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# MAPPING SOLENOIDS ONTO STRONGLY SELF-ENTWINED, CIRCLE-LIKE CONTINUA

J. T. Rogers, Jr.

**A circle-like continuum $C$ is self-entwined if there exists a sequence $\{C_i\}$ of circular chains which define $C$, a point $p$ in $C$, and a sequence $\{D_i\}$ such that, for each $i$, (1) either $D_i$ is a subchain of $C_i$, or $D_i = C_i$, (2) $D_{i+1}$ circles at least twice in $C_i$, (3) $C_{i+1}$ circles at least once in $C_i$, and (4) the point $p$ is in the first link of $D_i$. If, in addition, each $D_{i+1}$ circles more times in $C_i$ than $C_{i+1}$ circles in $C_i$, then $C$ is said to be strongly self-entwined.**

**The purpose of this paper is to prove the following.**

**THEOREM 1. No solenoid can be mapped onto a strongly self-entwined, circle-like continuum.**

We show that each self-entwined, circle-like, plane continuum is strongly self-entwined; hence Theorem 1 implies that no solenoid can be mapped onto a self-entwined, circle-like, plane continuum.

Theorem 1 has another interesting corollary. Let $n$ be a natural number greater than one. Let $V_n$ denote the circle-like plane continuum which is the common part of a descending sequence $\{C_i\}$ of circular chains such that $C_{i+1}$ circles $n$ times in $C_i$ in the positive direction and then $n-1$ times in the negative direction (see [1] for the definition of circling) and such that the first link of $C_i$ contains the closure of the first link of $C_{i+1}$. The continuum $V_n$ is obviously self-entwined, so no solenoid can be mapped onto $V_n$. This contrasts with a result [6] of J. W. Rogers, Jr., who has shown that each member of an analogous class of arc-like continua is a continuous image of each solenoid.

We assume the terminology and definitions of [3]. We use the equivalent definition of self-entwined, circle-like continuum given in [3]. We assume that each factor space of an inverse sequence is a triangulation of the unit circle $C$ and that each bonding map is a surjective, piecewise-linear map of nonnegative degree. We also assume that, under these maps, the image of each vertex is either a vertex or a midpoint of a one-simplex, and that adjacent vertices are mapped into a simplex. Such inverse sequences are called *barycentric inverse sequences*. Each circle-like continuum has such an inverse limit representation [4, Lemma 8].

We redefine strongly self-entwined, circle-like continua in the terminology of [3]. If $X = \lim \{X_i, f_i^{i+1}\}$ is a self-entwined, circle-like continuum (hence we may assume for each $i$ that $\deg(f_i^{i+1}) > 0$ and

$R(f_i^{i+1}) > 1$), then we say that $X$ is strongly self-entwined if

$$R(f_i^{i+1}) > \deg(f_i^{i+1}) \text{ for each } i .$$

A solenoid is a circle-like continuum which is the inverse limit of an inverse sequence such that each bonding map is one of the complex functions $\{w = z^n\}_{n=1}^{\infty}$. A pseudo-circle is a non-arc-like, hereditarily indecomposable, circle-like plane continuum [4].

**1. Mapping solenoids onto circle-like continua.** We proceed immediately to the main theorem.

THEOREM 1. *No solenoid can be mapped onto a strongly self-entwined, circle-like continuum.*

*Proof.* Let $X = \lim \{X_i, f_i^{i+1}\}$ be a strongly self-entwined, circle-like continuum. We may assume that $\deg(f_i^{i+1}) \geqq 1$ and

$$R(f_i^{i+1}) > \deg(f_i^{i+1}),\ i = 1, 2, \cdots .$$

Let $S = \lim \{S_i, g_i^{i+1}\}$ be the 2-solenoid; we may assume that each bonding map $g_i^{i+1}$ is the complex function $w = z^2$. We prove the theorem for $S$; the proof of the general case is similar.

Suppose that there exists a map $f$ of $S$ onto $X$. Let $\{\varepsilon_n\}$ be a decreasing sequence of positive numbers converging to zero and bounded above by 1/2. The existence of $f$ implies the existence of an infinite diagram

$$(1) \qquad \begin{array}{ccccccccc} S_{n(1)} & \longleftarrow & S_{n(2)} & \longleftarrow & \cdots & \longleftarrow & S_{n(k)} & \longleftarrow & \cdots \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & & & \downarrow{\scriptstyle h_k} & & \\ X_{m(1)} & \longleftarrow & X_{m(2)} & \longleftarrow & \cdots & \longleftarrow & X_{m(k)} & \longleftarrow & \cdots , \end{array}$$

where $\{m(k)\}$ and $\{n(k)\}$ are increasing sequences of positive integers and where every subdiagram

$$(2) \qquad \begin{array}{ccc} S_{n(k)} & \longleftarrow & S_{n(r)} \\ \downarrow{\scriptstyle h_k} & & \downarrow{\scriptstyle h_r} \\ X_{m(k)} & \longleftarrow & X_{m(r)} \end{array}$$

is $\varepsilon_k$-commutative for all $r \geqq k$. See [2, Theorem 1] for details.

