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## **DIFFERENTIABILITY OF MINIMAL SURFACES AT THE BOUNDARY**

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**Let  $\Gamma$  be a Jordan curve in  $R^3$  and  $F(z) = (u(z), v(z), w(z))$ :  $\{|z| \leq 1\} \rightarrow R^3$  be a solution of Plateau's problem for  $\Gamma$ , where  $z = x + iy$  are isothermal parameters. Then  $u, v, w$  are harmonic in  $\{|z| < 1\}$  and are the real parts of analytic functions  $\lambda, \mu, \nu$ . Using the Poisson integral and the defining properties of minimal surfaces, Kellogg's theorem for conformal mapping is generalized by proving:**

- 1. If  $\Gamma \in C^{1,\alpha}$ ,  $0 < \alpha < 1$ , then  $\lambda, \mu, \nu \in C^{1,\alpha}$  for  $|z| \leq 1$  and if  $\Gamma \in \mathcal{C}^{1,1}$  then  $\lambda', \mu', \nu'$  have modulus of continuity  $Kt \log 1/t$  for  $|z| \leq 1$ ;  $K$  and the Hölder constants depend only on the geometry of  $\Gamma$ .**
- 2. If  $\Gamma \in C^{n,\omega(t)}$ ,  $n \geq 2$ , where  $\omega(t)$  is a modulus of continuity satisfying a Dini condition, then  $\lambda, \mu, \nu \in C^{n,\omega^*(t)}$  for  $|z| \leq 1$ , where  $\omega^*(t)$  is a certain modulus of continuity. Once again  $\omega^*$  depends only on  $\Gamma$ .**

Let  $\Gamma$  be a closed Jordan curve in  $R^3$ . Then  $S$  is called a generalized minimal surface spanning  $\Gamma$  if  $S$  is represented by a triple of real valued functions

$$F(z) = (u(z), v(z), w(z)) : \{|z| \leq 1\} \rightarrow R^3 \quad (z = x + iy = re^{i\theta})$$

such that

- (a)  $u, v, w$  are harmonic in  $|z| < 1$  and continuous in  $|z| \leq 1$
- (b)  $x$  and  $y$  are isothermal parameters in  $z < 1$ , i.e.,

$$\begin{aligned} F_x^2 &= u_x^2 + v_x^2 + w_x^2 = u_y^2 + v_y^2 + w_y^2 = F_y^2 \\ F_x \cdot F_y &= u_x u_y + v_x v_y + w_x w_y = 0 \quad \text{for } |z| < 1 \end{aligned}$$

- (c)  $F(e^{i\theta})$  is a homeomorphism of  $|z| = 1$  with  $\Gamma$ .

A solution to Plateau's problem for  $\Gamma$  is a generalized minimal surface spanning  $\Gamma$ , and a solution may be normalized by specifying that three fixed points on  $|z| = 1$  correspond to three fixed points on  $\Gamma$ . We shall consider the solutions to be normalized, and we note that there may be more than one normalized surface spanning a given curve  $\Gamma$ .

Consider the analytic functions of which  $u, v, w$  are the real parts:

$$\lambda(z) = u(z) + iu^*(z) \quad \mu(z) = v(z) + iv^*(z) \quad \nu(z) = w(z) + iw^*(z).$$

Then the condition (b) is equivalent to

$$(1) \quad \lambda'^2(z) + \mu'^2(z) + \nu'^2(z) = 0 \quad |z| < 1.$$

This paper will deal with the differentiability of  $\lambda, \mu, \nu$  at the boundary  $|z| = 1$ , under given smoothness conditions on the curve  $\Gamma$ .

It was noted by Weierstrass that if the boundary  $\Gamma$  of a minimal surface  $S$  contains a straight line segment  $\alpha$ , then the surface may be extended analytically as a minimal surface across  $\alpha$ , by use of the reflection principle. In 1951 H. Lewy [5] proved that if  $\alpha$  is an analytic arc then the surface can be extended analytically across  $\alpha$ .

For an up-to-date account of the studies on the boundary behavior of minimal surfaces see the recent paper of J. C. C. Nitsche [7]. In that paper Nitsche proved among other results that if  $\Gamma \in C^{n,\alpha}$  for  $n \geq 1$  and  $0 < \alpha < 1$ , then  $F(z) \in C^{n,\alpha}$  in  $|z| \leq 1$  and the Hölder constant for the  $n$ th derivatives of  $F(z)$  is the same for all solutions of Plateau's problem, i.e., they depend only on the geometrical properties of  $\Gamma$ . In this connection see also [4], where a completely different proof of the first part of Nitsche's theorem is given.

In the following we shall say that a function  $f(z) \in C^{n,\omega(t)}$  for  $z$  in some domain if  $f^{(n)}$  exists and has modulus of continuity  $\omega(t)$ , i.e.,

$$|f^{(n)}(t_1) - f^{(n)}(t_2)| \leq \omega(|t_1 - t_2|) \quad \text{for } |t_1 - t_2| < \sigma,$$

where  $\omega(t)$  is a nondecreasing, non-negative function for  $0 \leq t \leq \sigma$  and  $\int_0^\sigma (\omega(t)/t)dt < \infty$ . We shall assume, as we may without loss of generality, that  $t = O(\omega(|t|))$  as  $t \rightarrow 0$ . In the following  $O(\varphi(t))$  shall mean  $O(\varphi(t))$  as  $t \rightarrow 0$ . Note that if  $\omega(t) = kt^\alpha$ ,  $0 < \alpha < 1$ ,  $k$  a constant, then  $f(t) \in C^{n,\alpha}$ . We shall denote by  $s(\theta) = s(F(e^{i\theta}))$  the arclength along  $\Gamma$  with  $s(0) = 0$ . Our principal results are the following.

**THEOREM 1.** *If  $\Gamma \in C^{1,\alpha}$ ,  $0 < \alpha \leq 1$  then each of  $\lambda, \mu, \nu$  is continuously differentiable in  $|z| \leq 1$ . In addition, there exists a constant  $c$  such that  $|s'(\theta)| \leq c$ ,  $-\pi \leq \theta \leq \pi$ , where  $c$  is dependent only on  $\Gamma$ .*

**THEOREM 2.** *Suppose  $\Gamma \in C^{1,\omega(t)}$  and  $\lambda, \mu, \nu$  are continuously differentiable for  $|z| \leq 1$ . Let  $c$  be a constant such that  $\max_{|\theta| \leq \pi} |s'(\theta)| \leq c$  and let  $\omega_0(t) = \omega(ct)$ . Then there exist constants  $K$  and  $K_1$  depending on  $c$  and on  $\omega(t)$ , such that  $\lambda'(e^{i\theta}), \mu'(e^{i\theta}), \nu'(e^{i\theta})$  have modulus of continuity*

$$\omega_0^*(\theta) = K \left( \int_0^\theta \frac{\omega_0(t)}{t} dt + \theta \int_\theta^\pi \frac{\omega_0(t)}{t^2} dt \right)$$

and  $\lambda'(z), \mu'(z), \nu'(z)$  have modulus of continuity  $K_1 \omega_0^*(\pi t)$  for  $|z| \leq 1$ .

Combining Theorems 1 and 2 we obtain: *If  $\Gamma \in C^{1,\alpha}$ ,  $0 < \alpha < 1$  then  $\lambda, \mu, \nu \in C^{1,\alpha}$  for  $|z| \leq 1$ . If  $\Gamma \in C^{1,1}$  then  $\lambda, \mu, \nu \in C^{1,\omega^*(t)}$  for  $\omega^*(t) = Kt \log 3\pi/t$  for some constant  $K$ . Furthermore there exists a constant  $c$  such that  $|s'(\theta)| \leq c$  for all  $|\theta| \leq \pi$ .  $K$  and  $c$  depend on*

$\Gamma$  only.

