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# DIFFERENTIABILITY OF MINIMAL SURFACES AT THE BOUNDARY

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**Let $\Gamma$ be a Jordan curve in $R^3$ and $F(z) = (u(z), v(z), w(z))$: $\{|z| \leqq 1\} \rightarrow R^3$ be a solution of Plateau's problem for $\Gamma$, where $z = x + iy$ are isothermal parameters. Then $u, v, w$ are harmonic in $\{|z| < 1\}$ and are the real parts of analytic functions $\lambda, \mu, \nu$. Using the Poisson integral and the defining properties of minimal surfaces, Kellogg's theorem for conformal mapping is generalized by proving: 1. If $\Gamma \in C^{1,\alpha}$, $0 < \alpha < 1$, then $\lambda, \mu, \nu \in C^{1,\alpha}$ for $|z| \leqq 1$ and if $\Gamma \in {}^{1,1}$ then $\lambda', \mu', \nu'$ have modulus of continuity $Kt \log 1/t$ for $|z| \leqq 1$; $K$ and the Hölder constants depend only on the geometry of $\Gamma$. 2. If $\Gamma \in C^{n,\omega(t)}$, $n \geqq 2$, where $\omega(t)$ is a modulus of continuity satisfying a Dini condition, then $\lambda, \mu, \nu \in C^{n,\omega^*(t)}$ for $|z| \leqq 1$, where $\omega^*(t)$ is a certain modulus of continuity. Once again $\omega^*$ depends only on $\Gamma$.**

Let $\Gamma$ be a closed Jordan curve in $R^3$. Then $S$ is called a generalized minimal surface spanning $\Gamma$ if $S$ is represented by a triple of real valued functions

$$F(z) = (u(z), v(z), w(z)) : \{|z| \leqq 1\} \rightarrow R^3 \quad (z = x + iy = re^{i\theta})$$

such that

(a) $u, v, w$ are harmonic in $|z| < 1$ and continuous in $|z| \leqq 1$

(b) $x$ and $y$ are isothermal parameters in $z < 1$, i.e.,

$$F_x^2 = u_x^2 + v_x^2 + w_x^2 = u_y^2 + v_y^2 + w_y^2 = F_y^2$$
$$F_x \cdot F_y = u_x u_y + v_x v_y + w_x w_y = 0 \quad \text{for} \quad |z| < 1$$

(c) $F(e^{i\theta})$ is a homeomorphism of $|z| = 1$ with $\Gamma$.

A solution to Plateau's problem for $\Gamma$ is a generalized minimal surface spanning $\Gamma$, and a solution may be normalized by specifying that three fixed points on $|z| = 1$ correspond to three fixed points on $\Gamma$. We shall consider the solutions to be normalized, and we note that there may be more than one normalized surface spanning a given curve $\Gamma$.

Consider the analytic functions of which $u, v, w$ are the real parts:

$$\lambda(z) = u(z) + iu^*(z) \quad \mu(z) = v(z) + iv^*(z) \quad \nu(z) = w(z) + iw^*(z) .$$

Then the condition (b) is equivalent to

$$(1) \qquad \lambda'^2(z) + \mu'^2(z) + \nu'^2(z) = 0 \quad |z| < 1 .$$

This paper will deal with the differentiability of $\lambda, \mu, \nu$ at the boundary $|z| = 1$, under given smoothness conditions on the curve $\Gamma$.

It was noted by Weierstrass that if the boundary $\Gamma$ of a minimal surface $S$ contains a straight line segment $\alpha$, then the surface may be extended analytically as a minimal surface across $\alpha$, by use of the reflection principle. In 1951 H. Lewy [5] proved that if $\alpha$ is an analytic arc then the surface can be extended analytically across $\alpha$.

For an up-to-date account of the studies on the boundary behavior of minimal surfaces see the recent paper of J. C. C. Nitsche [7]. In that paper Nitsche proved among other results that if $\Gamma \in C^{n,\alpha}$ for $n \geqq 1$ and $0 < \alpha < 1$, then $F(z) \in C^{n,\alpha}$ in $|z| \leqq 1$ and the Hölder constant for the $n$th derivatives of $F(z)$ is the same for all solutions of Plateau's problem, i.e., they depend only on the geometrical properties of $\Gamma$. In this connection see also [4], where a completely different proof of the first part of Nitsche's theorem is given.

In the following we shall say that a function $f(z) \in C^{n,\omega(t)}$ for $z$ in some domain if $f^{(n)}$ exists and has modulus of continuity $\omega(t)$, i.e.,

$$|f^{(n)}(t_1) - f^{(n)}(t_2)| \leqq \omega(|t_1 - t_2|) \qquad \text{for} \quad |t_1 - t_2| < \sigma ,$$

where $\omega(t)$ is a nondecreasing, non-negative function for $0 \leqq t \leqq \sigma$ and $\int_0^\sigma (\omega(t)/t)dt < \infty$. We shall assume, as we may without loss of generality, that $t = O(\omega(|t|))$ as $t \to 0$. In the following $O(\varphi(t))$ shall mean $O(\varphi(t))$ as $t \to 0$. Note that if $\omega(t) = kt^\alpha$, $0 < \alpha < 1$, $k$ a constant, then $f(t) \in C^{n,\alpha}$. We shall denote by $s(\theta) = s(F(e^{i\theta}))$ the arclength along $\Gamma$ with $s(0) = 0$. Our principal results are the following.

THEOREM 1. *If $\Gamma \in C^{1,\alpha}$, $0 < \alpha \leqq 1$ then each of $\lambda, \mu, \nu$ is continuously differentiable in $|z| \leqq 1$. In addition, there exists a constant $c$ such that $|s'(\theta)| \leqq c$, $-\pi \leqq \theta \leqq \pi$, where $c$ is dependent only on $\Gamma$.*

THEOREM 2. *Suppose $\Gamma \in C^{1,\omega(t)}$ and $\lambda, \mu, \nu$ are continuously differentiable for $|z| \leqq 1$. Let $c$ be a constant such that* $\max_{|\theta| \leqq \pi} |s'(\theta)| \leqq c$ *and let $\omega_0(t) = \omega(ct)$. Then there exist constants $K$ and $K_1$ depending on $c$ and on $\omega(t)$, such that $\lambda'(e^{i\theta}), \mu'(e^{i\theta}), \nu'(e^{i\theta})$ have modulus of continuity*

$$\omega_0^*(\theta) = K\left(\int_0^\theta \frac{\omega_0(t)}{t} dt + \theta \int_\theta^\pi \frac{\omega_0(t)}{t^2} dt\right)$$

and *$\lambda'(z), \mu'(z), \nu'(z)$ have modulus of continuity $K_1\omega_0^*(\pi t)$* for $|z| \leqq 1$.

