Pacific Journal of Mathematics

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Vol. 37, No. 1

January 1971

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A ring of left quotients $Q_{\mathcal{F}}$ of a ring R can be constructed relative to any hereditary torsion class \mathscr{T} of left R-modules. For Morita equivalent rings R and S we construct a one-toone correspondence between the hereditary torsion classes (strongly complete Serre classes) of $_R\mathfrak{M}$ and $_s\mathfrak{M}$ and describe the resulting correspondence between the strongly complete filters of left ideals of R and S. We show that the proper rings of left quotients of R and S relative to corresponding hereditary torsion classes are Morita equivalent. Applications are made to the maximal and the classical rings of left quotients and the corresponding torsion theories.

A torsion theory for the category $_{R}\mathfrak{M}$ of unitary left modules over an associative ring R with identity has been defined by Dickson [3] to be a pair $(\mathcal{T}, \mathcal{F})$ of classes of left R-modules such that

- (a) $\mathcal{T} \cap \mathcal{F} = \{0\}$
- (b) \mathcal{T} is closed under homomorphic images
- (c) \mathcal{T} is closed under submodules

(d) for every left R-module M there exists a submodule T(M) of M with $T(M) \in T$ and $M/T(M) \in \mathcal{F}$.

A class $\mathcal{T}(\mathcal{F})$ of left modules is called a *torsion (torsion-free)* class if there is a (necessarily unique) class $\mathcal{F}(\mathcal{T})$ such that $(\mathcal{T}, \mathcal{F})$ is a torsion theory. A torsion class closed under submodules is said to be *hereditary*. By [3, Theorem 2.3] a class \mathcal{T} is a hereditary torsion class if and only if it is closed under submodules, homomorphic images, extensions, and arbitrary direct sums. Walker and Walker [13] call such a class a *strongly complete Serre class*. Gabriel [4] has shown that for a ring R there is a one-to-one correspondence between the strongly complete Serre classes of $_{R}\mathfrak{M}$ and the strongly complete filters F of left ideals of R given by the mapping

$$\mathscr{T} \longrightarrow F(\mathscr{T}) = \{ I \leq R \mid R/I \in \mathscr{T} \}$$

where $I \leq R$ denotes that I is a left ideal of R. The inverse correspondence is given by

$$F \longrightarrow \mathscr{T}(F) = \{ M \in {}_{R}\mathfrak{M} \mid (0:m) \in F \text{ for all } m \in M \}$$

where $(0: m) = \{r \in R \mid rm = 0\}$. We say a strongly complete filter F of left ideals of R is faithful if $(0: r) \in F$ implies r = 0 for each $r \in R$. A strongly complete Serre class \mathcal{T} is called a faithful Serre

class if $F(\mathcal{T})$ is faithful. Viewing \mathcal{T} as a hereditary torsion class this is equivalent to the requirement that $_{R}R$ is torsion-free.

1. Rings of quotients. Throughout this section \mathscr{T} will denote a faithful Serre class of $_{R}\mathfrak{M}$ with associated filter F. Then $(\mathscr{T}, \mathscr{F})$ is a torsion theory for $_{R}\mathfrak{M}$ and $_{R}R \in \mathscr{F}$ where

$$\mathscr{F} = \{M \in {}_{\scriptscriptstyle R}\mathfrak{M} \mid \operatorname{Hom}_{\scriptscriptstyle R}(T, M) = 0 \text{ for all } T \in \mathscr{T}\}.$$

Let \mathscr{A} denote the quotient category of $_{\mathbb{R}}\mathfrak{M}$ relative to \mathscr{T} as defined in [4] and let

$$R_{\mathcal{F}} = \operatorname{Hom}_{\mathscr{A}}(R, R) = \varinjlim_{I \in F} \operatorname{Hom}_{R}(I, R)$$

the endomorphism ring of R as an object of \mathscr{N} . The opposite ring of $R_{\mathscr{F}}$ is denoted by $Q_{\mathscr{F}}$ and is called the *ring of left quotients* of R relative to \mathscr{T} . The natural ring anti-isomorphism of R and $\operatorname{Hom}_{R}(R, R)$ induces a one-to-one ring homomorphism $\varphi: R \to Q_{\mathscr{F}}$. We usually identify R as a unital subring of $Q_{\mathscr{F}}$. More generally, for each left R-module M let

$$M_{\mathscr{F}} = \operatorname{Hom}_{\mathscr{F}}(R, M) = \lim_{R/I, M' \in \mathscr{F}} \operatorname{Hom}_{R}(I, M/M')$$
.

