Pacific Journal of Mathematics

ON A PROBLEM OF DANZER

RAM PRAKASH BAMBAH AND ALAN C. WOODS

Vol. 37, No. 2 February 1971

PACIFIC JOURNAL OF MATHEMATICS
Vol. 37, No. 2, 1971

ON A PROBLEM OF DANZER

R. P. BAMBAH AND A. C. WOODS

By a Danzer set S we shall mean a subset of the n-dimensional Euclidean space R_n which has the property that every closed convex body of volume one in R_n contains a point of S. L. Danzer has asked if for $n \geqq 2$ there exist such sets S with a finite density. The answer to this question is still unknown. In this note our object is to prove two theorems about Danzer sets.

If Λ is a n-dimensional lattice, any translate $\Gamma = \Lambda + p$ of Λ will be called a grid Γ; Λ will be called the lattice of Γ and the determinant $d(\Lambda)$ of Λ will be called the determinant of Γ and will be denoted by $d(\Gamma)$. In § 2 we prove

THEOREM 1. **For $n \geqq 2$, a Danzer set cannot be the union of a finite number of grids.**

Let S be a Danzer set and $X > 0$ a positive real number. Let $N(S, X)$ be the number of points of S in the box $\max_{1 \leqq i \leqq n} |x_i| \leqq X$. Let $D(S, X) = N(S, X)/(2X)^n$. In § 3 we prove

THEOREM 2. **There exist Danzer sets S with $D(S, X) = 0((\log X)^{n-1})$ as $X \to \infty$.**

The case $n = 2$ of the theorem is known, although no proof seems to have been published. The referee has pointed out that a lower bound of 2 can easily be established for the density of a Danzer set in $n = 2$, but the authors are unaware of any further results in this direction.

2. Proof of Theorem 1. We shall assume throughout that $n \geqq 2$. It is obvious that if S is a Danzer set and T is a volume preserving affine transformation of R_n onto itself, then $T(S)$ is also a Danzer set.

Let $S_1, S_2, \cdots$ be a sequence of sets in R_n. Let S be the set of points X such that there exists a subsequence $S_{i_1}, S_{i_2}, \cdots$ of $\{S_r\}$ and points $X_{i_r} \in S_{i_r}$, such that $X_{i_r} \to X$ as $r \to \infty$. We write

$$S = \lim_{r \to \infty} S_r = \lim S_r .$$

LEMMA 1. *Let $\{S_r\}$ be a sequence of Danzer sets in R_n. Then $S = \lim S_r$ is also a Danzer set.*

Proof. Let K be a closed convex body of Volume 1. Then for each r, $K \cap S_r \neq \phi$, so that for each r, there exists $X_r \in K \cap S_r$. Since K is compact, $\{X_r\}$ has a convergent subsequence $\{X_{i_r}\}$ converging to a point X in $K \cap S$.

LEMMA 2. *Let* $S^{(j)} = \lim_{r\to\infty} S_r^{(j)}$, $j = 1, \cdots, k$. *Then*

$$\bigcup_{j=1}^{k} S^{(j)} = \lim_{r\to\infty}\left(\bigcup_{j=1}^{k} S_r^{(j)}\right).$$

Proof. $X \in \cup S^{(j)} \Rightarrow X \in S^{(j)}$ for some j, say $j = j_0 \Rightarrow$ there exist a subsequence $\{S_{i_r}^{(j_0)}\}$ of $\{S_{i_r}^{(j_0)}\}$ and points $X_{i_r} \in S_{i_r}^{(j_0)}$ such that $X_{i_r} \to X \Rightarrow X_{i_r} \in \cup S_{i_r}$ and $X_{i_r} \to X \Rightarrow X \in \lim_{r\to\infty}(\bigcup_{j=1}^{k} S_r^{(j)})$. Thus $\cup S^{(j)} \subset \lim(\bigcup_{j=1}^{k} S_r^{(j)})$. Let $X \in \lim(U_{j=1}^{k} S_r^{(i)})$. Then there exists a sequence $\{i_r\}$ of natural numbers and $X_{i_r} \in \bigcup_{j=1}^{k} S_{i_r}^{(j)}$ such that $X_{i_r} \to X$. Since k is finite, there exists a $j = j_0$ say, and an infinite subsequence k_r of i_r such that $X_{k_r} \in S_{k_r}^{(j_0)}$. Then $X_{k_r} \to X$ and $X \in S^{(j_0)}$, so that $X \in \cup S^{(j_0)}$ and $\lim_{r\to\infty}(\bigcup_{j=1}^{k} S_s^{(j)}) \subset \cup S^{(j)}$.