Combining Theorems 1 and 2 we obtain: *If $\Gamma \in C^{1,\alpha}$, $0 < \alpha < 1$ then $\lambda, \mu, \nu \in C^{1,\alpha}$ for $|z| \leqq 1$. If $\Gamma \in C^{1,1}$ then $\lambda, \mu, \nu \in C^{1,\omega^*(t)}$ for $\omega^*(t) = Kt \log 3\pi/t$ for some constant $K$. Furthermore there exists a constant $c$ such that $|s'(\theta)| \leqq c$ for all $|\theta| \leqq \pi$. $K$ and $c$ depend on*

*$\Gamma$ only.*

THEOREM 3. *Suppose that $\Gamma \in {}^{n,\omega(t)}$, $n \geqq 2$. Let $c$ be a constant such that $|s'(\theta)| \leqq c$, $|\theta| \leqq \pi$, and let $\omega_0(t) = \omega(ct)$ (such a constant $c$ which depends only on $\Gamma$ exists by Theorem* 1). Then:

(i) *$\lambda^{(n)}$, $\mu^{(n)}$, $\nu^{(n)}$ have continuous extensions to $|z| = 1$ and there exist constants $K$ and $K_1$, depending only on $\Gamma$ such that $\lambda^{(n)}(e^{i\theta})$, $\mu^{(n)}(e^{i\theta})$, $\nu^{(n)}(e^{i\theta})$ have modulus of continuity*

$$\omega_0^*(\theta) = K\left[\int_0^\theta \frac{\omega_0(t)}{t}dt + \theta\int_\theta^\pi \frac{\omega_0(t)}{t^2}dt\right]$$

and *$\lambda^{(n)}(z)$, $\mu^{(n)}(z)$, $\nu^{(n)}(z)$ have modulus of continuity $K_1\omega_0^*(\pi t)$ for $|z| \leqq 1$.*

(ii) *There exists a constant $c_n$ depending only on $\Gamma$, $n$ such that $|s^{(n)}(\theta)| \leqq c_n$ for $|\theta| \leqq \pi$.*

Conformal mappings in the plane are special cases of minimal surfaces and in the conformal mapping case the result for $\omega(t) = Kt^\alpha$, $0 < \alpha < 1$ is due to O. D. Kellogg. The extension of Kellogg's theorem to a modulus of continuity satisfying a Dini condition $\int_0^\sigma (\omega(t)/t)dt < \infty$, was given by S. E. Warschawski [8] for $n = 1$ (for $n > 1$ see [9]).

The case $\Gamma \in C^{1,\omega(t)}$, i.e., the proof of Theorem 3 for $n = 1$, does not seem to lend itself to the method we use in establishing our Theorem 1. However, Warschawski [10] has recently given a proof of this case along different lines.

We note that our results overlap to some extent with those of Nitsche [7]. They were obtained independently, although a basic device used in the proof of Theorem 1 (Lemmas 5 and 6) is the same. However, there are differences both in approach and in detail between the two proofs.

The results hold for minimal surfaces in $n$-space, in which case we have $n$ harmonic and $n$ analytic functions. Also, it will be apparent that the theorems are local in the sense that they are true for subarcs of $\Gamma$.

## 2. Auxiliary Results.

In the following we shall need a number of lemmas.

LEMMA 1. *Suppose that the function $f(z) = u(re^{it}) + iu^*(re^{it})$ is holomorphic in $|z| < 1$ and $u(re^{it})$ is continuous in $|z| \leqq 1$. Suppose also that for some integer $n \geqq 0$*

$$|u(e^{it})| \leqq A\,|t|^n\omega(|t|) \qquad \text{for } |t| \leqq \pi$$

*where $A$ is a constant and $\omega(t)$ is nondecreasing and nonnegative.*

*Then there exists a constant $M$, depending only on $A$ and on $n$, such that for $r \geqq 1/2$,*

$$|f^{(n+1)}(r)| \leqq M \int_{1-r}^{\pi} \frac{\omega(t)}{t^2} dt .$$

*Proof.* We begin with the Poisson Integral for $f$:

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{it}) \frac{e^{it} + z}{e^{it} - z} dt + iu^*(0) \quad |z| < 1 .$$

Differentiating, we obtain

$$f^{(n+1)}(z) = \frac{(n+1)!}{\pi} \int_{-\pi}^{\pi} \frac{u(e^{it})e^{it}}{(e^{it} - z)^{n+2}} dt$$

and in particular

$$\begin{aligned} |f^{(n+1)}(r)| &\leqq \frac{2A(n+1)!}{\pi} \int_0^{\pi} \frac{t^n \omega(t)}{[1 - 2r\cos t + r^2]^{n/2+1}} dt \\ &\leqq \frac{2A(n+1)!}{\pi} \int_0^{\pi} \frac{t^n \omega(t)}{\left[(1-r)^2 + 4r\frac{t^2}{\pi^2}\right]^{n/2+1}} dt \\ &\leqq \frac{2A(n+1)!}{\pi} \left[ \int_0^{1-r} \frac{t^n \omega(t)}{(1-r)^{n+2}} dt + \int_{1-r}^{\pi} \frac{t^n \omega(t)}{\left[4r\frac{t^2}{\pi^2}\right]^{n/2+1}} dt \right] \\ &\leqq \frac{2A(n+1)!}{\pi} \left[ \frac{\omega(1-r)}{(1-r)^{n+2}} \int_0^{1-r} t^n dt + \frac{\pi^{n+2}}{2^{n/2+1}} \int_{1-r}^{\pi} \frac{t^n \omega(t)}{t^{n+2}} dt \right] \end{aligned}$$

for $r \geqq 1/2$,

$$\leqq \frac{2An!}{\pi} \frac{\omega(1-r)}{1-r} + \frac{A(n+1)!\,\pi^{n+1}}{2^{n/2}} \int_{1-r}^{\pi} \frac{\omega(t)}{t^2} dt .$$

Now

$$\int_{1-r}^{\pi} \frac{\omega(t)}{t^2} dt \geqq \omega(1-r) \left[ \frac{1}{1-r} - \frac{1}{\pi} \right] > \frac{1}{2} \frac{\omega(1-r)}{1-r}$$

so that we may choose $M$ depending only on $A$ and on $n$ such that

$$|f^{(n+1)}(r)| \leqq M \int_{1-r}^{\pi} \frac{\omega(t)}{t^2} dt \quad \text{for} \quad r \geqq \frac{1}{2} .$$

In the case $n = 0$, $\omega(t) = t^\alpha$ $0 < \alpha < 1$ we have here a result of Hardy and Littlewood (see [2] p. 360–366): If the conditions on $u$ and $f$ are satisfied and if $|u(e^{it})| \leqq A|t|^\alpha$, $0 < \alpha \leqq 1$, $|t| < \pi$ then there exists a constant $M$ depending on $A$ such that for $r \geqq 1/2$,

$$|f'(r)| \leqq \frac{M}{(1-r)^{1-\alpha}} \qquad \text{if} \quad 0 < \alpha < 1 ,$$

and

$$|f'(r)| \leqq M \log \frac{\pi}{1-r} \qquad \text{if} \quad \alpha = 1 .$$

For our study of the higher derivatives it is useful to extend Lemma 1.