Using the composition of morphisms in \mathscr{A} each $M_{\mathscr{P}}$ is a right $R_{\mathscr{P}}$ -module and thus a left $Q_{\mathscr{P}}$ -module. The ring homomorphism \mathscr{P} induces a left *R*-module structure on $M_{\mathscr{P}}$ and there is a natural left *R*-homomorphism $\mathscr{P}_{M}: M \to M_{\mathscr{P}}$ given by $\mathscr{P}_{M}(m) = [\rho_{m}]$, the equivalence class of ρ_{m} in $M_{\mathscr{P}}$, where for each $m \in M$, $\rho_{m}: R \to M$ by $\rho_{m}(r) = rm$. As shown in [13] for each left *R*-module *M*, ker $\mathscr{P}_{M} = T(M) = \{m \in M \mid (0: m) \in F\}$.

A left R-module M is said to be \mathcal{T} -injective if for every exact sequence

$$0 \longrightarrow K \longrightarrow L \longrightarrow T \longrightarrow 0$$

of left *R*-modules with $T \in \mathcal{T}$, the associated sequence

$$0 \longrightarrow \operatorname{Hom}_{R}(T, M) \longrightarrow \operatorname{Hom}_{R}(L, M) \longrightarrow \operatorname{Hom}_{R}(K, M) \longrightarrow 0$$

is exact. By [13, Proposition 4.2] for each left *R*-module *M*

$$E_{\mathscr{T}}(M) = \{ x \in E(M) \mid (M: x) \in F \}$$

is \mathscr{T} -injective and is (up to isomorphism) the unique minimal \mathscr{T} -injective module containing M where E(M) is an injective envelope of M. We call $E_{\mathscr{T}}(M)$ a \mathscr{T} -injective envelope of M. The following lemmas are consequences of [4, Proposition 4, page 413] but the proof is included for the sake of completeness.

LEMMA 1.1. For each $M \in \mathscr{F}$, $E_{\mathscr{F}}(M) \cong M_{\mathscr{F}}$ as left R-modules.

Proof. For each $x \in E_{\mathscr{T}}(M)$, $(M: x) \in F$. Define $\lambda: E_{\mathscr{T}}(M) \to M_{\mathscr{T}}$ by $\lambda(x) = [\rho_x]$ for each $x \in E_{\mathscr{T}}(M)$ where $\rho_x(r) = rx$ for each $r \in (M: x)$. It is easily checked that λ is additive.

By [3, Theorem 2.9] \mathscr{F} is closed under injective envelopes. Thus E(M) and hence $E_{\mathscr{F}}(M) \in \mathscr{F}$. If $x \in E_{\mathscr{F}}(M)$ and $\lambda(x) = 0$, then Ix = 0 for some $I \in F$. Since $E_{\mathscr{F}}(M) \in \mathscr{F}$ this implies x = 0. Thus λ is one-to-one.

Let $[f] \in M_{\mathscr{T}}$ be represented by $f: I \to M$ with $I \in F$. Since $E_{\mathscr{T}}(M)$ is \mathscr{T} -injective and contains M, f extends to an R-homomorphism $\overline{f}: R$ $\longrightarrow E_{\mathscr{T}}(M)$. Let $x = \overline{f}(1) \in E_{\mathscr{T}}(M)$. Then $\lambda(x) = [f]$ so λ is onto. Finally, for $x \in E_{\mathscr{T}}(M)$ and $r \in R$ one checks that $\lambda(rx) = r\lambda(x)$. In the special case that $M = {}_{R}R$ we have the following.

LEMMA 1.2. As left R-modules, $Q_{\mathcal{T}} \cong E_{\mathcal{T}}(R)$.

From this we get the following proposition which will be used later in studying Morita equivalence of quotient rings.

PROPOSITION 1.3. If \mathscr{T} is any faithful Serre class of $_{\mathbb{R}}\mathfrak{M}$, then $Q_{\mathscr{T}} \cong \operatorname{End}_{\mathbb{R}}(E_{\mathscr{T}}(\mathbb{R}))^{\circ}$ as rings.