This completes the proof of the lemma.

LEMMA 3. *Let* $\Gamma_1, \Gamma_2, \cdots$ *be a sequence of grids in* R_n *with equal determinants* $d(\Gamma_r) = \Delta$. *Then* $\{\Gamma_r\}$ *has a subsequence* $\{\Gamma_{i_r}\}$, *such that* $\lim_{r\to\infty} \Gamma_{i_r}$ *is either a grid or is contained in a hyperplane.*

Proof. If $\lim_{r\to\infty} \Gamma_r = \phi$, there is nothing to prove. Assume, therefore, that $\Gamma = \lim_{r\to\infty} \Gamma_r \neq \phi$. Let $X \in \Gamma$. Then there exists a subsequence $\{i_r\}$ of natural numbers and points $X_{i_r} \in \Gamma_{i_r}$, such that $X_{i_r} \to X$. Then $\Lambda_{i_r} = \Gamma_{i_r} - X_{i_r}$ is a sequence of homogeneous lattices and $\lim \Gamma_{i_r} = X + \lim \Lambda_{i_r}$. Therefore, it is enough to prove the theorem for lattices.

Let $\{\Lambda_r\}$ be a sequence of lattices with determinants $d(\Lambda_r) = \Delta$, independent of r. Let $\mu_1(\Lambda_r), \cdots, \mu_n(\Lambda_r)$ be the successive minima of the Euclidean distance with respect to Λ_r, i.e., $\mu_i(\Lambda_r) = \inf \mu$: such that $|X| < \mu$ has i linearly independent points of Λ_r.

Suppose, first, that there exists $\delta > 0$, such that $\mu_1(\Lambda_r) \geqq \delta$ for infinitely many r. Then a subsequence satisfies the conditions of Mahler's compactness theorem and has a subsequence convergent in the sense of Mahler (see, e.g., Cassels [2]). The last subsequence converges to the limiting lattice in our sense also.

We may, therefore, assume $\mu_1(\Lambda_r) \to 0$ as $r \to \infty$. Since

$$\mu_1(\Lambda_r) \cdots \mu_n(\Lambda_r) \geqq \frac{2^n}{n!} \cdot \frac{1}{J_n},$$

where J_n is the volume of the sphere $|X| < 1$, (see, e.g., Cassels [2]), and since $n \geqq 2$, it follows that $\mu_n(A_r) \to \infty$ as $r \to \infty$. For each r, let $P_{r_1}, \cdots, P_{r_n}$ be points such that $|P_{r_i}| = \mu_i(A_r)$. Let π_r be the plane through $0, p_{r_1}, \cdots, p_{r_{n-1}}$. It is easily seen that there exists a subsequence $\{A_{i_r}\}$ of $\{A_r\}$ such that the sequence $\{\pi_{i_r}\}$ converges to a plane π. We assert that $\lim_{r\to\infty}\{A_{i_r}\} \subset \pi$. For, let $X \in \lim_{r\to\infty} A_{i_r}$. Then $X = \lim X_{k_r}$, where k_r is a subsequence of i_r and $X_{k_r} \in A_{k_r}$. There exists M independent of k_r, such that $|X_{k_r}| \leqq M$ for all k_r. Also

$$X_{k_r} = g_{r,1}P_{k_r,1} + \cdots + g_{r,n}P_{k_r,n},\ g_{r,i} \text{ real },$$

and if $g_{r,n} \neq 0$ then $|X_{k_r}| \geqq \mu_n(A_{k_r})$. Since $\mu_n(A_{k_r}) \to \infty$ as $r \to \infty$, $g_{r,n} = 0$ for all large r and $X \in \pi$. This proves the lemma