**THEOREM 3.** Suppose that  $\Gamma \in {}^{n,\omega(t)}$ ,  $n \geq 2$ . Let  $c$  be a constant such that  $|s'(\theta)| \leq c$ ,  $|\theta| \leq \pi$ , and let  $\omega_0(t) = \omega(ct)$  (such a constant  $c$  which depends only on  $\Gamma$  exists by Theorem 1). Then:

(i)  $\lambda^{(n)}, \mu^{(n)}, \nu^{(n)}$  have continuous extensions to  $|z| = 1$  and there exist constants  $K$  and  $K_1$ , depending only on  $\Gamma$  such that  $\lambda^{(n)}(e^{i\theta}), \mu^{(n)}(e^{i\theta}), \nu^{(n)}(e^{i\theta})$  have modulus of continuity

$$\omega_0^*(\theta) = K \left[ \int_0^\theta \frac{\omega_0(t)}{t} dt + \theta \int_\theta^\pi \frac{\omega_0(t)}{t^2} dt \right]$$

and  $\lambda^{(n)}(z), \mu^{(n)}(z), \nu^{(n)}(z)$  have modulus of continuity  $K_1 \omega_0^*(\pi t)$  for  $|z| \leq 1$ .

(ii) There exists a constant  $c_n$  depending only on  $\Gamma, n$  such that  $|s^{(n)}(\theta)| \leq c_n$  for  $|\theta| \leq \pi$ .

Conformal mappings in the plane are special cases of minimal surfaces and in the conformal mapping case the result for  $\omega(t) = Kt^\alpha$ ,  $0 < \alpha < 1$  is due to O. D. Kellogg. The extension of Kellogg's theorem to a modulus of continuity satisfying a Dini condition  $\int_0^\sigma (\omega(t)/t) dt < \infty$ , was given by S. E. Warschawski [8] for  $n = 1$  (for  $n > 1$  see [9]).

The case  $\Gamma \in C^{1,\omega(t)}$ , i.e., the proof of Theorem 3 for  $n = 1$ , does not seem to lend itself to the method we use in establishing our Theorem 1. However, Warschawski [10] has recently given a proof of this case along different lines.

We note that our results overlap to some extent with those of Nitsche [7]. They were obtained independently, although a basic device used in the proof of Theorem 1 (Lemmas 5 and 6) is the same. However, there are differences both in approach and in detail between the two proofs.

The results hold for minimal surfaces in  $n$ -space, in which case we have  $n$  harmonic and  $n$  analytic functions. Also, it will be apparent that the theorems are local in the sense that they are true for subarcs of  $\Gamma$ .

**2. Auxiliary Results.** In the following we shall need a number of lemmas.

**LEMMA 1.** Suppose that the function  $f(z) = u(re^{it}) + iu^*(re^{it})$  is holomorphic in  $|z| < 1$  and  $u(re^{it})$  is continuous in  $|z| \leq 1$ . Suppose also that for some integer  $n \geq 0$

$$|u(e^{it})| \leq A |t|^n \omega(|t|) \quad \text{for } |t| \leq \pi$$

where  $A$  is a constant and  $\omega(t)$  is nondecreasing and nonnegative.

Then there exists a constant  $M$ , depending only on  $A$  and on  $n$ , such that for  $r \geq 1/2$ ,

$$|f^{(n+1)}(r)| \leq M \int_{1-r}^{\pi} \frac{\omega(t)}{t^2} dt.$$

*Proof.* We begin with the Poisson Integral for  $f$ :

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{it}) \frac{e^{it} + z}{e^{it} - z} dt + iu^*(0) \quad |z| < 1.$$

Differentiating, we obtain

$$f^{(n+1)}(z) = \frac{(n+1)!}{\pi} \int_{-\pi}^{\pi} \frac{u(e^{it})e^{it}}{(e^{it} - z)^{n+2}} dt$$

and in particular

$$\begin{aligned} |f^{(n+1)}(r)| &\leq \frac{2A(n+1)!}{\pi} \int_0^{\pi} \frac{t^n \omega(t)}{[1 - 2r \cos t + r^2]^{n/2+1}} dt \\ &\leq \frac{2A(n+1)!}{\pi} \int_0^{\pi} \frac{t^n \omega(t)}{\left[(1-r)^2 + 4r \frac{t^2}{\pi^2}\right]^{n/2+1}} dt \\ &\leq \frac{2A(n+1)!}{\pi} \left[ \int_0^{1-r} \frac{t^n \omega(t)}{(1-r)^{n+2}} dt + \int_{1-r}^{\pi} \frac{t^n \omega(t)}{\left[4r \frac{t^2}{\pi^2}\right]^{n/2+1}} dt \right] \\ &\leq \frac{2A(n+1)!}{\pi} \left[ \frac{\omega(1-r)}{(1-r)^{n+2}} \int_0^{1-r} t^n dt + \frac{\pi^{n+2}}{2^{n/2+1}} \int_{1-r}^{\pi} \frac{t^n \omega(t)}{t^{n+2}} dt \right] \end{aligned}$$

for  $r \geq 1/2$ ,

$$\leq \frac{2An!}{\pi} \frac{\omega(1-r)}{1-r} + \frac{A(n+1)!}{2^{n/2}} \frac{\pi^{n+1}}{1-r} \int_{1-r}^{\pi} \frac{\omega(t)}{t^2} dt.$$

Now

$$\int_{1-r}^{\pi} \frac{\omega(t)}{t^2} dt \geq \omega(1-r) \left[ \frac{1}{1-r} - \frac{1}{\pi} \right] > \frac{1}{2} \frac{\omega(1-r)}{1-r}$$

so that we may choose  $M$  depending only on  $A$  and on  $n$  such that

$$|f^{(n+1)}(r)| \leq M \int_{1-r}^{\pi} \frac{\omega(t)}{t^2} dt \quad \text{for } r \geq \frac{1}{2}.$$

In the case  $n = 0$ ,  $\omega(t) = t^\alpha$   $0 < \alpha < 1$  we have here a result of Hardy and Littlewood (see [2] p. 360-366): If the conditions on  $u$  and  $f$  are satisfied and if  $|u(e^{it})| \leq A|t|^\alpha$ ,  $0 < \alpha \leq 1$ ,  $|t| < \pi$  then there exists a constant  $M$  depending on  $A$  such that for  $r \geq 1/2$ ,

$$|f'(r)| \leq \frac{M}{(1-r)^{1-\alpha}} \quad \text{if } 0 < \alpha < 1,$$

and

$$|f'(r)| \leq M \log \frac{\pi}{1-r} \quad \text{if } \alpha = 1.$$

For our study of the higher derivatives it is useful to extend Lemma 1.