Proof. Let $f \in \operatorname{End}_{R}(Q_{\mathcal{F}})$ and let $q, x \in Q_{\mathcal{F}}$. Then for each $r \in (R; q)$, r(qf(x) - f(qx)) = 0. But $(R; q) \in F$ and $Q_{\mathcal{F}} \in \mathscr{F}$. Thus qf(x) = f(qx). It follows that $\operatorname{End}_{R}(Q_{\mathcal{F}}) = \operatorname{End}_{Q_{\mathcal{F}}}(Q_{\mathcal{F}})$. Using the natural ring antiisomorphism and (1.2) we have

$$Q_{\mathscr{T}}\cong \operatorname{End}_{{\scriptscriptstyle Q}_{\mathscr{T}}}(Q_{\mathscr{T}})^{\scriptscriptstyle 0}=\operatorname{End}_{{\scriptscriptstyle R}}(Q_{\mathscr{T}})^{\scriptscriptstyle 0}\cong \operatorname{End}_{{\scriptscriptstyle R}}(E_{\mathscr{T}}(R))^{\scriptscriptstyle 0}$$
 .

We now investigate more closely the relationship between the ring of left quotients $Q_{\mathcal{T}}$ and the torsion theory $(\mathcal{T}, \mathcal{F})$. As previously noted ker $\varphi_M = T(M)$ for each left *R*-module *M* where φ_M is the natural *R*-homomorphism from *M* to $M_{\mathcal{T}}$. For each left *R*-module *M*, $\varphi_M = \theta_M \eta_M$ where

$$\eta_{\scriptscriptstyle M}: M \longrightarrow Q_{\scriptscriptstyle \mathscr{F}} oldsymbol{\otimes}_{\scriptscriptstyle R} M \ \ \, \mathrm{by} \ \ \, \eta_{\scriptscriptstyle M}(m) = 1 \otimes m$$

and

$$heta_{\scriptscriptstyle M}: Q_{\mathscr{T}} igotimes_{\scriptscriptstyle R} M \longrightarrow M_{\mathscr{T}} \quad ext{by} \quad heta_{\scriptscriptstyle M}(x igodot m) = x arphi_{\scriptscriptstyle M}(m)$$

for each $m \in M$ and each $x \in Q_{\mathscr{T}}$. Thus in general we have ker $\eta_M \subseteq T(M)$.

THEOREM 1.4. Let \mathscr{T} be a strongly complete Serre class of $_{\mathbb{R}}\mathfrak{M}$.

Then $T(M) = \ker \eta_M$ for every left R-module M if and only if $Q_{\mathscr{F}} \varphi(I) = Q_{\mathscr{F}}$ for all $I \in F = F(\mathscr{F})$. Moreover $Q_{\mathscr{F}}$ is flat as a right R-module whenever $T(M) = \ker \eta_M$ for all M.

Proof. If $Q_{\mathscr{P}} \varphi(I) = Q_{\mathscr{P}}$ for all $I \in F$, then θ_M is an isomorphism for each left *R*-module *M* by [13, Theorem 3.2]. Hence ker $\varphi_M = \ker \eta_M = T(M)$ for every *M*.

Conversely if ker $\eta_M = T(M)$ for every left *R*-module *M*, then $R/I = \ker \eta_{R/I}$ for each $I \in F$. Thus $Q_{\mathscr{T}} \bigotimes_R R/I = 0$ for every $I \in F$. Hence for each $I \in F$ the mapping $Q_{\mathscr{T}} \bigotimes_R I \longrightarrow Q_{\mathscr{T}} \bigotimes_R R$ is an isomorphism. Thus $Q_{\mathscr{T}} = Q_{\mathscr{T}} \mathscr{P}(I)$ for each $I \in F$. The last remark follows by [13, Corollary 3.3].

We conclude this section indicating two important special cases of this result.

A left ideal I of R is said to be *dense* if $(I:a)b \neq 0$ for all a, b in R with $b \neq 0$. The strongly complete faithful filter D of denseleft ideals of R is maximal among all the strongly complete faithful filters of left ideals of R. The corresponding faithful Serre class

$$\mathscr{T}' = \{ M \in {}_{\scriptscriptstyle R}\mathfrak{M} \mid (0:m) \in D \text{ for all } m \in M \}$$

is thus maximal among all the faithful Serre classes of $_{R}\mathfrak{M}$ and coincides with the E(R)-torsion class considered by Jans [6]. The ring of left quotients of R relative to \mathscr{T}' is called the maximal ring of left quotients of R and is denoted by $Q(_{R}R)$.