LEMMA 2. *Suppose that* $f(z) = u(re^{it}) + iu^*(re^{it})$ *satisfies the hypotheses of Lemma* 1 *and that for* $n \geqq 0$

$$(2) \qquad u(e^{it}) = \sum_{i=0}^{n} a_i t^i + O(|t|^n \omega(|t|)) \quad \textit{for} \ |t| \leqq \pi$$

*where* $\omega(t)$ *is nondecreasing, nonnegative and* $t = O(\omega(|t|))$. *Then there exists a constant* $M$ *depending only on* $n$, *on the* $\{a_i\}$ *and on the constant in the* $O(|t|^n \omega(|t|))$ *term such that for* $r \geqq 1/2$,

$$|f^{(n+1)}(r)| \leqq M \int_{1-r}^{\pi} \frac{\omega(t)}{t^2} dt .$$

*Proof.* Let

$$p_k(t) = \operatorname{Re} \frac{(e^{it}-1)^k}{i^k} = \operatorname{Re}\left[\frac{i^k t^k}{i^k} + \frac{k}{2}\frac{i^{k+1}t^{k+1}}{i^k} + \cdots\right]$$
$$= \sum_{j=k}^{n} a_{jk} t^j + O(|t|^{n+1}) \quad a_{kk} = 1 \quad 0 \leqq k \leqq n .$$

Then consider

$$(3) \qquad \sum_{k=0}^{n} x_k p_k(t) = \sum_{k=0}^{n} x_k \left[\sum_{j=k}^{n} a_{jk} t^j + O(|t|^{n+1})\right]$$

where the real constants $x_k$ are chosen so that

$$\sum_{k=0}^{n} x_k \left(\sum_{j=k}^{n} a_{jk} t^j\right) = \sum_{j=0}^{n} a_j t^j ;$$

this may be done as these $x_k$ are the solutions of the equation

$$\begin{pmatrix} a_{00} & 0 & 0 & \cdots & 0 \\ a_{10} & a_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ a_{n0} & a_{n1} & & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} .$$

We then set

$$p(z) = \sum_{k=0}^{n} x_k \frac{(z-1)^k}{i^k} .$$

Now let $g(z) = f(z) - p(z)$. Then $g$ is holomorphic for $|z| < 1$, continuous for $|z| \leqq 1$, $g^{(n+1)}(z) \equiv f^{(n+1)}(z)$ and

$$\begin{aligned} |\operatorname{Re} g(e^{it})| &= |\operatorname{Re} [f(e^{it}) - p(e^{it})]| \\ &= \left| u(e^{it}) - \sum_{k=0}^{n} x_k p_k(t) \right| \\ &= O(|t|^n \omega(|t|)) + O(|t|^{n+1}) = O(|t|^n \omega(|t|)) \end{aligned} \tag{4}$$

since $t = O(\omega(|t|))$. Thus by Lemma 1

$$|f^{(n+1)}(r)| = |g^{(n+1)}(r)| \leqq M \int_{1-r}^{\pi} \frac{\omega(t)}{t^2} dt$$

where the constant $M$ depends only on the constant in the $O$-term in (4). Now note that the $\{a_{jk}\}$ are totally independent of the function $u$, so the $\{x_i\}$ are dependent only on the $\{a_i\}$. The $\{x_i\}$ affect the constant in the $O(t^n \omega(|t|))$ term in (4) via (3) so that the constant in (4) depends only on the $\{a_i\}$ and the $O(|t|^n \omega(|t|))$ term in (2). Thus the value of $M$ depends only on these constants.

COROLLARY. *If the conditions of Lemma 2 are satisfied and if* $\int_0^{\pi} (\omega(t)/t) dt < \infty$, *then there exists a constant $A$ dependent only on the* $\{a_i\}$, $\omega(t)$, $n$, *and the constant in the $O$ term in* (2), *such that for* $r \geqq 1/2$

$$|f^{(n)}(r)| \leqq A .$$

*Proof.* Let $A_1$ be the constant in the $O$ term in (4). Then as in the proof of Lemma 1,

$$\begin{aligned} |f^{(n)}(r) - p^{(n)}(r)| &\leqq \frac{n!\, A_1}{\pi} \int_0^{\pi} \frac{t^n \omega(t)}{\left(\dfrac{4rt^2}{\pi^2}\right)^{(n+1)/2}} dt \\ &\leqq \frac{n!\, A_1 \pi^n}{2^{(n+1)/2}} \int_0^{\pi} \frac{\omega(t)}{t} d\theta = A_2 \end{aligned}$$

so that

$$|f^{(n)}(r)| \leqq A_2 + |p^{(n)}(r)| .$$

But $p^{(n)}(r) = n!\, x_n$ and $x_n$ depends on the $\{a_i\}$ so

$$|f^{(n)}(r)| \leqq A_2 + n!\, x_n = A .$$

LEMMA 3. *Suppose $f(z)$ is holomorphic in $|z| < 1$ and $f'(z)$ satisfies the condition*

$$|f'(re^{i\vartheta})| \leqq M \int_{1-r}^{\pi} \frac{\omega(t)}{t^2} dt \tag{5}$$

for all $|\theta| \leqq \pi$ and for all $0 < r < 1$. Here $M$ is a constant and $\omega(t)$ is nondecreasing, nonnegative, bounded for $0 \leqq t \leqq \pi$, and $\int_0^\pi (\omega(t)/t)dt < \infty$. Then,

(i) *$\lim_{r\to 1} f(re^{i\theta}) = f(e^{i\theta})$ exists and is finite for $|\theta| \leqq \pi$ and $f(e^{i\theta})$ has the modulus of continuity*

$$\omega^*(\theta) = 3M\left[\int_0^\theta \frac{\omega(t)}{t}dt + \theta\int_\theta^\pi \frac{\omega(t)}{t^2}dt\right].$$

(ii) *$f(z)$ is continuous in $|z| \leqq 1$ and has modulus of continuity $A\omega^*(\pi t)$ where $A$ is a constant depending only on the function $\omega^*(t)$. That is, for $|z_1|, |z_2| \leqq 1$,*

$$|f(z_2) - f(z_1)| \leqq A\omega^*(\pi|z_2 - z_1|).$$

*Here we define $\omega^*(t) = \omega^*(\pi)$ for $t \geqq \pi$.*

For the proof of part (i) see [10], Lemma 4; the proof of part (ii) is patterned after that of the more special theorem in [2], page 363.

In the case $\omega(t) = t^\alpha$, $0 < \alpha < 1$ this is another result of Hardy and Littlewood ([2] Pages 360–366):

If $f$ is as in Lemma 3 and if $|f'(re^{i\theta})| \leqq M/(1-r)^{1-\alpha}$ for all $|\theta| \leqq \pi$ then $f(e^{i\theta}) \in \mathrm{Lip}(\alpha)$ for $|\theta| \leqq \pi$. If $\omega(t) = t$ then $|f'(re^{i\theta})| \leqq M\log(\pi/(1-r))$ and the conclusion is that $f(e^{i\theta})$ has modulus of continuity $\omega^*(t) = 3Mt\log(3\pi/t)$.

We note that a result analogous to Lemma 3 can be obtained if (5) is satisfied for a subarc $\theta_1 \leqq \theta \leqq \theta_2$ of $|z| = 1$ for $0 < r < 1$. Then $f(e^{i\theta})$ has modulus of continuity $\omega^*(t)$ on this arc and $f(z)$ has modulus of continuity $A\omega^*(\pi t)$ in the sector $\theta_1 \leqq \theta \leqq \theta_2$, $0 \leqq r \leqq 1$, $A$ depending on $\omega^*$. Thus it will be evident that our theorems will hold for subarcs of $\Gamma$.