LEMMA 2. Suppose that  $f(z) = u(re^{it}) + iu^*(re^{it})$  satisfies the hypotheses of Lemma 1 and that for  $n \geq 0$

$$(2) \quad u(e^{it}) = \sum_{i=0}^n a_i t^i + O(|t|^n \omega(|t|)) \quad \text{for } |t| \leq \pi$$

where  $\omega(t)$  is nondecreasing, nonnegative and  $t = O(\omega(|t|))$ . Then there exists a constant  $M$  depending only on  $n$ , on the  $\{a_i\}$  and on the constant in the  $O(|t|^n \omega(|t|))$  term such that for  $r \geq 1/2$ ,

$$|f^{(n+1)}(r)| \leq M \int_{1-r}^{\pi} \frac{\omega(t)}{t^2} dt.$$

*Proof.* Let

$$\begin{aligned} p_k(t) &= \operatorname{Re} \frac{(e^{it} - 1)^k}{i^k} = \operatorname{Re} \left[ \frac{i^k t^k}{i^k} + \frac{k}{2} \frac{i^{k+1} t^{k+1}}{i^k} + \dots \right] \\ &= \sum_{j=k}^n a_{jk} t^j + O(|t|^{n+1}) \quad a_{kk} = 1 \quad 0 \leq k \leq n. \end{aligned}$$

Then consider

$$(3) \quad \sum_{k=0}^n x_k p_k(t) = \sum_{k=0}^n x_k \left[ \sum_{j=k}^n a_{jk} t^j + O(|t|^{n+1}) \right]$$

where the real constants  $x_k$  are chosen so that

$$\sum_{k=0}^n x_k \left( \sum_{j=k}^n a_{jk} t^j \right) = \sum_{j=0}^n a_j t^j;$$

this may be done as these  $x_k$  are the solutions of the equation

$$\begin{pmatrix} a_{00} & 0 & 0 & \dots & 0 \\ a_{10} & a_{11} & 0 & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ a_{n0} & a_{n1} & & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

We then set

$$p(z) = \sum_{k=0}^n x_k \frac{(z-1)^k}{i^k}.$$

Now let  $g(z) = f(z) - p(z)$ . Then  $g$  is holomorphic for  $|z| < 1$ , continuous for  $|z| \leq 1$ ,  $g^{(n+1)}(z) \equiv f^{(n+1)}(z)$  and

$$\begin{aligned}
 (4) \quad |\operatorname{Re} g(e^{it})| &= |\operatorname{Re} [f(e^{it}) - p(e^{it})]| \\
 &= \left| u(e^{it}) - \sum_{k=0}^n x_k p_k(t) \right| \\
 &= O(|t|^n \omega(|t|)) + O(|t|^{n+1}) = O(|t|^n \omega(|t|))
 \end{aligned}$$

since  $t = O(\omega(|t|))$ . Thus by Lemma 1

$$|f^{(n+1)}(r)| = |g^{(n+1)}(r)| \leq M \int_{1-r}^{\pi} \frac{\omega(t)}{t^2} dt$$

where the constant  $M$  depends only on the constant in the  $O$ -term in (4). Now note that the  $\{a_{jk}\}$  are totally independent of the function  $u$ , so the  $\{x_i\}$  are dependent only on the  $\{a_i\}$ . The  $\{x_i\}$  affect the constant in the  $O(t^n \omega(|t|))$  term in (4) via (3) so that the constant in (4) depends only on the  $\{a_i\}$  and the  $O(|t|^n \omega(|t|))$  term in (2). Thus the value of  $M$  depends only on these constants.

**COROLLARY.** *If the conditions of Lemma 2 are satisfied and if  $\int_0^{\pi} (\omega(t)/t) dt < \infty$ , then there exists a constant  $A$  dependent only on the  $\{a_i\}$ ,  $\omega(t)$ ,  $n$ , and the constant in the  $O$  term in (2), such that for  $r \geq 1/2$*

$$|f^{(n)}(r)| \leq A.$$

*Proof.* Let  $A_1$  be the constant in the  $O$  term in (4). Then as in the proof of Lemma 1,

$$\begin{aligned}
 |f^{(n)}(r) - p^{(n)}(r)| &\leq \frac{n! A_1}{\pi} \int_0^{\pi} \frac{t^n \omega(t)}{\left(\frac{4rt^2}{\pi^2}\right)^{(n+1)/2}} dt \\
 &\leq \frac{n! A_1 \pi^n}{2^{(n+1)/2}} \int_0^{\pi} \frac{\omega(t)}{t} d\theta = A_2
 \end{aligned}$$

so that

$$|f^{(n)}(r)| \leq A_2 + |p^{(n)}(r)|.$$

But  $p^{(n)}(r) = n! x_n$  and  $x_n$  depends on the  $\{a_i\}$  so

$$|f^{(n)}(r)| \leq A_2 + n! x_n = A.$$

**LEMMA 3.** *Suppose  $f(z)$  is holomorphic in  $|z| < 1$  and  $f'(z)$  satisfies the condition*

$$(5) \quad |f'(re^{i\theta})| \leq M \int_{1-r}^{\pi} \frac{\omega(t)}{t^2} dt$$

for all  $|\theta| \leq \pi$  and for all  $0 < r < 1$ . Here  $M$  is a constant and  $\omega(t)$  is nondecreasing, nonnegative, bounded for  $0 \leq t \leq \pi$ , and  $\int_0^\pi (\omega(t)/t) dt < \infty$ .

Then,

(i)  $\lim_{r \rightarrow 1} f(re^{i\theta}) = f(e^{i\theta})$  exists and is finite for  $|\theta| \leq \pi$  and  $f(e^{i\theta})$  has the modulus of continuity

$$\omega^*(\theta) = 3M \left[ \int_0^\theta \frac{\omega(t)}{t} dt + \theta \int_\theta^\pi \frac{\omega(t)}{t^2} dt \right].$$

(ii)  $f(z)$  is continuous in  $|z| \leq 1$  and has modulus of continuity  $A\omega^*(\pi t)$  where  $A$  is a constant depending only on the function  $\omega^*(t)$ . That is, for  $|z_1|, |z_2| \leq 1$ ,

$$|f(z_2) - f(z_1)| \leq A\omega^*(\pi |z_2 - z_1|).$$

Here we define  $\omega^*(t) = \omega^*(\pi)$  for  $t \geq \pi$ .

For the proof of part (i) see [10], Lemma 4; the proof of part (ii) is patterned after that of the more special theorem in [2], page 363.

In the case  $\omega(t) = t^\alpha$ ,  $0 < \alpha < 1$  this is another result of Hardy and Littlewood ([2] Pages 360-366):

If  $f$  is as in Lemma 3 and if  $|f'(re^{i\theta})| \leq M/(1-r)^{1-\alpha}$  for all  $|\theta| \leq \pi$  then  $f(e^{i\theta}) \in \text{Lip}(\alpha)$  for  $|\theta| \leq \pi$ . If  $\omega(t) = t$  then  $|f'(re^{i\theta})| \leq M \log(\pi/(1-r))$  and the conclusion is that  $f(e^{i\theta})$  has modulus of continuity  $\omega^*(t) = 3Mt \log(3\pi/t)$ .

We note that a result analogous to Lemma 3 can be obtained if (5) is satisfied for a subarc  $\theta_1 \leq \theta \leq \theta_2$  of  $|z| = 1$  for  $0 < r < 1$ . Then  $f(e^{i\theta})$  has modulus of continuity  $\omega^*(t)$  on this arc and  $f(z)$  has modulus of continuity  $A\omega^*(\pi t)$  in the sector  $\theta_1 \leq \theta \leq \theta_2$ ,  $0 \leq r \leq 1$ ,  $A$  depending on  $\omega^*$ . Thus it will be evident that our theorems will hold for subarcs of  $\Gamma$ .

The first link between the geometry of  $\Gamma$  and the function  $F$  is given by the following Lemma, (see [8] pp. 615-17 and [6] p. 238).