For each left *R*-module $_{R}M$ we let $Z(_{R}M)$ denote the set of all elements of $_{R}M$ whose annihilator is an essential left ideal of *R*. Then $Z(_{R}M)$ is a submodule of $_{R}M$ called the *singular submodule* of $_{R}M$. For a ring *R* with $Z(_{R}R) = 0$, a left ideal is dense if and only if it is essential. For such rings $Q(_{R}R)$ is von Neumann regular. (See [7]) Moreover for a ring *R* with $Z(_{R}R) = 0$, $Q(_{R}R)$ is semisimple (with minimum condition) if and only if $Q(_{R}R)I = Q(_{R}R)$ for all essential left ideals of *R* by [11, Theorem 1.6] or [13, Theorem 4.19]. Combining these facts with (1.4) we get the following results of Sandomierski [11].

PROPOSITION 1.5. Let R be a ring with $Z(_{R}R) = 0$. Then Z(M) =ker η_{M} where $\eta_{M}: M \longrightarrow Q(_{R}R) \bigotimes_{R} M$ via $\eta_{M}(m) = 1 \bigotimes m$ for every left R-module M if and only if $Q(_{R}R)$ is semisimple. Moreover, if $Q(_{R}R)$ is semisimple it is flat as a right R-module.

Let U denote the set of two-sided nonzero divisors of R, let $F_c = \{I \leq R \mid I \cap U \neq \emptyset\}$ and let

$$\mathscr{T}_{\scriptscriptstyle C} = \{M \in {}_{\scriptscriptstyle R}\mathfrak{M} \mid (0; m) \in F_{\scriptscriptstyle C} \quad \text{for all} \quad m \in M\}$$
.

A ring R is said to be *left* Ore if for all $a \in R$ and $d \in U$ there exist

 $a' \in R$ and $d' \in U$ such that d'a = a'd. One checks that F_c is a strongly complete faithful filter of left ideals of R and \mathscr{T}_c is a faithful Serre class of $_R\mathfrak{M}$ if and only if R is left Ore. For any left Ore ring R, the ring of left quotients of R relative to \mathscr{T}_c is denoted by $Q_c(R)$ and is called the *classical ring of left quotients* of R. For a left Ore ring R, $Q_c(R)$ has the following properties:

(a) $d \in U$ implies d^{-1} exists in $Q_c(R)$

(b) for each $q \in Q_c(R)$, there exists $a \in R$ and $d \in U$ with $q = d^{-1}a$. For a left Ore ring R, every $I \in F_c$ contains an invertible element of $Q_c(R)$. Hence $Q_c(R)I = Q_c(R)$ for every $I \in F_c$. Applying (1.4) we have the following results of Levy [8].

PROPOSITION 1.6. Let R be a left Ore ring. Then for each left R-module M, the kernel of the mapping $\eta_M: M \longrightarrow Q_c(R) \bigotimes_R M$ defined by $\eta_M(m) = 1 \bigotimes m$ is $T_c(M) = \{m \in M \mid (0:m) \in F_c\}$. Moreover $Q_c(R)$ is flat as a right R-module.

2. Morita equivalence of quotient rings. Morita has shown that two rings R and S have equivalent categories of unitary left modules if and only if $S \cong \operatorname{End}_R(P_R)$ for some right R-progenerator P_R where a right R-module P_R is called a progenerator if it is finitely generated projective and if the right regular module R_R is isomorphic to a direct summand of a direct sum of copies of P_R . (See [1] or [10]) Two such rings are said to be Morita equivalent. Throughout this paper we assume $S = \operatorname{End}_R(P_R)$ with P_R a progenerator. Then the functors

$$G = P \bigotimes_{R} (): {}_{R} \mathfrak{M} \longrightarrow {}_{S} \mathfrak{M}$$

and

$$H = P^* \bigotimes_{s} (): {}_{s}\mathfrak{M} \longrightarrow {}_{R}\mathfrak{M}$$

are inverse category equivalences where $P^* = \operatorname{Hom}_R(P, R)$ is a left *R*-progenerator.

If $\mathcal{J}(R)$ is any strongly complete Serre class of $_{R}\mathfrak{M}$, then