The first link between the geometry of $\Gamma$ and the function $F$ is given by the following Lemma, (see [8] pp. 615–17 and [6] p. 238).

LEMMA 4. *Suppose $\Gamma$ is a closed Jordan curve in $R^3$ and $F(z)$ is a solution to Plateau's problem for $\Gamma$. For two points $p_1, p_2 \in \Gamma$, let $\Delta s(p_1p_2)$ denote the length of the shorter arc between $p_1$ and $p_2$. Suppose there exist constants $c > 1$ and $\delta > 0$ such that $\Delta s(p_1p_2)/\overline{p_1p_2} < c$ for $\Delta s(p_1p_2) < \delta$. Then there exist constants $K > 0$, $\delta_1 > 0$, depending on $\Gamma$ only, such that for $|\theta - \theta_0| < \delta_1$*

$$|F(e^{i\theta}) - F(e^{i\theta_0})| \leqq |s(\theta) - s(\theta_0)| \leqq K|\theta - \theta_0|^\beta$$

*where $s(\theta)$ for $|\theta| \leqq \pi$ is arclength along $\Gamma$ and where $\beta = 2/(1+c)^2$ so that $0 < \beta < 1/2$.*

*Proof.* Let $D[F] = 1/2\iint_{|z|<1}(F_x^2 + F_y^2)\,dx\,dy$, the Dirichlet integral of $F$.

If there exists a constant $B$ such that for each solution $F$ to Plateau's problem, $D[F] \leqq B$, then Lemma 3.2 of [1] implies that the family of solutions is equicontinuous. Since $x$ and $y$ are isothermal coordinates $D[F] = A[F]$, the area of the minimal surface, and by the isoperimetric inequality for minimal surfaces, $A[F] \leqq L^2/4\pi$ where $L$ is the length of $\Gamma$. Thus $D[F] \leqq L^2/4\pi = B$ for all minimal surfaces spanning $\Gamma$ which satisfy the three point condition and, as the modulus of continuity of the vectors $\{F(e^{i\theta})\}$ depends only on $B$, it depends only on $\Gamma$. Thus the family of arclength functions $\{s(\theta)\}$ associated with the minimal surfaces has a uniform modulus of continuity which depends only on $\Gamma$.

Let $D$ be the diameter of $\Gamma$ and let $\delta' > 0$ be such that $|\theta - \theta'| < \delta'$ implies $|s(\theta) - s(\theta')| < \min(\delta, D/2)$ for all minimal surface spanning $\Gamma$.

Let $k_\rho = \{z : |z - e^{i\theta_0}| = \rho, |z| < 1\}$ where $\rho < \min(\delta'/4, 1)$ and let $e^{i\theta_2}$ and $e^{i\theta_1}$ be the endpoints of $k_\rho$ which are on $|z| = 1$. Then $|\theta_2 - \theta_1| < \delta'$ so $|s(\theta_2) - s(\theta_1)| < \min(\delta, D/2)$. Thus $F(e^{i\theta_0})$ must be on the shorter arc between $F(e^{i\theta_2})$ and $F(e^{i\theta_1})$. This is true for all solutions to the Plateau problem for $\Gamma$.

Now let $l_\rho$ = length of $F(k_\rho)$. Then, for $z_0 = e^{i\theta_0}$

$$l_\rho = \int_{k_\rho} |F_\varphi(z_0 + \rho e^{i\varphi})|\,d\varphi$$

and by Schwarz's inequality

$$l_\rho^2 \leqq \pi \int_{k_\rho} |F_\varphi(z_0 + \rho e^{i\varphi})|^2\,d\varphi$$

so that

$$\frac{l_\rho^2}{\rho} \leqq \pi \int_{k_\rho} \frac{1}{\rho^2}|F_\varphi(z_0 + \rho e^{i\varphi})|^2\,\rho\,d\varphi\,.$$

Since $F$ is a minimal surface $1/\rho^2 \cdot F_\varphi^2 = F_\rho^2$ so that $1/\rho^2 \cdot F_\varphi^2 = 1/2(F_\rho^2 + 1/\rho^2 \cdot F_\varphi^2)$ and thus

$$\int_0^r \frac{l_\rho^2}{\rho}\,d\rho \leqq \frac{\pi}{2}\int_0^r\int_{k_\rho}\left(F_\rho^2 + \frac{1}{\rho^2}F_\varphi^2\right)\rho\,d\varphi\,d\rho\,.$$

Letting $\Delta_r = F(\{z : |z - e^{i\theta_0}| \leqq r, |z| < 1\})$ and $A(r)$ = area of $\Delta_r$, we have

$$\mathscr{F}(r) := \int_0^r \frac{l_\rho^2}{\rho}\,d\rho \leqq \pi A(r)\,.$$

Let $L$ denote the length of the boundary of $\Delta_r$. By the isoperimetric inequality $A(r) \leqq L^2/4\pi$. By our first remarks letting $p_1 = F(e^{i\theta_1})$

and $p_2 = F(e^{i\theta_2})$, we have

$$L = l_r + \Delta s(p_1p_2) \leqq l_r + c\,\overline{p_1p_2} \leqq (1+c)l_r$$

so that

$$\mathscr{F}(r) \leqq \frac{\pi L^2}{4\pi} = \frac{L^2}{4} \leqq \frac{l_r^2(1+c)^2}{4}\,.$$

Now $\mathscr{F}'(r) = l_r^2/r$ a.e., so $r\mathscr{F}'(r) = l_r^2$ and $\mathscr{F}(r) \leqq (1+c)^2/4\cdot r\mathscr{F}'(r)$. Then for $\rho < \rho_0 = \min(\delta'/4, 1)$

$$\frac{4}{(1+c)^2}\int_\rho^{\rho_0}\frac{dr}{r} \leqq \int_\rho^{\rho_0}\frac{\mathscr{F}'(r)}{\mathscr{F}(r)}dr$$

so that

$$\left(\frac{\rho_0}{\rho}\right)^{4/(1+c)^2} \leqq \frac{\mathscr{F}(\rho_0)}{\mathscr{F}(\rho)}\,.$$

Choose $M$ so that $\mathscr{F}(\rho)/(\rho^{4/(1+c)^2}) \leqq \mathscr{F}(\rho_0)/(\rho_0{}^{4/(1+c)^2}) = M-1$. $M$ depends only on $\Gamma$ since $\mathscr{F}(\rho_0) \leqq \pi A(\rho_0) \leqq \pi A[F] \leqq L^2/4\pi = B$ and $\rho_0$ depends only on $\delta'$. Then $\mathscr{F}(\rho) < M\rho^{4/(1+c)^2}$ so that

$$\int_{\rho/2}^{\rho}\frac{l_r^2}{r}\,dr \leqq \int_0^\rho \frac{l_r^2}{r}\,dr < M\rho^{4/(1+c)^2}\,.$$

