

# Pacific Journal of Mathematics

**MAPPING SOLENOIDS ONTO STRONGLY SELF-ENTWINED,  
CIRCLE-LIKE CONTINUA**

JAMES TED ROGERS JR.

## MAPPING SOLENOIDS ONTO STRONGLY SELF-ENTWINED, CIRCLE-LIKE CONTINUA

J. T. Rogers, Jr.

**A circle-like continuum  $C$  is self-entwined if there exists a sequence  $\{C_i\}$  of circular chains which define  $C$ , a point  $p$  in  $C$ , and a sequence  $\{D_i\}$  such that, for each  $i$ , (1) either  $D_i$  is a subchain of  $C_i$ , or  $D_i = C_i$ , (2)  $D_{i+1}$  circles at least twice in  $C_i$ , (3)  $C_{i+1}$  circles at least once in  $C_i$ , and (4) the point  $p$  is in the first link of  $D_i$ . If, in addition, each  $D_{i+1}$  circles more times in  $C_i$  than  $C_{i+1}$  circles in  $C_i$ , then  $C$  is said to be strongly self-entwined.**

**The purpose of this paper is to prove the following.**

**THEOREM 1. No solenoid can be mapped onto a strongly self-entwined, circle-like continuum.**

We show that each self-entwined, circle-like, plane continuum is strongly self-entwined; hence Theorem 1 implies that no solenoid can be mapped onto a self-entwined, circle-like, plane continuum.

Theorem 1 has another interesting corollary. Let  $n$  be a natural number greater than one. Let  $V_n$  denote the circle-like plane continuum which is the common part of a descending sequence  $\{C_i\}$  of circular chains such that  $C_{i+1}$  circles  $n$  times in  $C_i$  in the positive direction and then  $n - 1$  times in the negative direction (see [1] for the definition of circling) and such that the first link of  $C_i$  contains the closure of the first link of  $C_{i+1}$ . The continuum  $V_n$  is obviously self-entwined, so no solenoid can be mapped onto  $V_n$ . This contrasts with a result [6] of J. W. Rogers, Jr., who has shown that each member of an analogous class of arc-like continua is a continuous image of each solenoid.

We assume the terminology and definitions of [3]. We use the equivalent definition of self-entwined, circle-like continuum given in [3]. We assume that each factor space of an inverse sequence is a triangulation of the unit circle  $C$  and that each bonding map is a surjective, piecewise-linear map of nonnegative degree. We also assume that, under these maps, the image of each vertex is either a vertex or a midpoint of a one-simplex, and that adjacent vertices are mapped into a simplex. Such inverse sequences are called *barycentric inverse sequences*. Each circle-like continuum has such an inverse limit representation [4, Lemma 8].

We redefine strongly self-entwined, circle-like continua in the terminology of [3]. If  $X = \lim \{X_i, f_i^{i+1}\}$  is a self-entwined, circle-like continuum (hence we may assume for each  $i$  that  $\deg(f_i^{i+1}) > 0$  and

$R(f_i^{i+1}) > 1$ ), then we say that  $X$  is strongly self-entwined if

$$R(f_i^{i+1}) > \text{deg}(f_i^{i+1}) \text{ for each } i .$$

A solenoid is a circle-like continuum which is the inverse limit of an inverse sequence such that each bonding map is one of the complex functions  $\{w = z^n\}_{n=1}^\infty$ . A pseudo-circle is a non-arc-like, hereditarily indecomposable, circle-like plane continuum [4].

1. Mapping solenoids onto circle-like continua. We proceed immediately to the main theorem.

**THEOREM 1.** *No solenoid can be mapped onto a strongly self-entwined, circle-like continuum.*

*Proof.* Let  $X = \lim \{X_i, f_i^{i+1}\}$  be a strongly self-entwined, circle-like continuum. We may assume that  $\text{deg}(f_i^{i+1}) \geq 1$  and

$$R(f_i^{i+1}) > \text{deg}(f_i^{i+1}), i = 1, 2, \dots .$$

Let  $S = \lim \{S_i, g_i^{i+1}\}$  be the 2-solenoid; we may assume that each bonding map  $g_i^{i+1}$  is the complex function  $w = z^2$ . We prove the theorem for  $S$ ; the proof of the general case is similar.

Suppose that there exists a map  $f$  of  $S$  onto  $X$ . Let  $\{\varepsilon_n\}$  be a decreasing sequence of positive numbers converging to zero and bounded above by  $1/2$ . The existence of  $f$  implies the existence of an infinite diagram

$$(1) \quad \begin{array}{ccccccc} S_{n(1)} & \longleftarrow & S_{n(2)} & \longleftarrow & \dots & \longleftarrow & S_{n(k)} & \longleftarrow & \dots \\ h_1 \downarrow & & h_2 \downarrow & & & & h_k \downarrow & & \\ X_{m(1)} & \longleftarrow & X_{m(2)} & \longleftarrow & \dots & \longleftarrow & X_{m(k)} & \longleftarrow & \dots \end{array} ,$$

where  $\{m(k)\}$  and  $\{n(k)\}$  are increasing sequences of positive integers and where every subdiagram

$$(2) \quad \begin{array}{ccc} S_{n(k)} & \longleftarrow & S_{n(r)} \\ h_k \downarrow & & h_r \downarrow \\ X_{m(k)} & \longleftarrow & X_{m(r)} \end{array}$$

is  $\varepsilon_k$ -commutative for all  $r \geq k$ . See [2, Theorem 1] for details.