$$\mathscr{T}(S) = \{ M \in {}_{s}\mathfrak{M} \mid H(M) \in \mathscr{T}(R) \}$$

is a strongly complete Serre class of ${}_{S}\mathfrak{M}$ since H preserves exactness and direct sums. The mapping pairing each $\mathscr{T}(R)$ with $\mathscr{T}(S)$ as defined above gives a one-to-one correspondence between the strongly complete Serre classes of ${}_{R}\mathfrak{M}$ and ${}_{S}\mathfrak{M}$. Henceforth $\mathscr{T}(R)$ and $\mathscr{T}(S)$ will denote corresponding strongly complete Serre classes of ${}_{R}\mathfrak{M}$ and ${}_{S}\mathfrak{M}$ respectively. By our introductory remarks there are (unique) classes $\mathscr{F}(R)$ and $\mathscr{F}(S)$ such that ($\mathscr{T}(R), \mathscr{F}(R)$) and ($\mathscr{T}(S), \mathscr{F}(S)$) are hereditary torsion theories for ${}_{R}\mathfrak{M}$ and ${}_{S}\mathfrak{M}$ respectively. Moreover, $\mathscr{F}(S) = \{M \in {}_{s}\mathfrak{M} \mid H(M) \in \mathscr{F}(R)\}$.

PROPOSITION 2.1. $\mathcal{T}(R)$ is faithful if and only if $\mathcal{T}(S)$ is faithful.

Proof. If $\mathscr{T}(R)$ is faithful, then $_{R}R \in \mathscr{F}(R)$. Hence by [3, Theorem 2.3] every finitely generated projective left *R*-module is in $\mathscr{F}(R)$. But $H(_{s}S) \cong _{R}P^{*}$ is a finitely generated projective left *R*-module, so $H(_{s}S) \in \mathscr{F}(R)$. Thus $_{s}S \in \mathscr{F}(S)$, so $\mathscr{T}(S)$ is faithful. The converse follows by a dual argument.

Throughout the remainder of this paper unless otherwise noted we restrict our attention to the case where $\mathscr{T}(R)$ and $\mathscr{T}(S)$ and faithful.

We let $Q_{\mathscr{T}(R)}$ and $Q_{\mathscr{T}(S)}$ denote the rings of left quotients of R and S relative to $\mathscr{T}(R)$ and $\mathscr{T}(S)$ respectively as defined in § 1. Before examining the Morita equivalence of $Q_{\mathscr{T}(R)}$ and $Q_{\mathscr{T}(S)}$ we need a few observations on \mathscr{T} -injectivity. Using routine arguments with the category equivalences G and H one gets the following.

LEMMA 2.2. Let M be a left R-module. Then M is $\mathcal{T}(R)$ - injective if and only if G(M) is $\mathcal{T}(S)$ -injective.

PROPOSITION 2.3. Let M be a left R-module with $\mathcal{T}(R)$ -injective envelope $E_{\mathcal{T}(R)}(M)$. Then $G(E_{\mathcal{T}(R)}(M)$ is a $\mathcal{T}(S)$ -injective envelope of G(M).

Proof. By the lemma, $G(E_{\mathscr{T}(R)}(M))$ is a $\mathscr{T}(S)$ -injective extension of G(M). Using the fact that G induces an isomorphism between the lattices of submodules of $E_{\mathscr{T}(R)}(M)$ and $G(E_{\mathscr{T}(R)}(M))$ one checks that $G(E_{\mathscr{T}(R)}(M))$ is a minimal $\mathscr{T}(S)$ -injective extension of G(M).

Two left *R*-modules *M* and *N* are said to be *similar* if each is isomorphic to a direct summand of a finite direct sum of copies of the other. Observing that finite direct sums of $\mathscr{T}(R)$ -injective modules are $\mathscr{T}(R)$ -injective one checks that similar left *R*-modules have similar $\mathscr{T}(R)$ -injective envelopes. Since the left *R*-module $_{R}P^{*}$ is a progenerator and is thus similar to $_{R}R$ we have $E_{\mathscr{T}(R)}(_{R}P^{*})$ is similar to $E_{\mathscr{T}(R)}(_{R}R)$.

To simplify our notation we let $E_{\mathscr{T}}(R) = E_{\mathscr{T}(R)}(_{\mathbb{R}}R)$, $E_{\mathscr{T}}(P^*) = E_{\mathscr{T}(R)}(_{\mathbb{R}}P^*)$ and $E_{\mathscr{T}}(S) = E_{\mathscr{T}(S)}(_{S}S)$. Then using (2.3) and the fact that $G(P^*) \cong {}_{S}S$, we have

$$\operatorname{End}_{R}(E_{\mathscr{F}}(P^{*})) \cong \operatorname{End}_{S}(G(E_{\mathscr{F}}(P^{*})))$$

 $\cong \operatorname{End}_{S}(E_{\mathscr{F}}(G(P^{*})))$
 $\cong \operatorname{End}_{S}(E_{\mathscr{F}}(S))$.

