STRUCTURE OF SEMIPRIME \((p, q)\) RADICALS

THOMAS L. GOULDING AND AUGUSTO H. ORTIZ
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In this note, the structure of the semiprime \((p, q)\) radicals is investigated. Let \(p(x)\) and \(q(x)\) be polynomials over the integers. An element \(a\) of an arbitrary associative ring \(R\) is called \((p, q)\)-regular if \(a \in \psi(a) \cdot R \cdot q(a)\). A ring \(R\) is \((p, q)\)-regular if every element of \(R\) is \((p, q)\)-regular. It is easy to prove that \((p, q)\)-regularity is a radical property and also that it is a semiprime radical property (meaning that the radical of a ring is a semiprime ideal of the ring) if and only if the constant coefficients of \(p(x)\) and \(q(x)\) are \(\pm 1\). It is shown that every \((p, q)\)-semisimple ring is isomorphic to a subdirect sum of rings which are either right primitive or left primitive.

Our results follow the ideas in [1]. However, a direct application of the results of [1] is not possible here because condition \(P_1\) [1, p. 302] is not always satisfied in the present case.

Let \(R\) be an arbitrary associative ring. Let \(p(x) = 1 + n_1x + \cdots + n_kx^k\) be a polynomial over the integers. For each element \(a \in R\), let \(F_{p}(a) = \psi(a) \cdot R\). In what follows we take \(q(x) = 1\). Thus an element \(a\) of \(R\) is called \((p, 1)\)-regular if \(a \in F_{p}(a)\). A ring \(R\) is called \((p, 1)\)-regular if every element in \(R\) is \((p, 1)\)-regular. We shall denote the \((p, 1)\) radical property by \(F_{p}\).

A right ideal \(I\) of \(R\) will be called \((p, 1)\)-modular if there exists an element \(e \in R\) such that \(F_{p}(e) + eI \subseteq I\). In order to specify the element \(e\) we shall sometimes say that \(I\) is \((p, 1)\)-modular. An ideal \(P\) of \(R\) will be called \((p, 1)\)-primitive if \(P\) is the largest two sided ideal contained in some maximal \((p, 1)\)-modular right ideal for some \(e\). For a right ideal \(M\) of \(R\), let \((M: R) = \{a \in R \mid Ra \subseteq M\}\) and let \(p_{\delta}(x) = p(x) - 1\) throughout this paper.

**Lemma 1.** An ideal \(P\) of \(R\) is \((p, 1)\)-primitive if and only if there exists \(e \in R\) and a maximal \((p, 1)\)-modular right ideal \(M\) such that \(P = (M: R)\).

**Proof.** It is clear that \((M: R)\) is a two sided ideal of \(R\). Moreover if \(a \in (M: R)\), then \(a = p(e) \cdot a - p_{\delta}(e) \cdot a \in F_{p}(e) + Ra \subseteq M\). Finally if \(K\) is an ideal contained in \(M\), then \(RK \subseteq K \subseteq M\). Hence \(K \subseteq (M: R)\). Thus \((M: R)\) is the largest two sided ideal contained in \(M\).

**Lemma 2.** If \(I\) is a \((p, 1)\)-modular right ideal of \(R\) and if \(b \in I\), then
\[
F_{p}(e + b) \subseteq I.
\]
Proof. \( p(e + b) \cdot r = p(e) \cdot r + b r_1 + e b r_2 + \cdots + e^{k-1} b r_k \in F_R(e) + I + eI + \cdots + e^{k-1} I \subseteq I. \)

**Theorem 3.** If \( P \) is a \((p, 1)\)-primitive ideal of \( R \), then \( R/P \) is \( F \)-semisimple.

**Proof.** Let \( W/P \) be a nonzero \((p, 1)\)-regular ideal of \( R/P \), where \( P \) is \((p, 1)\)-semiprimitive, say \( P = (M: R) \). Since \( P \) is the largest ideal in \( M \), \( W + M \) contains \( M \) properly. But \( e(W + M) \subset W + M \). Hence \( e \notin W + M \), since otherwise \( W + M \) would be \((p, 1)\)-modular, violating the maximality of \( M \). Thus, say, \( e = w + m \). Since \( W/P \) is \((p, 1)\)-regular,

\[ w + P \in F_{R/P}(w + P) = [F_R(w) + P]/P. \]

Now \( F_R(w) = F_R(e - m) \subseteq M \), using Lemma 2. Thus \( w \in M + P \subseteq M \). But then \( e = w + m \in M \), a contradiction. Therefore \( W/P \) must be 0.