Now there exists a $\rho_1$ with $\rho/2 \leqq \rho_1 \leqq \rho$ such that

$$l_{\rho_1}^2\int_{\rho/2}^{\rho}\frac{dr}{r} < M\rho^{4/(1+c)^2}$$

so that

$$l_{\rho_1}^2 \log 2 < M\rho^{4/(1+c)^2}$$

and thus

$$l_{\rho_1} < \sqrt{\frac{M}{\log 2}}\,\rho^{2/(1+c)^2} = \sqrt{\frac{M}{\log 2}}\,\rho^{\beta}\,.$$

Thus if $|e^{i\theta} - e^{i\theta_0}| = \rho/2$ and if $p_1 = F(e^{i\theta_1})$ and $p_2 = F(e^{i\theta_2})$ are the endpoints of $k_{\rho_1}$

$$\begin{aligned}|F(e^{i\theta}) - F(e^{i\theta_0})| &\leqq |s(\theta) - s(\theta_0)| \leqq c\,\overline{p_1p_2}\\ &\leqq c\sqrt{\frac{M}{\log 2}}\,\rho^\beta \leqq c\sqrt{\frac{M}{\log 2}}\,2^\beta\,|\theta-\theta_0|^\beta\,.\end{aligned}$$

Letting $K = c\sqrt{\dfrac{M}{\log 2}}\,2^\beta$ we have

$$|F(e^{i\theta}) - F(e^{i\theta_0})| \leqq |s(\theta) - s(\theta_0)| \leqq K\,|\theta-\theta_0|^\beta\,.$$

This is true for $|\theta - \theta_0| < 1/3\min(\delta'/4, 1) = \delta_1$, for we may then choose $\rho$ so that $\rho = 2\,|e^{i\theta} - e^{i\theta_0}| < 2\,|\theta-\theta_0| < \rho_0 = \min(\delta'/4, 1)$.

Since $s(\theta)$ is bounded we may find a constant $K_1$ such that $|s(\theta) - s(\theta_0)| \leqq K_1\,|\theta-\theta_0|^\beta$ for all $\theta, \theta_0 \in [-\pi, \pi]$. It is in this form that we shall use Lemma 4. ($K_1$ clearly depends on $\Gamma$ only.)

For the hypothesis of Lemma 4 to hold, it is sufficient that $\Gamma$ be continuously differentiable with respect to arclength. Then $c$ may be taken as close to 1 as we like, so that $\beta$ is as close to 1/2 as we like. The constant $K_1$ will depend on $c$, but will be uniform for all solutions to the Plateau problem for $\Gamma$.

**3. The first derivative.** We first prove Theorem 1. From Lemma 4 we know that $F(e^{i\theta}) \in \mathrm{Lip}\,(\beta)$ for any $0 < \beta < 1/2$. Our first step is to improve the Hölder exponent by a "bootstrap" technique involving the Hardy-Littlewood forms of Lemmas 1 and 3.

LEMMA 5. *Suppose $\Gamma$ is a smooth closed Jordan curve and $F(z)$ is a minimal surface spanning $\Gamma$. Suppose $F(1) = (0, 0, 0)$ and the tangent to $\Gamma$ at $F(1)$ is along the positive $u$ axis. Let $\mathscr{F}(s) = (U(s), V(s), W(s))$ be the parametrization of $\Gamma$ with respect to arclength $s$. Let $s(\theta) = s(F(e^{i\theta}))$ and $s(0) = 0$, so that $\mathscr{F}(0) = F(1) = (0, 0, 0)$ and $\mathscr{F}'(0)$ is along the positive $u$ axis.*

*Suppose that $\mathscr{F}(s) \in C^{1,\alpha}$ for some $0 < \alpha \leqq 1$ and that $F(e^{i\theta}) \in \mathrm{Lip}\,(\beta)$ for some $\beta > 0$, with Hölder constant $K_\beta$.*

*Then there exists a constant $K$, depending only on $\Gamma$, $K_\beta$, and $\beta$, such that for $|\theta| \leqq \pi$*

$$|v(e^{i\theta})| \leqq K|\theta|^{\beta(1+\alpha)} \quad |w(e^{i\theta})| \leqq K|\theta|^{\beta(1+\alpha)} .$$

*Proof.* Since $V(s) \in C^{1,\alpha}$ and $V_s(0) = 0$ we have, for some constant $K_0$

$$|V_s(s)| \leqq K_0|s|^\alpha .$$

Since $V(0) = 0$ we integrate to obtain

$$|V(s)| \leqq \frac{K_0}{1+\alpha}|s|^{1+\alpha} . \tag{6}$$

$F(\theta) \in \mathrm{Lip}\,(\beta)$ implies that $s(\theta) \in \mathrm{Lip}\,(\beta)$ so that there exists $K_\beta'$ (depending on $K_\beta$ and $\Gamma$) such that

$$|s(\theta)| \leqq K_\beta'|\theta|^\beta; \tag{7}$$

combining (6) and (7) one obtains

$$|v(e^{i\theta})| = |V(s(\theta))| \leqq \frac{K_0}{1+\alpha}(K_\beta')^{1+\alpha}|\theta|^{\beta(1+\alpha)} = K|\theta|^{\beta(1+\alpha)} .$$

The proof for $w(e^{i\theta})$ is analogous.

We now apply Lemma 5 to raise the Hölder exponent for $F(e^{i\theta})$.

LEMMA 6. *Suppose $\Gamma$ is a closed Jordan curve and $F(z)$ is a minimal surface spanning $\Gamma$. Suppose $\Gamma \in C^{1,\alpha}$ for $0 < \alpha \leqq 1$ and* *that* $F(e^{i\theta}) \in \mathrm{Lip}(\beta)$ *with Hölder constant* $K_\beta$, *where* $\beta(1+\alpha) < 1$. *Then* $(F(e^{i\theta}) \in \mathrm{Lip}(\beta(1+\alpha))$ *with the Hölder constant depending only on* $K_\beta$ *and* $\Gamma$.

*Proof.* First assume that $\Gamma, F$ are in the position of Lemma 5. Then $|v(e^{i\theta})| \leqq K|\theta|^{\beta(1+\alpha)}$ and $|w(e^{i\theta})| \leqq K|\theta|^{\beta(1+\alpha)}$.

Consider now $\mu(z) = v(z) + iv^*(z)$ and $\nu(z) = w(z) + iw^*(z)$. Then by Lemma 1 ($n = 0$), there exists a constant $M$ depending only on $K$ such that for $b = \beta(1+\alpha)$

$$|\mu'(r)| \leqq \frac{M}{(1-r)^{1-b}} \quad \text{and} \quad |\nu'(r)| \leqq \frac{M}{(1-r)^{1-b}}.$$

Letting $\lambda(z) = u(z) + iu^*(z)$ and applying (1) we have

$$|\lambda'(z)|^2 \leqq |\mu'(z)|^2 + |\nu'(z)|^2$$

and hence

$$|\lambda'(r)| \leqq \frac{\sqrt{2}M}{(1-r)^{1-b}}.$$

We would now like to apply Lemma 3 to conclude that $\lambda, \mu, \nu \in \mathrm{Lip}(\beta(1+\alpha))$.