Thus by (1.3)

$$Q_{\mathscr{T}(R)}\cong \operatorname{End}_{R}(E_{\mathscr{T}}(R))^{\mathfrak{o}}$$

and

$$Q_{\mathscr{T}(S)}\cong \operatorname{End}_{S}(E_{\mathscr{T}}(S))^{\circ}\cong \operatorname{End}_{R}(E_{\mathscr{T}}(P^{*}))^{\circ}.$$

Hirata [5, Theorem 1.5] has shown that for similar left *R*-modules M and N, the rings $E = \operatorname{End}_R(M)^\circ$ and $E' = \operatorname{End}_R(N)^\circ$ are Morita equivalent. (The opposite rings arise from our convention of regarding mappings as operating on the left.) Moreover $\operatorname{Hom}_R(M, N)$ is a progenerator both as a left *E*-module and as a right *E'*-module. Similarly $\operatorname{Hom}_R(N, M)$ is a progenerator both as a left *E*-module and as a left *E'*-module and as a right *E'*-module.

Letting $M = E_{\mathscr{F}}(R)$ and $N = E_{\mathscr{F}}(P^*)$ we conclude that the rings $Q_{\mathscr{F}(R)}$ and $Q_{\mathscr{F}(S)}$ are Morita equivalent and that $\operatorname{Hom}_R(E_{\mathscr{F}}(P^*), E_{\mathscr{F}}(R))$ is a progenerator both as a left $Q_{\mathscr{F}(S)}$ -module and as a right $Q_{\mathscr{F}(R)}$ -module.

Since $P \bigotimes_{\mathbb{R}} E_{\mathscr{T}}(\mathbb{R})$ is $\mathscr{T}(S)$ -injective and

$$0 \longrightarrow S \longrightarrow E_{\mathscr{T}}(S) \longrightarrow E_{\mathscr{T}}(S) / S \longrightarrow 0$$

is an exact sequence of left S-modules with $E_{\mathcal{T}}(S)/S \in \mathcal{T}(S)$,

$$0 \longrightarrow \operatorname{Hom}_{S}(E_{\mathscr{F}}(S)/S, P \bigotimes_{R} E_{\mathscr{F}}(R)) \longrightarrow \operatorname{Hom}_{S}(E_{\mathscr{F}}(S), P \bigotimes_{R} E_{\mathscr{F}}(R)) \longrightarrow \operatorname{Hom}_{S}(S, P \bigotimes_{R} E_{\mathscr{F}}(R)) \longrightarrow 0$$

is an exact sequence of right $Q_{\mathscr{F}(R)}$ -modules. But $\operatorname{Hom}_{S}(E_{\mathscr{F}}(S)/S, P\bigotimes_{R}E_{\mathscr{F}}(R)) = 0$ since $E_{\mathscr{F}}(S)/S \in \mathscr{F}(S)$ and $P\bigotimes_{R}E_{\mathscr{F}}(R) \in \mathscr{F}(S)$. Hence as a right $Q_{\mathscr{F}(R)}$ -module

$$\operatorname{Hom}_{R}(E_{\mathscr{F}}(P^{*}), E_{\mathscr{F}}(R)) \cong \operatorname{Hom}_{S}(E_{\mathscr{F}}(S), P\bigotimes_{R} E_{\mathscr{F}}(R))$$
$$\cong \operatorname{Hom}_{S}(S, P\bigotimes_{R} E_{\mathscr{F}}(R))$$
$$\cong P\bigotimes_{R} E_{\mathscr{F}}(R) \cong P\bigotimes_{R} Q_{\mathscr{F}(R)} .$$

Summarizing, we have the following theorem.

THEOREM 2.4. Let $\mathscr{T}(R)$ be a faithful Serre class of $_{\mathbb{R}}\mathbb{M}$ and let $\mathscr{T}(S)$ be the corresponding faithful Serre class of $_{s}\mathbb{M}$. Then the rings of left quotients $Q_{\mathscr{T}(R)}$ and $Q_{\mathscr{T}(S)}$ are Morita equivalent. Moreover $P\bigotimes_{\mathbb{R}}Q_{\mathscr{T}(R)}$ is a right $Q_{\mathscr{T}(R)}$ -progenerator with

$$Q_{\mathcal{F}(S)} \cong \operatorname{End}_{Q_{\mathcal{F}(R)}}(P \bigotimes_{R} Q_{\mathcal{F}(R)})$$
.