LEMMA 4. Suppose  $\Gamma$  is a closed Jordan curve in  $R^3$  and  $F(z)$  is a solution to Plateau's problem for  $\Gamma$ . For two points  $p_1, p_2 \in \Gamma$ , let  $\Delta s(p_1, p_2)$  denote the length of the shorter arc between  $p_1$  and  $p_2$ . Suppose there exist constants  $c > 1$  and  $\delta > 0$  such that  $\Delta s(p_1, p_2)/|\overline{p_1 p_2}| < c$  for  $\Delta s(p_1, p_2) < \delta$ . Then there exist constants  $K > 0$ ,  $\delta_1 > 0$ , depending on  $\Gamma$  only, such that for  $|\theta - \theta_0| < \delta_1$

$$|F(e^{i\theta}) - F(e^{i\theta_0})| \leq |s(\theta) - s(\theta_0)| \leq K|\theta - \theta_0|^\beta$$

where  $s(\theta)$  for  $|\theta| \leq \pi$  is arclength along  $\Gamma$  and where  $\beta = 2/(1+c)^2$  so that  $0 < \beta < 1/2$ .



*Proof.* Let  $D[F] = 1/2 \iint_{|z| < 1} (F_x^2 + F_y^2) dx dy$ , the Dirichlet integral of  $F$ .

If there exists a constant  $B$  such that for each solution  $F$  to Plateau's problem,  $D[F] \leq B$ , then Lemma 3.2 of [1] implies that the family of solutions is equicontinuous. Since  $x$  and  $y$  are isothermal coordinates  $D[F] = A[F]$ , the area of the minimal surface, and by the isoperimetric inequality for minimal surfaces,  $A[F] \leq L^2/4\pi$  where  $L$  is the length of  $\Gamma$ . Thus  $D[F] \leq L^2/4\pi = B$  for all minimal surfaces spanning  $\Gamma$  which satisfy the three point condition and, as the modulus of continuity of the vectors  $\{F(e^{i\theta})\}$  depends only on  $B$ , it depends only on  $\Gamma$ . Thus the family of arclength functions  $\{s(\theta)\}$  associated with the minimal surfaces has a uniform modulus of continuity which depends only on  $\Gamma$ .

Let  $D$  be the diameter of  $\Gamma$  and let  $\delta' > 0$  be such that  $|\theta - \theta'| < \delta'$  implies  $|s(\theta) - s(\theta')| < \min(\delta, D/2)$  for all minimal surface spanning  $\Gamma$ .

Let  $k_\rho = \{z : |z - e^{i\theta_0}| = \rho, |z| < 1\}$  where  $\rho < \min(\delta'/4, 1)$  and let  $e^{i\theta_2}$  and  $e^{i\theta_1}$  be the endpoints of  $k_\rho$  which are on  $|z| = 1$ . Then  $|\theta_2 - \theta_1| < \delta'$  so  $|s(\theta_2) - s(\theta_1)| < \min(\delta, D/2)$ . Thus  $F(e^{i\theta_0})$  must be on the shorter arc between  $F(e^{i\theta_2})$  and  $F(e^{i\theta_1})$ . This is true for all solutions to the Plateau problem for  $\Gamma$ .

Now let  $l_\rho = \text{length of } F(k_\rho)$ . Then, for  $z_0 = e^{i\theta_0}$

$$l_\rho = \int_{k_\rho} |F_\varphi(z_0 + \rho e^{i\varphi})| d\varphi$$

and by Schwarz's inequality

$$l_\rho^2 \leq \pi \int_{k_\rho} |F_\varphi(z_0 + \rho e^{i\varphi})|^2 d\varphi$$

so that

$$\frac{l_\rho^2}{\rho} \leq \pi \int_{k_\rho} \frac{1}{\rho^2} |F_\varphi(z_0 + \rho e^{i\varphi})|^2 \rho d\varphi.$$

Since  $F$  is a minimal surface  $1/\rho^2 \cdot F_\varphi^2 = F_\rho^2$  so that  $1/\rho^2 \cdot F_\varphi^2 = 1/2(F_\rho^2 + 1/\rho^2 \cdot F_\varphi^2)$  and thus

$$\int_0^r \frac{l_\rho^2}{\rho} d\rho \leq \frac{\pi}{2} \int_0^r \int_{k_\rho} \left( F_\rho^2 + \frac{1}{\rho^2} F_\varphi^2 \right) \rho d\varphi d\rho.$$

Letting  $\mathcal{A}_r = F(\{z : |z - e^{i\theta_0}| \leq r, |z| < 1\})$  and  $A(r) = \text{area of } \mathcal{A}_r$ , we have

$$\mathcal{F}(r) := \int_0^r \frac{l_\rho^2}{\rho} d\rho \leq \pi A(r).$$

Let  $L$  denote the length of the boundary of  $\mathcal{A}_r$ . By the isoperimetric inequality  $A(r) \leq L^2/4\pi$ . By our first remarks letting  $p_1 = F(e^{i\theta_1})$

and  $p_2 = F(e^{i\theta_2})$ , we have

$$L = l_r + \Delta s(p_1 p_2) \leq l_r + c \overline{p_1 p_2} \leq (1 + c)l_r$$

so that

$$\mathcal{F}(r) \leq \frac{\pi L^2}{4\pi} = \frac{L^2}{4} \leq \frac{l_r^2(1 + c)^2}{4}.$$

Now  $\mathcal{F}'(r) = l_r^2/r$  a.e., so  $r\mathcal{F}'(r) = l_r^2$  and  $\mathcal{F}(r) \leq (1 + c)^2/4 \cdot r\mathcal{F}'(r)$ . Then for  $\rho < \rho_0 = \min(\delta'/4, 1)$

$$\frac{4}{(1 + c)^2} \int_\rho^{\rho_0} \frac{dr}{r} \leq \int_\rho^{\rho_0} \frac{\mathcal{F}'(r)}{\mathcal{F}(r)} dr$$

so that

$$\left(\frac{\rho_0}{\rho}\right)^{4/(1+c)^2} \leq \frac{\mathcal{F}(\rho_0)}{\mathcal{F}(\rho)}.$$

Choose  $M$  so that  $\mathcal{F}(\rho)/(\rho^{4/(1+c)^2}) \leq \mathcal{F}(\rho_0)/(\rho_0^{4/(1+c)^2}) = M - 1$ .  $M$  depends only on  $\Gamma$  since  $\mathcal{F}(\rho_0) \leq \pi A(\rho_0) \leq \pi A[F] \leq L^2/4\pi = B$  and  $\rho_0$  depends only on  $\delta'$ . Then  $\mathcal{F}(\rho) < M\rho^{4/(1+c)^2}$  so that

$$\int_{\rho/2}^\rho \frac{l_r^2}{r} dr \leq \int_0^\rho \frac{l_r^2}{r} dr < M\rho^{4/(1+c)^2}.$$

Now there exists a  $\rho_1$  with  $\rho/2 \leq \rho_1 \leq \rho$  such that

$$l_{\rho_1}^2 \int_{\rho/2}^\rho \frac{dr}{r} < M\rho^{4/(1+c)^2}$$

so that

$$l_{\rho_1}^2 \log 2 < M\rho^{4/(1+c)^2}$$

and thus

$$l_{\rho_1} < \sqrt{\frac{M}{\log 2}} \rho^{2/(1+c)^2} = \sqrt{\frac{M}{\log 2}} \rho^\beta.$$

Thus if  $|e^{i\theta} - e^{i\theta_0}| = \rho/2$  and if  $p_1 = F(e^{i\theta_1})$  and  $p_2 = F(e^{i\theta_2})$  are the endpoints of  $k_{\rho_1}$

$$\begin{aligned} |F(e^{i\theta}) - F(e^{i\theta_0})| &\leq |s(\theta) - s(\theta_0)| \leq c \overline{p_1 p_2} \\ &\leq c \sqrt{\frac{M}{\log 2}} \rho^\beta \leq c \sqrt{\frac{M}{\log 2}} 2^\beta |\theta - \theta_0|^\beta. \end{aligned}$$

Letting  $K = c \sqrt{\frac{M}{\log 2}} 2^\beta$  we have

$$|F(e^{i\theta}) - F(e^{i\theta_0})| \leq |s(\theta) - s(\theta_0)| \leq K |\theta - \theta_0|^\beta.$$

This is true for  $|\theta - \theta_0| < 1/3 \min(\delta'/4, 1) = \delta_1$ , for we may then choose  $\rho$  so that  $\rho = 2|e^{i\theta} - e^{i\theta_0}| < 2|\theta - \theta_0| < \rho_0 = \min(\delta'/4, 1)$ .