Since each  $\varepsilon_k < 1/2$ , Diagram (2) and Lemma 4 of [4] assure us that

$$(3) \quad \text{deg}(h_k \circ g_{n(k)}^{n(r)}) = \text{deg}(f_{m(k)}^{m(r)} \circ h_r) \quad (r > k) .$$

We show (as in Theorem 5 of [3]) that the revolving number of

$h_k \circ g_n^{n(r)}$  is less than that of  $f_m^{m(r)} \circ h_r$ . Now it is not necessary that the two revolving numbers be equal, since the two composite maps may differ by  $\varepsilon_k$ ; since  $\varepsilon_k < 1/2$ , however, the revolving numbers can differ by no more than two (one at each end of a defining interval). For this reason, we add two to  $R(h_k \circ g_n^{n(r)})$  in the last inequality.

Because the bonding maps of the solenoid are so smooth, the inequality of Theorem 1 of [3] is actually an equality. Therefore,

$$\begin{aligned} R(h_1 \circ g_n^{n(r)}) &= R(g_n^{n(r)}) \cdot \deg(h_1) - \deg(h_1) + R(h_1) \\ &= \deg(g_n^{n(r)}) \cdot \deg(h_1) - \deg(h_1) + R(h_1) \\ &= \deg(h_1 \circ g_n^{n(r)}) - \deg(h_1) + R(h_1). \end{aligned}$$

On the other hand, repeated applications of Theorem 1 of [3] imply that

$$R(f_m^{m(r)}) \geq \sum_{i=3}^r [R(f_m^{m(i)}) \cdot \deg(f_m^{m(i-1)}) - \deg(f_m^{m(i-1)})] + R(f_m^{m(2)}).$$

Since

$$R(f_m^{m(i)}) \geq 1 + \deg(f_m^{m(i)}) \text{ and } \deg(f_m^{m(i)}) \geq 1,$$

we have

$$\begin{aligned} R(f_m^{m(r)}) &\geq \sum_{i=2}^r \deg(f_m^{m(i)}) \\ &\geq \deg(f_m^{m(r)}) + \sum_{i=2}^{r-1} (\deg f_m^{m(i)}) \\ &\geq \deg(f_m^{m(r)}) + (r - 2). \end{aligned}$$

Again applying Theorem 1 of [3], we find that

$$\begin{aligned} R(f_m^{m(r)} \circ h_r) &\geq R(h_r) \cdot \deg(f_m^{m(r)}) - \deg(f_m^{m(r)}) + R(f_m^{m(r)}) \\ &\geq \deg(h_r) \cdot \deg(f_m^{m(r)}) - \deg(f_m^{m(r)}) \\ &\quad + \deg(f_m^{m(r)}) + (r - 2) \\ &\geq \deg(f_m^{m(r)} \circ h_r) + r - 2 \\ &\geq \deg(h_1 \circ g_n^{n(r)}) + r - 2 \\ &\geq R(h_1 \circ g_n^{n(r)}) + \deg(h_1) - R(h_1) + r - 2. \end{aligned}$$

If we choose  $r$  to exceed  $R(h_1) - \deg(h_1) + 5$ , then we obtain

$$R(f_m^{m(r)} \circ h_r) > R(h_1 \circ g_n^{n(r)}) + 2.$$

This contradiction establishes the theorem.

**COROLLARY 1.** *No solenoid can be mapped onto a self-entwined, circle-like, plane continuum.*

*Proof.* It suffices to show that each self-entwined, circle-like, plane continuum  $C$  is strongly self-entwined. Since  $C$  is self-entwined,  $C$  is the inverse limit of an inverse sequence  $\{C_i, f_i^{i+1}\}$ , where  $R(f_i^{i+1}) > 1$  and  $\deg(f_i^{i+1}) > 0$ , for each  $i$ . By Theorem 3 of [1], we may assume, by choosing a subsequence if necessary, that  $\deg(f_i^{i+1}) = 1$ , for all  $i$ . Therefore,  $R(f_i^{i+1}) > \deg(f_i^{i+1})$ , and  $C$  is strongly self-entwined.

**COROLLARY 2.** *No solenoid can be mapped onto a  $V_n$ .*

**COROLLARY 3.** *No solenoid can be mapped onto the pseudo-circle. [4].*

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