Let F_R be a free right *R*-module of rank *n*. Then $\operatorname{End}_R(F_R) \cong R_n$ and $\operatorname{End}_{Q_{\mathscr{T}(R)}}(F\bigotimes_R Q_{\mathscr{T}(R)}) \cong (Q_{\mathscr{T}(R)})_n$.

COROLLARY 2.5. Let
$$\mathcal{T}(R)$$
 be a faithful Serre class of $_{\mathbb{R}}\mathfrak{M}$ and

let $\mathscr{T}(R_n)$ be the corresponding faithful Serre class of $_{R_n}\mathfrak{M}$. Then $Q_{\mathscr{T}(R_n)}\cong (Q_{\mathscr{T}(R)})_n$.

Previously in this section we described a one-to-one correspondence between the strongly complete Serre classes of $_{R}\mathfrak{M}$ and $_{S}\mathfrak{M}$. We conclude this section by describing the resulting correspondence between the strongly complete filters of left ideals of R and S.

By hypothesis $S = \operatorname{End}_{R}(P_{R})$ with P_{R} a progenerator. Since P_{R} is finitely generated and projective, by the Dual Basis Lemma [2, Proposition VII, 3.1] there exist $x_{1}, \dots, x_{n} \in P$ and $f_{1}, \dots, f_{n} \in P^{*}$ such that

$$x = \sum x_i f_i(x)$$
 and $f = \sum f(x_i) f_i$

for all $x \in P$ and all $f \in P^*$.

For each left ideal I of R, let

 $\overline{I} = \{s \in S \mid s(x_i) \in PI \text{ for all } i = 1, \dots, n\} = \cap (0; s\overline{x}_i)$

where \bar{x}_i is the canonical image in P/PI of x_i . Similarly, for each left ideal J of S, let

$$ar{J}=\{r\in R\mid rf_i\in P^*J ext{ for all } i=1,\,\cdots,\,n\}=\ \cap\ (0\colon_{\scriptscriptstyle R}ar{f}_i)$$

where \overline{f}_i is the canonical image in P^*/P^*J of f_i .

If $I \in F(R)$, the strongly complete filter of left ideals corresponding to $\mathscr{T}(R)$, then $G(R/I) = P \bigotimes_{\mathbb{R}} R/I \cong P/PI \in \mathscr{T}(S)$. Thus $(0: {}_{S}\overline{x}_{i}) \in F(S)$, the strongly complete filter of left ideals corresponding to $\mathscr{T}(S)$, for all $i = 1, \dots, n$. It follows that $\overline{I} \in F(S)$.

Similarly, if $J \in F(S)$, then $H(S/J) = P^* \bigotimes_s S/J \cong P^*/P^*J \in \mathscr{T}(R)$. Thus (0: $_R \overline{f}_i) \in F(R)$ for all $i = 1, \dots, n$. Thus $\overline{J} \in F(R)$.

Finally, if $J \in F(S)$ and $I = \overline{J}$ one checks that $\overline{I} \leq J$. Thus we have shown the following.

PROPOSITION 2.6. Let $\mathscr{T}(R)$ and $\mathscr{T}(S)$ be corresponding strongly complete Serre classes of $_{\mathbb{R}}\mathfrak{M}$ and $_{S}\mathfrak{M}$ with associated filters of left ideals F(R) and F(S) and let J be a left ideal of S. Then $J \in F(S)$ if and only if there exists an $I \in F(R)$ with $\overline{I} \leq J$.

3. Applications. In this section the results of the preceding section and applied to the maximal and the classical rings of left quotients.

Let $\mathcal{T}'(R)$ and $\mathcal{T}'(S)$ denote the maximal faithful Serre classes of $_{\mathbb{R}}\mathfrak{M}$ and $_{S}\mathfrak{M}$. By virtue of their maximality $\mathcal{T}'(R)$ and $\mathcal{T}'(S)$ correspond as in §2. Hence as a special case of (2.4) we have the following.

THEOREM 3.1. The maximal rings of left quotients of Morita

equivalent rings are Morita equivalent.

COROLLARY 3.2. Let R and S be Morita equivalent rings. Then $Q(_{R}R)$ is von Neumann regular if and only if $Q(_{s}S)$ is von Neumann regular. Consequently, $Z(_{R}R) = 0$ if and only if $Z(_{s}S) = 0$.

In the following let R be a left Ore ring and let $\mathscr{T}_{c}(R)$ and $F_{c}(R)$ be as defined in §1. As usual let $S = \operatorname{End}_{R}(P_{R})$ with P_{R} a right R-progenerator. It is unknown whether S is necessarily left Ore. Indeed, we do not know whether the ring of $n \times n$ matrices over a left Ore ring is left Ore for n > 1 unless additional requirements are placed on $Q_{c}(R)$. (See Small [12, Theorem 2.28]) As a partial result we shall show that S is left Ore if R is commutative.