Since  $s(\theta)$  is bounded we may find a constant  $K_1$  such that  $|s(\theta) - s(\theta_0)| \leq K_1 |\theta - \theta_0|^\beta$  for all  $\theta, \theta_0 \in [-\pi, \pi]$ . It is in this form that we shall use Lemma 4. ( $K_1$  clearly depends on  $\Gamma$  only.)

For the hypothesis of Lemma 4 to hold, it is sufficient that  $\Gamma$  be continuously differentiable with respect to arclength. Then  $c$  may be taken as close to 1 as we like, so that  $\beta$  is as close to  $1/2$  as we like. The constant  $K_1$  will depend on  $c$ , but will be uniform for all solutions to the Plateau problem for  $\Gamma$ .

**3. The first derivative.** We first prove Theorem 1. From Lemma 4 we know that  $F(e^{i\theta}) \in \text{Lip}(\beta)$  for any  $0 < \beta < 1/2$ . Our first step is to improve the Hölder exponent by a "bootstrap" technique involving the Hardy-Littlewood forms of Lemmas 1 and 3.

**LEMMA 5.** *Suppose  $\Gamma$  is a smooth closed Jordan curve and  $F(z)$  is a minimal surface spanning  $\Gamma$ . Suppose  $F(1) = (0, 0, 0)$  and the tangent to  $\Gamma$  at  $F(1)$  is along the positive  $u$  axis. Let  $\mathcal{F}(s) = (U(s), V(s), W(s))$  be the parametrization of  $\Gamma$  with respect to arclength  $s$ . Let  $s(\theta) = s(F(e^{i\theta}))$  and  $s(0) = 0$ , so that  $\mathcal{F}(0) = F(1) = (0, 0, 0)$  and  $\mathcal{F}'(0)$  is along the positive  $u$  axis.*

*Suppose that  $\mathcal{F}(s) \in C^{1,\alpha}$  for some  $0 < \alpha \leq 1$  and that  $F(e^{i\theta}) \in \text{Lip}(\beta)$  for some  $\beta > 0$ , with Hölder constant  $K_\beta$ .*

*Then there exists a constant  $K$ , depending only on  $\Gamma$ ,  $K_\beta$ , and  $\beta$ , such that for  $|\theta| \leq \pi$*

$$|v(e^{i\theta})| \leq K |\theta|^{\beta(1+\alpha)} \quad |w(e^{i\theta})| \leq K |\theta|^{\beta(1+\alpha)}.$$

*Proof.* Since  $V(s) \in C^{1,\alpha}$  and  $V_s(0) = 0$  we have, for some constant  $K_0$

$$|V_s(s)| \leq K_0 |s|^\alpha.$$

Since  $V(0) = 0$  we integrate to obtain

$$(6) \quad |V(s)| \leq \frac{K_0}{1+\alpha} |s|^{1+\alpha}.$$

$F(\theta) \in \text{Lip}(\beta)$  implies that  $s(\theta) \in \text{Lip}(\beta)$  so that there exists  $K'_\beta$  (depending on  $K_\beta$  and  $\Gamma$ ) such that

$$(7) \quad |s(\theta)| \leq K'_\beta |\theta|^\beta;$$

combining (6) and (7) one obtains

$$|v(e^{i\theta})| = |V(s(\theta))| \leq \frac{K_0}{1+\alpha} (K'_\beta)^{1+\alpha} |\theta|^{\beta(1+\alpha)} = K |\theta|^{\beta(1+\alpha)}.$$

The proof for  $w(e^{i\theta})$  is analogous.

We now apply Lemma 5 to raise the Hölder exponent for  $F(e^{i\theta})$ .

**LEMMA 6.** *Suppose  $\Gamma$  is a closed Jordan curve and  $F(z)$  is a minimal surface spanning  $\Gamma$ . Suppose  $\Gamma \in C^{1,\alpha}$  for  $0 < \alpha \leq 1$  and*

that  $F(e^{i\theta}) \in \text{Lip}(\beta)$  with Hölder constant  $K_\beta$ , where  $\beta(1 + \alpha) < 1$ . Then  $(F(e^{i\theta})) \in \text{Lip}(\beta(1 + \alpha))$  with the Hölder constant depending only on  $K_\beta$  and  $\Gamma$ .

*Proof.* First assume that  $\Gamma, F$  are in the position of Lemma 5. Then  $|v(e^{i\theta})| \leq K|\theta|^{\beta(1+\alpha)}$  and  $|w(e^{i\theta})| \leq K|\theta|^{\beta(1+\alpha)}$ .

Consider now  $\mu(z) = v(z) + iw^*(z)$  and  $\nu(z) = w(z) + iw^*(z)$ . Then by Lemma 1 ( $n = 0$ ), there exists a constant  $M$  depending only on  $K$  such that for  $b = \beta(1 + \alpha)$

$$|\mu'(r)| \leq \frac{M}{(1-r)^{1-b}} \quad \text{and} \quad |\nu'(r)| \leq \frac{M}{(1-r)^{1-b}}.$$

Letting  $\lambda(z) = u(z) + iw^*(z)$  and applying (1) we have

$$|\lambda'(z)|^2 \leq |\mu'(z)|^2 + |\nu'(z)|^2$$

and hence

$$|\lambda'(r)| \leq \frac{\sqrt{2}M}{(1-r)^{1-b}}.$$

We would now like to apply Lemma 3 to conclude that  $\lambda, \mu, \nu \in \text{Lip}(\beta(1 + \alpha))$ .

For any  $F(e^{i\theta})$  on  $\Gamma$ , let  $(u^\theta, v^\theta, w^\theta)$  be a new coordinate system centered at  $F(e^{i\theta})$  and such that the  $u^\theta$  axis is tangent to  $\Gamma$  at  $F(e^{i\theta})$ . Then  $(u^\theta(z), v^\theta(z), w^\theta(z)) = F^\theta(z)$  is a minimal surface and by a rotation of the unit circle we may assume that  $F^\theta(1) = F(e^{i\theta})$ . It is clear that  $F^\theta(e^{i\theta}) \in \text{Lip}(\beta)$  with the same Hölder constant as  $F(e^{i\theta})$ . Thus  $\Gamma, F^\theta$  are as in Lemma 5, so that we may use the preceding argument to see that

$$|(\mu^\theta)'(r)| \leq \frac{M}{(1-r)^{1-b}} \quad \text{and} \quad |(\nu^\theta)'(r)| \leq \frac{M}{(1-r)^{1-b}}$$

where  $\mu^\theta(z), \nu^\theta(z), \lambda^\theta(z)$  are the analytic functions with real parts  $v^\theta(z), w^\theta(z)$  and  $u^\theta(z)$ , respectively and  $\mu^\theta(1) = \nu^\theta(1) = \lambda^\theta(1) = 0$  so that  $|(\lambda^\theta)'(r)| \leq \sqrt{2}M/(1-r)^{1-b}$ .