For any $F(e^{i\theta})$ on $\Gamma$, let $(u^\theta, v^\theta, w^\theta)$ be a new coordinate system centered at $F(e^{i\theta})$ and such that the $u^\theta$ axis is tangent to $\Gamma$ at $F(e^{i\theta})$. Then $(u^\theta(z), v^\theta(z), w^\theta(z)) = F^\theta(z)$ is a minimal surface and by a rotation of the unit circle we may assume that $F^\theta(1) = F(e^{i\theta})$. It is clear that $F^\theta(e^{it}) \in \mathrm{Lip}(\beta)$ with the same Hölder constant as $F(e^{it})$. Thus $\Gamma, F^\theta$ are as in Lemma 5, so that we may use the preceding argument to see that

$$|(\mu^\theta)'(r)| \leqq \frac{M}{(1-r)^{1-b}} \quad \text{and} \quad |(\nu^\theta)'(r)| \leqq \frac{M}{(1-r)^{1-b}}$$

where $\mu^\theta(z), \nu^\theta(z), \lambda^\theta(z)$ are the analytic functions with real parts $v^\theta(z), w^\theta(z)$ and $u^\theta(z)$, respectively and $\mu^\theta(1) = \nu^\theta(1) = \lambda^\theta(1) = 0$ so that $|(\lambda^\theta)'(r)| \leqq \sqrt{2}M/(1-r)^{1-b}$.

$M$ is dependent only on $\Gamma, \beta$ and $K_\beta$. If $(a_{ij})$, $1 \leqq i, j \leqq 3$, is the orthogonal matrix of the coordinate transformation, we have

$$(8) \qquad \begin{cases} \lambda(re^{i\theta}) = a_{11}(\theta)\lambda^\theta(r) + a_{12}(\theta)\mu^\theta(r) + a_{13}(\theta)\nu^\theta(r) + \lambda(e^{i\theta}) \\ \mu(re^{i\theta}) = a_{21}(\theta)\lambda^\theta(r) + a_{22}(\theta)\mu^\theta(r) + a_{23}(\theta)\nu^\theta(r) + \mu(e^{i\theta}) \\ \nu(re^{i\theta}) = a_{31}(\theta)\lambda^\theta(r) + a_{32}(\theta)\mu^\theta(r) + a_{33}(\theta)\nu^\theta(r) + \nu(e^{i\theta}) \end{cases}$$

and therefore by the inequality of Schwarz and the orthogonality of the matrix $(a_{ij})$

$$|\lambda'(re^{i\theta})| \leqq \frac{2M}{(1-r)^{1-b}} \qquad \text{for } |\theta| \leqq 2\pi$$

and by Lemma 3, $\lambda \in \mathrm{Lip}(b)$. The same holds for $\mu$ and $\nu$, and the Hölder constant is as claimed.

LEMMA 7. *With $\Gamma, F$ defined as in Lemma 5, there exists an $\varepsilon > 0$ such that $v(e^{i\theta}) = O(\theta^{1+\varepsilon})$, $w(e^{i\theta}) = O(\theta^{1+\varepsilon})$ where the constant in $O$ depends only on $\Gamma$.*

*Proof.* Choose $0 < \beta < 1/2$ such that for all integers $n$, $(1 + \alpha)^n \neq 1/\beta$. Then there exists an integer $n$ such that $(1 + \alpha)^n\beta = 1 + \varepsilon > 1$ but $(1 + \alpha)^{n-1}\beta < 1$. Apply Lemma 6 $n - 1$ times to obtain $F(e^{i\theta}) \in \mathrm{Lip}(\beta(1 + \alpha)^{n-1})$ and then apply Lemma 5 to see that there exists $K$ constant such that $|v(\theta)| \leqq K|\theta|^{1+\varepsilon}$ and $|w(\theta)| \leqq K|\theta|^{1+\varepsilon}$.

*Proof of Theorem* 1. First suppose $\Gamma, F$ are as in Lemma 5. Then we claim $\lim_{r\to 1} \mu'(r) = \mu'(1)$, $\lim_{r\to 1} \nu'(r) = \nu'(1)$, $\lim_{r\to 1} \lambda'(r) = \lambda'(1)$ all exist and are finite. By Lemma 7 $v(\theta) = O(\theta^{1+\varepsilon})$, hence by Lemma 1 $|\mu''(r)| \leqq M/(1 - r)^{1-\varepsilon}$, for $r \leqq 1/2$. Then for $1/2 \leqq r_1 < r_2 < 1$

$$|\mu'(r_2) - \mu'(r_1)| = \left|\int_{r1}^{r2} \mu''(r)\,dr\right| \leqq \int_{r1}^{r2} \frac{M}{(1 - r)^{1-\varepsilon}}dr$$

$$\leqq \frac{M}{\varepsilon}|r_2 - r_1|^{\varepsilon}$$

so that $\lim_{r\to 1} \mu'(r) = \mu'(1)$ exists and is finite. Likewise $\lim_{r\to 1} \nu'(r) = \nu'(1)$ exists and is finite.

Since $\lambda'^2(r) = -(\mu'^2(r) + \nu'^2(r))$, we see $\lim_{r\to 1} \lambda'(r) = \lambda'(1)$ exists and is finite.

From (8) it is clear that each of $\lambda'(re^{i\theta})$, $\mu'(re^{i\theta})$, $\nu'(re^{i\theta})$ have radial limits for all $|\theta| \leqq \pi$ and the convergence is uniform for all $\theta$. Thus defining $\lambda'(e^{i\theta}) = \lim_{r\to 1} \lambda'(re^{i\theta})$, the function $\lambda'(e^{i\theta})$ is continuous. This, together with the uniform convergences of $\lambda'(re^{i\theta})$ to $\lambda'(e^{i\theta})$ implies that $\lambda'(z)$ is continuous for $|z| \leqq 1$. From this it follows that $\lambda(z)$ is differentiable at each $e^{i\theta}$, *ie.*

$$\lim_{z\to e^{i\theta}} \frac{\lambda(z) - \lambda(e^{i\theta})}{z - e^{i\theta}} = \lambda'(e^{i\theta}) .$$

The same facts are true for $\mu'(z)$ and $\nu'(z)$.

Finally, recall that if $\Gamma, F$ are as in Lemma 5 then there exist $\varepsilon > 0$ and $K > 0$ such that $|v(e^{i\theta})| \leqq K|\theta|^{1+\varepsilon}$ and $|w(e^{i\theta})| \leqq K|\theta|^{1+\varepsilon}$, where $K$ depends only on $\Gamma$.

Thus, by the corollary to Lemma 2 there exists a constant $K_1$ such that $|\mu'(1)| \leqq K_1$ and $|\nu'(1)| \leqq K_1$; hence $|\lambda'(1)| \leqq \sqrt{2}\,K_1$. By the equations (8) one sees that $|\lambda'(e^{i\theta})|, |\mu'(e^{i\theta})|, |\nu'(e^{i\theta})|$ are bounded by $2K_1$ for all $\theta$. Thus $|s'(\theta)| \leqq 2\sqrt{3}\,K_1 = c$ for $|\theta| \leqq \pi$, and $c$ is the same for any solution to Plateau's problem for $\Gamma$.