Since each $\varepsilon_k < 1/2$, Diagram (2) and Lemma 4 of [4] assure us that

$$(3) \qquad \deg(h_k \circ g_{n(k)}^{n(r)}) = \deg(f_{m(k)}^{m(r)} \circ h_r)\ (r > k) .$$

We show (as in Theorem 5 of [3]) that the revolving number of

$h_k \circ g^{n(r)}_{n(k)}$ is less than that of $f^{m(r)}_{m(k)} \circ h_r$. Now it is not necessary that the two revolving numbers be equal, since the two composite maps may differ by $\varepsilon_k$; since $\varepsilon_k < 1/2$, however, the revolving numbers can differ by no more than two (one at each end of a defining interval). For this reason, we add two to $R(h_k \circ g^{n(r)}_{n(k)})$ in the last inequality.

Because the bonding maps of the solenoid are so smooth, the inequality of Theorem 1 of [3] is actually an equality. Therefore,

$$\begin{aligned} R(h_1 \circ g^{n(r)}_{n(1)}) &= R(g^{n(r)}_{n(1)}) \cdot \deg(h_1) - \deg(h_1) + R(h_1) \\ &= \deg(g^{n(r)}_{n(1)}) \cdot \deg(h_1) - \deg(h_1) + R(h_1) \\ &= \deg(h_1 \circ g^{n(r)}_{n(1)}) - \deg(h_1) + R(h_1) . \end{aligned}$$

On the other hand, repeated applications of Theorem 1 of [3] imply that

$$R(f^{m(r)}_{m(1)}) \geqq \sum_{i=3}^{r} [R(f^{m(i)}_{m(i-1)}) \cdot \deg(f^{m(i-1)}_{m(1)}) - \deg(f^{m(i-1)}_{m(1)}] + R(f^{m(2)}_{m(1)}) .$$

Since

$$R(f^{m(i)}_{m(i-1)}) \geqq 1 + \deg(f^{m(i)}_{m(i-1)}) \text{ and } \deg(f^{m(i)}_{m(i-1)}) \geqq 1 ,$$

we have

$$\begin{aligned} R(f^{m(r)}_{m(1)}) &\geqq \sum_{i=2}^{r} \deg(f^{m(i)}_{m(1)}) \\ &\geqq \deg(f^{m(r)}_{m(1)}) + \sum_{i=2}^{r-1} (\deg f^{m(i)}_{m(1)}) \\ &\geqq \deg(f^{m(r)}_{m(1)} + (r-2) . \end{aligned}$$

Again applying Theorem 1 of [3], we find that

$$\begin{aligned} R(f^{m(r)}_{m(1)} \circ h_r) &\geqq R(h_r) \cdot \deg(f^{m(r)}_{m(1)}) - \deg(f^{m(r)}_{m(1)}) + R(f^{m(r)}_{m(1)}) \\ &\geqq \deg(h_r) \cdot \deg(f^{m(r)}_{m(1)}) - \deg(f^{m(r)}_{m(1)}) \\ &\quad + \deg(f^{m(r)}_{m(1)}) + (r-2) \\ &\geqq \deg(f^{m(r)}_{m(1)} \circ h_r) + r - 2 \\ &\geqq \deg(h_1 \circ g^{n(r)}_{n(1)}) + r - 2 \\ &\geqq R(h_1 \circ g^{n(r)}_{n(1)}) + \deg(h_1) - R(h_1) + r - 2 . \end{aligned}$$

If we choose $r$ to exceed $R(h_1) - \deg(h_1) + 5$, then we obtain

$$R(f^{m(r)}_{m(1)} \circ h_r) > R(h_1 \circ g^{n(r)}_{n(1)}) + 2 .$$

This contradiction establishes the theorem.

COROLLARY 1. *No solenoid can be mapped onto a self-entwined, circle-like, plane continuum.*

*Proof.* It suffices to show that each self-entwined, circle-like, plane continuum $C$ is strongly self-entwined. Since $C$ is self-entwined, $C$ is the inverse limit of an inverse sequence $\{C_i, f_i^{i+1}\}$, where $R(f_i^{i+1}) > 1$ and deg $(f_i^{i+1}) > 0$, for each $i$. By Theorem 3 of [1], we may assume, by choosing a subsequence if necessary, that $\deg(f_i^{i+1}) = 1$, for all $i$. Therefore, $R(f_i^{i+1}) > \deg(f_i^{i+1})$, and $C$ is strongly self-entwined.

COROLLARY 2. *No solenoid can be mapped onto a* $V_n$.

COROLLARY 3. *No solenoid can be mapped onto the pseudo-circle.* [4].

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