$M$  is dependent only on  $\Gamma, \beta$  and  $K_\beta$ . If  $(a_{ij}), 1 \leq i, j \leq 3$ , is the orthogonal matrix of the coordinate transformation, we have

$$(8) \quad \begin{cases} \lambda(re^{i\theta}) = a_{11}(\theta)\lambda^\theta(r) + a_{12}(\theta)\mu^\theta(r) + a_{13}(\theta)\nu^\theta(r) + \lambda(e^{i\theta}) \\ \mu(re^{i\theta}) = a_{21}(\theta)\lambda^\theta(r) + a_{22}(\theta)\mu^\theta(r) + a_{23}(\theta)\nu^\theta(r) + \mu(e^{i\theta}) \\ \nu(re^{i\theta}) = a_{31}(\theta)\lambda^\theta(r) + a_{32}(\theta)\mu^\theta(r) + a_{33}(\theta)\nu^\theta(r) + \nu(e^{i\theta}) \end{cases}$$

and therefore by the inequality of Schwarz and the orthogonality of the matrix  $(a_{ij})$

$$|\lambda'(re^{i\theta})| \leq \frac{2M}{(1-r)^{1-b}} \quad \text{for } |\theta| \leq 2\pi$$

and by Lemma 3,  $\lambda \in \text{Lip}(b)$ . The same holds for  $\mu$  and  $\nu$ , and the Hölder constant is as claimed.

**LEMMA 7.** *With  $\Gamma, F$  defined as in Lemma 5, there exists an  $\varepsilon > 0$  such that  $v(e^{i\theta}) = O(\theta^{1+\varepsilon})$ ,  $w(e^{i\theta}) = O(\theta^{1+\varepsilon})$  where the constant in  $O$  depends only on  $\Gamma$ .*

*Proof.* Choose  $0 < \beta < 1/2$  such that for all integers  $n$ ,  $(1 + \alpha)^n \neq 1/\beta$ . Then there exists an integer  $n$  such that  $(1 + \alpha)^n \beta = 1 + \varepsilon > 1$  but  $(1 + \alpha)^{n-1} \beta < 1$ . Apply Lemma 6  $n - 1$  times to obtain  $F(e^{i\theta}) \in \text{Lip}(\beta(1 + \alpha)^{n-1})$  and then apply Lemma 5 to see that there exists  $K$  constant such that  $|v(\theta)| \leq K|\theta|^{1+\varepsilon}$  and  $|w(\theta)| \leq K|\theta|^{1+\varepsilon}$ .

*Proof of Theorem 1.* First suppose  $\Gamma, F$  are as in Lemma 5. Then we claim  $\lim_{r \rightarrow 1} \mu'(r) = \mu'(1)$ ,  $\lim_{r \rightarrow 1} \nu'(r) = \nu'(1)$ ,  $\lim_{r \rightarrow 1} \lambda'(r) = \lambda'(1)$  all exist and are finite. By Lemma 7  $v(\theta) = O(\theta^{1+\varepsilon})$ , hence by Lemma 1  $|\mu'(r)| \leq M/(1 - r)^{1-\varepsilon}$ , for  $r \leq 1/2$ . Then for  $1/2 \leq r_1 < r_2 < 1$

$$\begin{aligned} |\mu'(r_2) - \mu'(r_1)| &= \left| \int_{r_1}^{r_2} \mu''(r) dr \right| \leq \int_{r_1}^{r_2} \frac{M}{(1 - r)^{1-\varepsilon}} dr \\ &\leq \frac{M}{\varepsilon} |r_2 - r_1|^\varepsilon \end{aligned}$$

so that  $\lim_{r \rightarrow 1} \mu'(r) = \mu'(1)$  exists and is finite. Likewise  $\lim_{r \rightarrow 1} \nu'(r) = \nu'(1)$  exists and is finite.

Since  $\lambda'^2(r) = -(\mu'^2(r) + \nu'^2(r))$ , we see  $\lim_{r \rightarrow 1} \lambda'(r) = \lambda'(1)$  exists and is finite.

From (8) it is clear that each of  $\lambda'(re^{i\theta})$ ,  $\mu'(re^{i\theta})$ ,  $\nu'(re^{i\theta})$  have radial limits for all  $|\theta| \leq \pi$  and the convergence is uniform for all  $\theta$ . Thus defining  $\lambda'(e^{i\theta}) = \lim_{r \rightarrow 1} \lambda'(re^{i\theta})$ , the function  $\lambda'(e^{i\theta})$  is continuous. This, together with the uniform convergences of  $\lambda'(re^{i\theta})$  to  $\lambda'(e^{i\theta})$  implies that  $\lambda'(z)$  is continuous for  $|z| \leq 1$ . From this it follows that  $\lambda(z)$  is differentiable at each  $e^{i\theta}$ , i.e.

$$\lim_{z \rightarrow e^{i\theta}} \frac{\lambda(z) - \lambda(e^{i\theta})}{z - e^{i\theta}} = \lambda'(e^{i\theta}).$$

The same facts are true for  $\mu'(z)$  and  $\nu'(z)$ .

Finally, recall that if  $\Gamma, F$  are as in Lemma 5 then there exist  $\varepsilon > 0$  and  $K > 0$  such that  $|v(e^{i\theta})| \leq K|\theta|^{1+\varepsilon}$  and  $|w(e^{i\theta})| \leq K|\theta|^{1+\varepsilon}$ , where  $K$  depends only on  $\Gamma$ .

Thus, by the corollary to Lemma 2 there exists a constant  $K_1$  such that  $|\mu'(1)| \leq K_1$  and  $|\nu'(1)| \leq K_1$ ; hence  $|\lambda'(1)| \leq \sqrt{2} K_1$ . By the equations (8) one sees that  $|\lambda'(e^{i\theta})|$ ,  $|\mu'(e^{i\theta})|$ ,  $|\nu'(e^{i\theta})|$  are bounded by  $2K_1$  for all  $\theta$ . Thus  $|s'(\theta)| \leq 2\sqrt{3} K_1 = c$  for  $|\theta| \leq \pi$ , and  $c$  is

the same for any solution to Plateau's problem for  $\Gamma$ .

We now prove a lemma preparatory to the proof of Theorem 2.

**LEMMA 8.** *Suppose  $\Gamma, F$  are positioned as in Lemma 5. Suppose also that  $\lambda', \mu', \nu'$  are continuous in  $|z| \leq 1$  and  $\Gamma \in C^{1, \omega(t)}$ . Let  $|s'(\theta)| \leq c, |\theta| \leq \pi$ , and let  $\omega_0(\theta) = \omega(c\theta)$ . Then*

$$|v(e^{i\theta})| \leq K |\theta \omega_0(|\theta|)|, |w(e^{i\theta})| \leq K |\theta \omega_0(|\theta|)|, |u^*(e^{i\theta})| \leq K |\theta \omega_0(|\theta|)|$$

for  $|\theta| \leq \pi$ , where the constant  $K$  depends only on  $c$  and  $\Gamma$ .

*Proof.* By the argument of Lemma 5 we have  $|V(s)| \leq |s| \omega(s)$  and since  $|s(\theta)| \leq c |\theta|, |v(e^{i\theta})| \leq c |\theta| \omega_0(|\theta|)$ ; likewise  $|w(e^{i\theta})| \leq c |\theta| \omega_0(|\theta|)$ .