We now prove a lemma preparatory to the proof of Theorem 2.

LEMMA 8. *Suppose $\Gamma$, $F$ are positioned as in Lemma 5. Suppose also that $\lambda'$, $\mu'$, $\nu'$ are continuous in $|z| \leqq 1$ and $\Gamma \in C^{1,\omega(t)}$. Let $|s'(\theta)| \leqq c$, $|\theta| \leqq \pi$, and let $\omega_0(\theta) = \omega(c\theta)$. Then*

$$|v(e^{i\theta})| \leqq K|\theta\omega_0(|\theta|)|, \; |w(e^{i\theta})| \leqq K|\theta\omega_0(|\theta|)|, \; |u^*(e^{i\theta})| \leqq K|\theta\omega_0(|\theta|)|$$

*for $|\theta| \leqq \pi$, where the constant $K$ depends only on $c$ and $\Gamma$.*

*Proof.* By the argument of Lemma 5 we have $|V(s)| \leqq |s|\,\omega(s)$ and since $|s(\theta)| \leqq c|\theta|$, $|v(e^{i\theta})| \leqq c|\theta|\,\omega_0(|\theta|)$; likewise $|w(e^{i\theta})| \leqq c|\theta|\,\omega_0(|\theta|)$.

By Lemma 4, $U_s(s(\theta))$ is uniformly continuous for $|\theta| \leqq \pi$ and $U_s(s(0)) = 1$. Therefore there exists a $\delta > 0$ (depending only on $\Gamma$) such that $|\theta| < \delta$ implies $U_s(s(\theta)) > 1/2$. Now $ds(\theta)/d\theta \neq 0$ for almost every $\theta$ and $U_s s_\theta = u_\theta$ and $V_s s_\theta = v_\theta$ so that

$$\frac{v_\theta(e^{i\theta})}{u_\theta(e^{i\theta})} = \frac{V_s(s(\theta))s_\theta(\theta)}{U_s(s(\theta))s_\theta(\theta)} = \frac{V_s(s(\theta))}{U_s(s(\theta))} \qquad \text{a.e.} \quad |\theta| < \delta\,.$$

But

$$\left|\frac{V_s(s)}{U_s(s)}\right| \leqq 2\omega(|s|) \leqq 2\omega_0(|\theta|)$$

so that

$$\left|\frac{v_\theta(e^{i\theta})}{u_\theta(e^{i\theta})}\right| \leqq 2\omega_0(|\theta|) \qquad \text{a.e.} \quad |\theta| < \delta\,;$$

likewise

$$\left|\frac{w_\theta(e^{i\theta})}{u_\theta(e^{i\theta})}\right| \leqq 2\omega_0(|\theta|) \qquad \text{a.e.} \quad |\theta| < \delta\,.$$

In polar coordinates the minimal surface condition implies that $u_r u_\theta + v_r v_\theta + w_r w_\theta = 0$ and therefore

$$-u_\theta^* = -u_r = v_r \frac{v_\theta}{u_\theta} + w_r \frac{w_\theta}{u_\theta}$$

but $|v_r(e^{i\theta})|$ and $|w_r(e^{i\theta})|$ are both bounded by $c$ for all $\theta$ so that $|u_\theta^*(e^{i\theta})| \leqq 4c\omega_0(|\theta|)$ a.e. $|\theta| < \delta$. Taking $u^*(e^{i0}) = 0$ we may integrate to obtain

$$|u^*(e^{i\theta})| \leqq 4c|\theta|\,\omega_0(|\theta|) \quad |\theta| < \delta\,.$$

Since $\delta$ was dependent only on $\Gamma$ it is clear that $K$ may be chosen to complete the proof of the lemma.

*Proof of Theorem* 2. Suppose first that $\Gamma$, $F$ are as in Lemma 5. Then the conclusion of Lemma 8 holds. Applying Lemma 1 to $-i\lambda(z)$, for instance, we obtain

$$|\lambda''(r)| \leqq M \int_{1-r}^{\pi} \frac{\omega_0(t)}{t^2} dt \quad \text{for} \quad r \geqq \frac{1}{2}$$

and analogous inequalities for $|\mu''(r)|$ and $|\nu''(r)|$. Since $M$ depends only on $\Gamma$ we see by applying the transformation (8) that

$$|\lambda''(re^{i\theta})| \leqq \sqrt{3}\, M \int_{1-r}^{\pi} \frac{\omega_0(t)}{t^2} dt \quad |\theta| \leqq \pi .$$

Analogous inequalities hold for $|\mu''(re^{i\theta})|$ and $|\nu''(re^{i\theta})|$. The conclusion of Theorem 2 then follows from Lemma 3.

**4. The higher derivatives.** In proving Theorem 3 for a given $n \geqq 2$, the result for $n-1$ is assumed, so that $\Gamma \in C^{n,\omega(t)}$ implies $\Gamma \in C^{n-1,1}$ and thus $s^{(n-1)}(\theta)$ has modulus of continuity $kt \log 3\pi/t$.

We shall make extensive use of the following fact: If $f(x) \in C^{n,\omega(t)}$ for $|x| \leqq \delta$, then

$$f(x) = \sum_{i=0}^{n} f^{(i)}(0) \frac{x^i}{i!} + O(|x^n|\, \omega(|x|)) .$$

We now prove a lemma analogous to Lemma 8.

LEMMA 9. *Suppose $\Gamma \in C^{n,\omega(t)}$, $n \geqq 2$, and that $\Gamma$, $F$ are positioned as in Lemma 5. Suppose $c \geqq |s'(\theta)|$ for $|\theta| \leqq \pi$ and that $\omega_0(\theta) = \omega(c\theta)$. Such a $c$ exists and is dependent only on $\Gamma$ by Theorem 1. Then there exist constants $\{b_i\}$, $\{c_i\}$, $\{a_i\}$ $2 \leqq i \leqq n$ such that*

$$(9) \qquad \begin{cases} v(e^{i\theta}) = \sum_{i=2}^{n} b_i \theta^i + O(|\theta|^n\, \omega_1(|\theta|)) \\ w(e^{i\theta}) = \sum_{i=2}^{n} c_i \theta^i + O(|\theta|^n\, \omega_1(|\theta|)) \\ u^*(e^{i\theta}) = \sum_{i=2}^{n} a_i \theta^i + O(|\theta|^n\, \omega_1(|\theta|)) \end{cases}$$

*where $\omega_1(|\theta|) = |\theta| \log 3\pi/|\theta| + \omega_0(|\theta|)$ and the constants in the $O(|\theta|^n\, \omega_1(|\theta|))$ terms depend only on $\Gamma$ and the constants $\{a_i\}$, $\{b_i\}$, $\{c_i\}$ are uniformly bounded by a constant depending only on $\Gamma$.*