As indicated in §2,

 $\mathscr{T}(S) = \{ M \in {}_{S}\mathfrak{M} \mid H(M) \in \mathscr{T}_{C}(R) \}$

is a faithful Serre class of ${}_{S}\mathfrak{M}$ with associated filter F(S) given by

$$F(S) = \{J \leq S \mid ar{I} \leq J \;\; ext{ for some } \;\; I \in {F_{\scriptscriptstyle C}(R)}\}$$
 .

Let

$$F_c(S) = \{J \leq S \mid J \cap U(S) \neq \emptyset\}$$

where U(S) denotes the set of nonzero divisors of S and let

$${\mathscr T}_{\scriptscriptstyle C}(S)=\{M\in {}_{\scriptscriptstyle S}\mathfrak{M}\mid (0{\rm :}\;m)\in {F}_{\scriptscriptstyle C}(S)\quad {\rm for \; all}\quad m\in M\} \;.$$

If $\mathscr{T}_c(S) = \mathscr{T}(S)$ or equivalently if $F_c(S) = F(S)$, then S is left Ore and $Q_c(R)$ and $Q_c(S)$ are Morita equivalent.

THEOREM 3.3. If R is commutative, then S is left Ore and $Q_c(R)$ and $Q_c(S)$ are Morita equivalent.

Proof. We show $F_c(S) = F(S)$. Let $J \in F(S)$. Then there exists $I \in F_c(R)$ with $\overline{I} \leq J$. Let $d \in I \cap U(R)$ and define $\rho_d \in S$ by $\rho_d(x) = xd$ for each $x \in P$. Then $\rho_d \in \overline{I}$ since $\rho_d(x) \in PI$ for all $x \in P$. For all $s \in S$ and all $x \in P$, $\rho_d s(x) = s\rho_d s(x) = s(x)d$. If $\rho_d s = 0$ then $f_i(s(x))d = 0$ for $i = 1, \dots, n$. Since $d \in U(R)$ and $f_i(s(x)) \in R$ this implies that $f_i(s(x)) = 0$ for $i = 1, \dots, n$. Therefore $s(x) = \sum x_i f_i(s(x)) = 0$ for all $x \in P$. Hence s = 0 so $\rho_d \in U(S)$. Thus $\rho_d \in J \cap U(S)$ so $J \in F_c(S)$.

Conversely, let $J \in F_c(S)$ and let $s \in J \cap (S)$. Let F_R be a free right *R*-module of rank *n* with $F_R = P_R \bigoplus P_R'$ for some P_R' and let $\Lambda: \operatorname{End}_R(F_R) \to R_n$ be a unital ring isomorphism. Using the fact that P_R is a progenerator one checks that $\overline{s} \in \operatorname{End}_R(F_R)$ defined by $\overline{s}(p, p') =$ (s(p), p') is a nonzero divisor of $\operatorname{End}_R(F_R)$. Since $\Lambda(\overline{s})$ is a nonzero divisor of R_n and R is commutative, det $\Lambda(\bar{s}) \in U(R)$. (See McCoy [9]). Thus letting I = Rd, we have $I \in F_c(R)$. Let s' denote the restriction of Λ^{-1} (adj $\Lambda(\bar{s})$) to P_R . Then $s's = \rho_d$ where $\rho_d(x) = xd$ for each $x \in P$ and since $s \in J$, $\rho_d \in J$. Let $t \in \bar{I}$. Define $t' \in S$ by

$$t'(x) = \sum_{i,j=1}^n x_j r_{ij} f_i(x)$$
 for each $x \in P$ where
 $t(x_i) = \sum_{j=1}^n x_j r_{ij} d \in PI$ for $i = 1, \dots, n$.

Then one checks that $t = t'\rho_d$ and since $\rho_d \in J$, $t \in J$. Hence $\overline{I} \leq J$ so $J \in F(S)$ by (2.6). Therefore $F_c(S) \subseteq F(S)$. Thus we have shown that $F_c(S) = F(S)$ and by our previous remarks the theorem follows.

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Received December 4, 1969. Portions of this paper are from the author's doctoral dissertation written at the University of Oregon under the direction of F. W. Anderson.

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Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics Vol. 37, No. 1 January, 1971

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