M$. But then $e = w + m \in M$, a contradiction. Therefore W/P must be 0.

THEOREM 4. *Let F be any semiprime $(p, 1)$ radical property. Then for all rings R, $F(R)$ is the intersection of all $(p, 1)$-primitive ideals of R.*

Proof. If P is a $(p, 1)$-primitive ideal of R, then R/P is F-semisimple, thus $P \supset F(R)$.

On the other hand suppose that the intersection K of all $(p, 1)$-primitive ideals of R is not $(p, 1)$-regular. That is, there is $e \in K$ such that $e \notin F_K(e)$. Then $e \notin F_R(e)$. But $F_R(e)$ is a $(p, 1)_e$-modular right ideal of R. Let M be a maximal $(p, 1)_e$-modular right ideal of R. Then $e \notin M \supset (M\colon R) \supset K$, a contradiction. Therefore K is $(p, 1)$-regular and thus $K \subset F(R)$.

COROLLARY 5. *Every F-semisimple ring is isomorphic to a sub-direct sum of $(p, 1)$-primitive rings.*

This, together with the next theorem, give the structure of the F-semisimple rings.

THEOREM 6. *Every $(p, 1)$-primitive ideal is primitive.*

Proof. Let P be a $(p, 1)$-primitive ideal of R. Then $P = (M\colon R)$ for some maximal $(p, 1)_e$-modular right ideal M. Then M is a modular (in the sense of [3]) right ideal. Thus M is contained in a modular maximal right ideal N. Thus $(M\colon R) \subset (N\colon R)$. Now if $(N\colon R) \not\subset (M\colon R)$, then there exists $a \in R$ such that $Ra \subset N$ but $Ra \not\subset M$. Thus $M + Ra + RaR$ is a right ideal which contains M properly. Since $e(M +$

$Ra + RaR) \subset M + Ra + RaR$, and since M is a maximal $(p, 1)_e$-modular right ideal of R, $M + Ra + RaR = R$. But each term M, Ra, and RaR is contained in N. Thus $N = R$, a contradiction. Therefore $P = (N: R)$ and P is primitive (in the Jacobson sense).

COROLLARY 7. *Every $(p, 1)$-regular radical F contains the Jacobson radical.*

THEOREM 8. *A semiprime $(p, 1)$-regular radical coincides with the Jacobson radical if the sum $p(1)$ or the alternate sum $p(-1)$ of the coefficients of $p(x)$ is* 0.

Proof. Let P be a primitive ideal of R, say $P = (M: R)$, where M is a modular [3] maximal right ideal of R. Suppose that $F(R) \not\subset P$. Then there exists $r \in R$ such that $r \cdot F(R) \not\subset M$. Thus $M + r \cdot F(R) = R$. In particular, there exists $a \in F(R)$ such that $r = ra \bmod M$. Since a is $(p, 1)$-regular, there is $a' \in R$ such that $a = p(a) \cdot a'$. Hence, supposing that $p(1) = 0$, $ra = r \cdot p(a) \cdot a' = p(1) \cdot raa' = 0$. But then $r \in M$, a contradiction. The case when $p(-1) = 0$ is analogous.

Since each $(p, 1)$-primitive ideal P of R is prime and R/P is F-semisimple, $F(R)$ is the intersection of all ideals I of R such that R/I is prime and F-semisimple. Since F is also hereditary, we have [2, p. 149] that F is a special radical.

The generalization of our results to all semiprime (p, q) radicals is as follows: Define $(1, q)_e$-modular left ideals and left $(1, q)$-primitive ideals in an analogous fashion. Next show that a $(1, q)$-semisimple ring is isomorphic to a subdirect sum of left primitive [3] rings. Finally, use Theorem 3 of [4] to prove, for $p(0) = \pm 1$ and $q(0) = \pm 1$, the following:

THEOREM 9. *For any semiprime (p, q) radical, every (p, q)-semisimple ring is isomorphic to a subdirect sum of rings which are either right primitive or left primitive.*

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Received April 2, 1970. This work was done while the second author held a Post Doctoral Fellowship at the University of Florida.

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PACIFIC JOURNAL OF MATHEMATICS

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 37, No. 1 January, 1971