By Lemma 4,  $U_s(s(\theta))$  is uniformly continuous for  $|\theta| \leq \pi$  and  $U_s(s(0)) = 1$ . Therefore there exists a  $\delta > 0$  (depending only on  $\Gamma$ ) such that  $|\theta| < \delta$  implies  $U_s(s(\theta)) > 1/2$ . Now  $ds(\theta)/d\theta \neq 0$  for almost every  $\theta$  and  $U_{s_{\theta}} = u_{\theta}$  and  $V_{s_{\theta}} = v_{\theta}$  so that

$$\frac{v_{\theta}(e^{i\theta})}{u_{\theta}(e^{i\theta})} = \frac{V_s(s(\theta))s_{\theta}(\theta)}{U_s(s(\theta))s_{\theta}(\theta)} = \frac{V_s(s(\theta))}{U_s(s(\theta))} \quad \text{a.e. } |\theta| < \delta.$$

But

$$\left| \frac{V_s(s)}{U_s(s)} \right| \leq 2\omega(|s|) \leq 2\omega_0(|\theta|)$$

so that

$$\left| \frac{v_{\theta}(e^{i\theta})}{u_{\theta}(e^{i\theta})} \right| \leq 2\omega_0(|\theta|) \quad \text{a.e. } |\theta| < \delta;$$

likewise

$$\left| \frac{w_{\theta}(e^{i\theta})}{u_{\theta}(e^{i\theta})} \right| \leq 2\omega_0(|\theta|) \quad \text{a.e. } |\theta| < \delta.$$

In polar coordinates the minimal surface condition implies that  $u_r u_{\theta} + v_r v_{\theta} + w_r w_{\theta} = 0$  and therefore

$$-u_{\theta}^* = -u_r = v_r \frac{v_{\theta}}{u_{\theta}} + w_r \frac{w_{\theta}}{u_{\theta}}$$

but  $|v_r(e^{i\theta})|$  and  $|w_r(e^{i\theta})|$  are both bounded by  $c$  for all  $\theta$  so that  $|u_{\theta}^*(e^{i\theta})| \leq 4c\omega_0(|\theta|)$  a.e.  $|\theta| < \delta$ . Taking  $u^*(e^{i\theta}) = 0$  we may integrate to obtain

$$|u^*(e^{i\theta})| \leq 4c |\theta| \omega_0(|\theta|) \quad |\theta| < \delta.$$

Since  $\delta$  was dependent only on  $\Gamma$  it is clear that  $K$  may be chosen to complete the proof of the lemma.

*Proof of Theorem 2.* Suppose first that  $\Gamma, F$  are as in Lemma 5. Then the conclusion of Lemma 8 holds. Applying Lemma 1 to  $-i\lambda(z)$ ,

for instance, we obtain

$$|\lambda''(r)| \leq M \int_{1-r}^{\pi} \frac{\omega_0(t)}{t^2} dt \quad \text{for } r \geq \frac{1}{2}$$

and analogous inequalities for  $|\mu''(r)|$  and  $|\nu''(r)|$ . Since  $M$  depends only on  $\Gamma$  we see by applying the transformation (8) that

$$|\lambda''(re^{i\theta})| \leq \sqrt{3} M \int_{1-r}^{\pi} \frac{\omega_0(t)}{t^2} dt \quad |\theta| \leq \pi.$$

Analogous inequalities hold for  $|\mu''(re^{i\theta})|$  and  $|\nu''(re^{i\theta})|$ . The conclusion of Theorem 2 then follows from Lemma 3.

4. **The higher derivatives.** In proving Theorem 3 for a given  $n \geq 2$ , the result for  $n-1$  is assumed, so that  $\Gamma \in C^{n, \omega(t)}$  implies  $\Gamma \in C^{n-1, 1}$  and thus  $s^{(n-1)}(\theta)$  has modulus of continuity  $kt \log 3\pi/t$ .

We shall make extensive use of the following fact: If  $f(x) \in C^{n, \omega(t)}$  for  $|x| \leq \delta$ , then

$$f(x) = \sum_{i=0}^n f^{(i)}(0) \frac{x^i}{i!} + O(|x|^n \omega(|x|)).$$

We now prove a lemma analogous to Lemma 8.

**LEMMA 9.** Suppose  $\Gamma \in C^{n, \omega(t)}$ ,  $n \geq 2$ , and that  $\Gamma, F$  are positioned as in Lemma 5. Suppose  $c \geq |s'(\theta)|$  for  $|\theta| \leq \pi$  and that  $\omega_0(\theta) = \omega(c\theta)$ . Such a  $c$  exists and is dependent only on  $\Gamma$  by Theorem 1. Then there exist constants  $\{b_i\}, \{c_i\}, \{a_i\}$   $2 \leq i \leq n$  such that

$$(9) \quad \begin{cases} v(e^{i\theta}) = \sum_{i=2}^n b_i \theta^i + O(|\theta|^n \omega_1(|\theta|)) \\ w(e^{i\theta}) = \sum_{i=2}^n c_i \theta^i + O(|\theta|^n \omega_1(|\theta|)) \\ u^*(e^{i\theta}) = \sum_{i=2}^n a_i \theta^i + O(|\theta|^n \omega_1(|\theta|)) \end{cases}$$

where  $\omega_1(|\theta|) = |\theta| \log 3\pi/|\theta| + \omega_0(|\theta|)$  and the constants in the  $O(|\theta|^n \omega_1(|\theta|))$  terms depend only on  $\Gamma$  and the constants  $\{a_i\}, \{b_i\}, \{c_i\}$  are uniformly bounded by a constant depending only on  $\Gamma$ .

*Proof.* We have

$$(10) \quad s(\theta) = \sum_{i=1}^{n-1} s^{(i)}(0) \frac{\theta^i}{i!} + O\left(|\theta|^n \log \frac{3\pi}{|\theta|}\right)$$

for  $|\theta| \leq \pi$ . By the induction hypothesis, there exists a constant  $K$  such that  $|s^{(i)}(\theta)| \leq K$  for  $1 \leq i \leq n-1$  and  $|\theta| \leq \pi$ , and such that

the constant in the  $O$  term is bounded by  $K$ . We also have

$$V(s) = \sum_{i=2}^n V^{(i)}(0) \frac{s^i}{i!} + O(|s|^n \omega(s))$$

so that

$$\begin{aligned} v(e^{i\theta}) = V(s(\theta)) &= \sum_{i=2}^n \frac{V^{(i)}(0)}{i!} \left[ \sum_{j=1}^{n-1} s^{(j)}(0) \frac{\theta^j}{j!} + O\left(|\theta|^n \log \frac{3\pi}{|\theta|}\right) \right]^i \\ &\quad + O\left( \left[ \sum_{j=1}^{n-1} s^{(j)}(0) \frac{\theta^j}{j!} + O\left(|\theta|^n \log \frac{3\pi}{|\theta|}\right) \right]^n \omega_0(|\theta|) \right) \\ &= \sum_{i=2}^n b_i \theta^i + O(|\theta|^n \omega_1(|\theta|)). \end{aligned}$$

The corresponding expression for  $w(e^{i\theta})$  is obtained similarly. Now, as in Lemma 8

$$-u_\theta^*(e^{i\theta}) = v_r(e^{i\theta}) \frac{v_\theta(e^{i\theta})}{u_\theta(e^{i\theta})} + w_r(e^{i\theta}) \frac{w_\theta(e^{i\theta})}{u_\theta(e^{i\theta})}$$

where  $v_\theta/u_\theta = V_s/U_s$  for  $|\theta| < \delta^1$ . But  $V_s(s)/U_s(s) \in C^{n-1,\omega}$  for  $|\theta| < \delta$  so that

$$\frac{V_s(s)}{U_s(s)} = \sum_{i=1}^{n-1} d_i s^i + O(|s|^{n-1} \omega(|s|)) \quad \text{for } |\theta| < \delta$$

and so using (10)

$$\frac{v_\theta(e^{i\theta})}{u_\theta(e^{i\theta})} = \sum_{i=1}^{n-1} f_i \theta^i + O(|\theta|^{n-1} \omega_1(|\theta|)).$$