*Proof.* We have

$$(10) \qquad s(\theta) = \sum_{i=1}^{n-1} s^{(i)}(0) \frac{\theta^i}{i!} + O\left(|\theta|^n \log \frac{3\pi}{|\theta|}\right)$$

for $|\theta| \leqq \pi$. By the induction hypothesis, there exists a constant $K$ such that $|s^{(i)}(\theta)| \leqq K$ for $1 \leqq i \leqq n-1$ and $|\theta| \leqq \pi$, and such that the constant in the $O$ term is bounded by $K$. We also have

$$V(s) = \sum_{i=2}^{n} V^{(i)}(0)\frac{s^i}{i!} + O(|s|^n \omega(s))$$

so that

$$\begin{aligned} v(e^{i\theta}) = V(s(\theta)) &= \sum_{i=2}^{n} \frac{V^{(i)}(0)}{i!}\Big[\sum_{j=1}^{n-1} s^{(j)}(0)\,\frac{\theta^j}{j!} + O\Big(|\theta|^n \log\frac{3\pi}{|\theta|}\Big)\Big]^i \\ &\quad + O\Big(\Big[\sum_{j=1}^{n-1} s^{(j)}(0)\,\frac{\theta^j}{j!} + O\Big(|\theta|^n \log\frac{3\pi}{|\theta|}\Big)\Big]^n \omega_0(|\theta|)\Big) \\ &= \sum_{i=2}^{n} b_i\theta^i + O(|\theta|^n \omega_1(|\theta|)) \,. \end{aligned}$$

The corresponding expression for $w(e^{i\theta})$ is obtained similarly. Now, as in Lemma 8

$$-u_\theta^*(e^{i\theta}) = v_r(e^{i\theta})\frac{v_\theta(e^{i\theta})}{u_\theta(e^{i\theta})} + w_r(e^{i\theta})\frac{w_\theta(e^{i\theta})}{u_\theta(e^{i\theta})}$$

where $v_\theta/u_\theta = V_s/U_s$ for $|\theta| < \delta$[1]. But $V_s(s)/U_s(s) \in C^{n-1,\omega}$ for $|\theta| < \delta$ so that

$$\frac{V_s(s)}{U_s(s)} = \sum_{i=1}^{n-1} d_i s^i + O(|s|^{n-1}\omega(|s|)) \quad \text{for} \quad |\theta| < \delta$$

and so using (10)

$$\frac{v_\theta(e^{i\theta})}{u_\theta(e^{i\theta})} = \sum_{i=1}^{n-1} f_i\theta^i + O(|\theta|^{n-1}\omega_1(|\theta|)) \,.$$

Since $\Gamma \in C^{n-1,1}$, $v_r(e^{i\theta}) \in C^{n-2,\omega_2(t)}$ where $\omega_2(t) = Kt\,(\log 3\pi/t)$, so that

$$\begin{aligned} v_r(e^{i\theta})\frac{v_\theta(e^{i\theta})}{u_\theta(e^{i\theta})} &= \Big[\sum_{i=0}^{n-2} g_i\theta^i + O\Big(|\theta|^{n-1}\log\frac{3\pi}{|\theta|}\Big)\Big] \\ &\quad \cdot\Big[\sum_{i=1}^{n-1} f_i\theta^i + O(|\theta|^{n-1}\omega_1(|\theta|))\Big] \\ &= \sum_{i=1}^{n-1} h_i\theta^i + O(|\theta|^{n-1}\omega_1(|\theta|)) \,. \end{aligned}$$

A similar expansion holds for $w_r(e^{i\theta})w_\theta(e^{i\theta})/u_\theta(e^{i\theta})$ so that

$$u_\theta^*(e^{i\theta}) = \sum_{i=1}^{n-1} m_i\theta^i + O(|\theta|^{n-1}\omega_1(|\theta|)) \quad \text{for} \quad |\theta| < \delta$$

and

$$u^*(e^{i\theta}) = \sum_{i=2}^{n} a_i\theta^i + O(|\theta|^n \omega_1(|\theta|)) \quad \text{for} \quad |\theta| \leqq \pi \,.$$

In each case the coefficients of the expansions and the constants in the $O$ terms are bounded uniformly, the bound depending only on $\Gamma$.

[1] At points $\theta_0$ where $ds/d\theta = 0$ we mean by $v_\theta(e^{i\theta_0})/u_\theta(e^{i\theta_0})$ the limit as $\theta \to \theta_0$.

*Proof of Theorem* 3. Let us first suppose that $\Gamma, F$ are as in Lemma 5. Then by Lemma 9, (9) holds. We may then apply lemma 2 to $i\lambda(z)$, $\mu(z)$ and $\nu(z)$ to conclude that

$$|\lambda^{(n+1)}(r)| \leqq M_n \int_{1-r}^{\pi} \frac{\omega_1(t)}{t^2} dt \quad (0 < r < 1)$$

$$|\mu^{(n+1)}(r)| \leqq M_n \int_{1-r}^{\pi} \frac{\omega_1(t)}{t^2} dt$$

and

$$|\nu^{(n+1)}(r)| \leqq M_n \int_{1-r}^{\pi} \frac{\omega_1(t)}{t^2} dt .$$

Since the constants involved in (9) are bounded by a constant depending only on $\Gamma$, $M_n$ depends only on $\Gamma$. Thus, for all $|\theta| \leqq \pi$ we have

$$|\lambda^{(n+1)}(re^{i\theta})| \leqq \sqrt{3}\, M_n \int_{1-r}^{\pi} \frac{\omega_1(t)}{t^2} dt$$

and the corresponding inequalities obtain for $\mu$ and $\nu$.

Part (i) of the theorem then follows from Lemma 3, with $\omega_1$ rather than $\omega_0$.

Furthermore, by the corollary to Lemma 2, if $\Gamma$ is positioned as in Lemma 5 then there exists a constant $K$ depending only on $\Gamma$, such that $|\lambda^{(m)}(1)| \leqq K$, $|\mu^{(m)}(1)| \leqq K$ and $|\nu^{(m)}(1)| \leqq K$ for $m = 1, 2 \cdots, n$. By the equations (8) one sees that $|\lambda^{(m)}(e^{i\theta})|$, $|\mu^{(m)}(e^{i\theta})|$, and $|\nu^{(m)}(e^{i\theta})|$ are bounded by $\sqrt{3}\, K$ for all $\theta$ and each $m$, $1 \leqq m \leqq n$. From this it follows that $|s^{(n)}(\theta)|$ is bounded for all $\theta$ by a constant $c_n$ depending only on $\Gamma$.

We may now see that Lemma 9 and Theorem 3 are true with $\omega_0(|\theta|)$ in place of $\omega_1(\theta)$.

Since $s^{(n)}(\theta)$ is continuous and bounded, $s(\theta) \in C^{n-1,1}$ i.e.,

$$s(\theta) = \sum_{i=1}^{n-1} s^{(i)}(0) \frac{\theta^i}{i!} + O(|\theta|^{n+1}) \tag{11}$$

where the coefficients and the constant in the $O$ term are bounded by some constant $K$. Then, using (11) instead of (10) in the proof of Lemma 9, we obtain (9) with $\omega_0(|\theta|)$ instead of $\omega_1(|\theta|)$. Then Theorem 3 may be proved with $\omega_0(|\theta|)$ instead of $\omega_1(|\theta|)$.

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