**Theorem 4.** Let \( F \) be any semiprime \((p, 1)\) radical property. Then for all rings \( R \), \( F(R) \) is the intersection of all \((p, 1)\)-primitive ideals of \( R \).

**Proof.** If \( P \) is a \((p, 1)\)-primitive ideal of \( R \), then \( R/P \) is \( F \)-semisimple, thus \( P \supset F(R) \).

On the other hand suppose that the intersection \( K \) of all \((p, 1)\)-primitive ideals of \( R \) is not \((p, 1)\)-regular. That is, there is \( e \in K \) such that \( e \notin F_K(e) \). Then \( e \notin F_K(e) \). But \( F_R(e) \) is a \((p, 1)\)-semiprimitive right ideal of \( R \). Let \( M \) be a maximal \((p, 1)\)-semiprimitive right ideal of \( R \). Then \( e \notin M \supset (M: R) \supset K \), a contradiction. Therefore \( K \) is \((p, 1)\)-regular and thus \( K \subseteq F(R) \).

**Corollary 5.** Every \( F \)-semisimple ring is isomorphic to a subdirect sum of \((p, 1)\)-primitive rings.

This, together with the next theorem, give the structure of the \( F \)-semisimple rings.

**Theorem 6.** Every \((p, 1)\)-primitive ideal is primitive.

**Proof.** Let \( P \) be a \((p, 1)\)-primitive ideal of \( R \). Then \( P = (M: R) \) for some maximal \((p, 1)\)-semiprimitive right ideal \( M \). Then \( M \) is a modular (in the sense of [3]) right ideal. Thus \( M \) is contained in a modular maximal right ideal \( N \). Thus \( (M: R) \subseteq (N: R) \). Now if \( (N: R) \cap (M: R) \), then there exists \( a \in R \) such that \( Ra \subseteq N \) but \( Ra \not\subseteq M \). Thus \( M + Ra + RaR \) is a right ideal which contains \( M \) properly. Since \( e(M + \)


Ra + RaR) ⊂ M + Ra + RaR, and since M is a maximal (p, 1)-modular right ideal of R, M + Ra + RaR = R. But each term M, Ra, and RaR is contained in N. Thus N = R, a contradiction. Therefore P = (N: R) and P is primitive (in the Jacobson sense).

**Corollary 7.** Every (p, 1)-regular radical F contains the Jacobson radical.

**Theorem 8.** A semiprime (p, 1)-regular radical coincides with the Jacobson radical if the sum p(1) or the alternate sum p(−1) of the coefficients of p(x) is 0.

**Proof.** Let P be a primitive ideal of R, say P = (M: R), where M is a modular [3] maximal right ideal of R. Suppose that F(R) ⊄ P. Then there exists r ∈ R such that r · F(R) ⊄ M. Thus M + r · F(R) = R. In particular, there exists a ∈ F(R) such that r = ra mod M. Since a is (p, 1)-regular, there is a' ∈ R such that a = p(a) · a'. Hence, supposing that p(1) = 0, ra = r · p(a) · a' = p(1) · raa' = 0. But then r ∈ M, a contradiction. The case when p(−1) = 0 is analogous.

Since each (p, 1)-primitive ideal P of R is prime and R/P is F-semisimple, F(R) is the intersection of all ideals I of R such that R/I is prime and F-semisimple. Since F is also hereditary, we have [2, p. 149] that F is a special radical.

The generalization of our results to all semiprime (p, q) radicals is as follows: Define (1, q)-modular left ideals and left (1, q)-primitive ideals in an analogous fashion. Next show that a (1, q)-semisimple ring is isomorphic to a subdirect sum of left primitive [3] rings. Finally, use Theorem 3 of [4] to prove, for p(0) = ±1 and q(0) = ±1, the following:

**Theorem 9.** For any semiprime (p, q) radical, every (p, q)-semisimple ring is isomorphic to a subdirect sum of rings which are either right primitive or left primitive.

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