Since  $\Gamma \in C^{n-1,1}$ ,  $v_r(e^{i\theta}) \in C^{n-2,\omega_2(t)}$  where  $\omega_2(t) = Kt(\log 3\pi/t)$ , so that

$$\begin{aligned} v_r(e^{i\theta}) \frac{v_\theta(e^{i\theta})}{u_\theta(e^{i\theta})} &= \left[ \sum_{i=0}^{n-2} g_i \theta^i + O\left(|\theta|^{n-1} \log \frac{3\pi}{|\theta|}\right) \right] \\ &\quad \cdot \left[ \sum_{i=1}^{n-1} f_i \theta^i + O(|\theta|^{n-1} \omega_1(|\theta|)) \right] \\ &= \sum_{i=1}^{n-1} h_i \theta^i + O(|\theta|^{n-1} \omega_1(|\theta|)). \end{aligned}$$

A similar expansion holds for  $w_r(e^{i\theta})w_\theta(e^{i\theta})/u_\theta(e^{i\theta})$  so that

$$u_j^*(e^{i\theta}) = \sum_{i=1}^{n-1} m_i \theta^i + O(|\theta|^{n-1} \omega_1(|\theta|)) \quad \text{for } |\theta| < \delta$$

and

$$u^*(e^{i\theta}) = \sum_{i=2}^n a_i \theta^i + O(|\theta|^n \omega_1(|\theta|)) \quad \text{for } |\theta| \leq \pi.$$

In each case the coefficients of the expansions and the constants in the  $O$  terms are bounded uniformly, the bound depending only on  $\Gamma$ .

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<sup>1</sup> At points  $\theta_0$  where  $ds/d\theta = 0$  we mean by  $v_\theta(e^{i\theta_0})/u_\theta(e^{i\theta_0})$  the limit as  $\theta \rightarrow \theta_0$ .



*Proof of Theorem 3.* Let us first suppose that  $\Gamma, F$  are as in Lemma 5. Then by Lemma 9, (9) holds. We may then apply lemma 2 to  $i\lambda(z), \mu(z)$  and  $\nu(z)$  to conclude that

$$|\lambda^{(n+1)}(r)| \leq M_n \int_{1-r}^{\pi} \frac{\omega_1(t)}{t^2} dt \quad (0 < r < 1)$$

$$|\mu^{(n+1)}(r)| \leq M_n \int_{1-r}^{\pi} \frac{\omega_1(t)}{t^2} dt$$

and

$$|\nu^{(n+1)}(r)| \leq M_n \int_{1-r}^{\pi} \frac{\omega_1(t)}{t^2} dt.$$

Since the constants involved in (9) are bounded by a constant depending only on  $\Gamma$ ,  $M_n$  depends only on  $\Gamma$ . Thus, for all  $|\theta| \leq \pi$  we have

$$|\lambda^{(n+1)}(re^{i\theta})| \leq \sqrt{3} M_n \int_{1-r}^{\pi} \frac{\omega_1(t)}{t^2} dt$$

and the corresponding inequalities obtain for  $\mu$  and  $\nu$ .

Part (i) of the theorem then follows from Lemma 3, with  $\omega_1$  rather than  $\omega_0$ .

Furthermore, by the corollary to Lemma 2, if  $\Gamma$  is positioned as in Lemma 5 then there exists a constant  $K$  depending only on  $\Gamma$ , such that  $|\lambda^{(m)}(1)| \leq K$ ,  $|\mu^{(m)}(1)| \leq K$  and  $|\nu^{(m)}(1)| \leq K$  for  $m = 1, 2, \dots, n$ . By the equations (8) one sees that  $|\lambda^{(m)}(e^{i\theta})|$ ,  $|\mu^{(m)}(e^{i\theta})|$ , and  $|\nu^{(m)}(e^{i\theta})|$  are bounded by  $\sqrt{3} K$  for all  $\theta$  and each  $m$ ,  $1 \leq m \leq n$ . From this it follows that  $|s^{(n)}(\theta)|$  is bounded for all  $\theta$  by a constant  $c_n$  depending only on  $\Gamma$ .

We may now see that Lemma 9 and Theorem 3 are true with  $\omega_0(|\theta|)$  in place of  $\omega_1(\theta)$ .

Since  $s^{(n)}(\theta)$  is continuous and bounded,  $s(\theta) \in C^{n-1,1}$  i.e.,

$$(11) \quad s(\theta) = \sum_{i=1}^{n-1} s^{(i)}(0) \frac{\theta^i}{i!} + O(|\theta|^{n+1})$$

where the coefficients and the constant in the  $O$  term are bounded by some constant  $K$ . Then, using (11) instead of (10) in the proof of Lemma 9, we obtain (9) with  $\omega_0(|\theta|)$  instead of  $\omega_1(|\theta|)$ . Then Theorem 3 may be proved with  $\omega_0(|\theta|)$  instead of  $\omega_1(|\theta|)$ .

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# Pacific Journal of Mathematics

Vol. 37, No. 1

January, 1971

Gregory Frank Bachelis and Haskell Paul Rosenthal, <i>On unconditionally converging series and biorthogonal systems in a Banach space</i> .....	1
Richard William Beals, <i>On spectral theory and scattering for elliptic operators with singular potentials</i> .....	7
J. Lennart (John) Berggren, <i>Solvable and supersolvable groups in which every element is conjugate to its inverse</i> .....	21
Lindsay Nathan Childs, <i>On covering spaces and Galois extensions</i> .....	29
William Jay Davis, David William Dean and Ivan Singer, <i>Multipliers and unconditional convergence of biorthogonal expansions</i> .....	35
Leroy John Derr, <i>Triangular matrices with the isoclinal property</i> .....	41
Paul Erdős, Robert James McEliece and Herbert Taylor, <i>Ramsey bounds for graph products</i> .....	45
Edward Graham Evans, Jr., <i>On epimorphisms to finitely generated modules</i> .....	47
Hector O. Fattorini, <i>The abstract Goursat problem</i> .....	51
Robert Dutton Fray and David Paul Roselle, <i>Weighted lattice paths</i> .....	85
Thomas L. Goulding and Augusto H. Ortiz, <i>Structure of semiprime <math>(p, q)</math> radicals</i> .....	97
E. W. Johnson and J. P. Lediaev, <i>Structure of Noether lattices with join-principal maximal elements</i> .....	101
David Samuel Kinderlehrer, <i>The regularity of minimal surfaces defined over slit domains</i> .....	109
Alistair H. Lachlan, <i>The transcendental rank of a theory</i> .....	119
Frank David Lesley, <i>Differentiability of minimal surfaces at the boundary</i> ....	123
Wolfgang Liebert, <i>Characterization of the endomorphism rings of divisible torsion modules and reduced complete torsion-free modules over complete discrete valuation rings</i> .....	141
Lawrence Carlton Moore, <i>Strictly increasing Riesz norms</i> .....	171
Raymond Moos Redheffer, <i>An inequality for the Hilbert transform</i> .....	181
James Ted Rogers Jr., <i>Mapping solenoids onto strongly self-entwined, circle-like continua</i> .....	213
Sherman K. Stein, <i>B-sets and planar maps</i> .....	217
Darrell R. Turnidge, <i>Torsion theories and rings of quotients of Morita equivalent rings</i> .....	225
Fred Ustina, <i>The Hausdorff means of double Fourier series and the principle of localization</i> .....	235
Stanley Joseph Wertheimer, <i>Quasi-compactness and decompositions for arbitrary relations</i> .....	253
Howard Henry Wicke and John Mays Worrell Jr., <i>On the open continuous images of paracompact Čech complete spaces</i> .....	265