LEMMA 4. *Let $\{\pi_i\}$ be a sequence of hyperplanes. Then $\{\pi_i\}$ has a subsequence $\{\pi_{i_\mu}\}$ whose limit lies in a hyperplane.*

Proof. If $\pi = \lim_{i\to\infty} \pi_i = \phi$ then there is nothing to prove. Assume, therefore, $X \in \pi$. Then there is a subsequence $\{k_r\}$ of natural numbers and points $X_{k_r} \in \pi_{k_r}$ such that $X_{k_r} \to X$. The planes $\hat{\pi}_{k_r} = \pi_{k_r} - X_{k_r}$ pass through 0 and have a subsequence $\hat{\pi}_{i_r}$ which converges to a plane $\hat{\pi}$ say. Then $\lim_{r\to\infty} \pi_{i_r} = \hat{\pi} + X$. This proves the lemma.

Proof of Theorem 1. We shall prove more, namely, *a Danzer set cannot be the union of a finite number of hyperplanes and a finite number of grids.*

Let $S = \bigcup_{i=1}^{r} \pi_i \bigcup_{j=1}^{t} \Gamma_j$ be a Danzer set, such that π_i are hyperplanes and Γ_j are grids. Let $t \geqq 1$. Let $X \neq Y$, X, $Y \in \Gamma_1$. For each positive integer k, let T_k be a volume preserving affine transformation such that $T_k(X) = X$ and $|T_k(Y) - X| = k^{-1}|Y - X|$. Since $n \geqq 2$, such transformations exist. For each k, $T_k(S)$ is a Danzer set, and by Lemma 1, so is the limit of every subsequence of $\{T_k(S)\}$. By Lemmas 3 and 4 we can choose a subsequence $\{T_{k_r}\}$ of $\{T_k\}$ such that each $\lim_{t\to\infty} T_{k_r}(\pi_i)$ lies in a hyperplane, while each $\lim_{t\to\infty} T_{k_r}(\Gamma_j)$ is either a grid or lies in a hyperplane. Since

$$\lim_{r\to\infty} T_{k_r}(S) = \bigcup_{i=1}^{t} \lim T_{k_r}(\pi_i) \bigcup_{j=1}^{t} \lim T_{k_r}(\Gamma_j)$$

and $\lim T_{k_r}(\Gamma_1)$ is in a hyperplane, the Danzer set $\lim T_{k_r}(S)$ lies in the union of a finite number of hyperplanes and $t_1 < t$ grids, so that we have (by increasing $T_{k_r}(S)$ if necessary) a Danzer set consisting of a finite number of hyperplanes and $t_1 < t$ grids. Repeating this process a number of times we obtain a Danzer set that is the union of a finite number of hyperplanes. This can easily be seen to lead to a contradiction which proves the theorem.

3. Proof of Theorem 2. Let K be a closed convex body in R_n. The set $S \subset R_n$ is said to be a covering set for K if $R_n \subset \bigcup_{A \in S} (K + A)$. The set S contains a point of each translate of K if and only if S is a covering set for $-K$. Clearly a set S is a Danzer set if and only if it is a covering set for each closed convex body of volume one. Therefore, in order to prove a given set S is a Danzer set, it is enough to prove that for every closed convex body K of volume one, S contains a covering set for K.

If Γ is a grid with lattice Λ, then it is easy to see that Γ is a covering set for K if and only if Λ is.

Let π be a parallelepiped. Let A_0 be one of its vertices and $A_1, \cdots, A_n$ be the n vertices joined to A_0 by edges of π. Let Λ be the lattice generated by $A_1 - A_0, \cdots, A_n - A_0$. By the grid generated by π we shall mean the grid $\Lambda + A_0$. It is easily seen that if a closed convex body K contains a parallelepiped which generates a grid Γ, then Γ is a covering set for K.

A lattice Λ will be called rectangular if it consists of points $(\alpha_1 u_1, \cdots, \alpha_n u_n)$, where α_i are fixed positive real numbers and u_i take integral values. A grid Γ will be called rectangular if its lattice is rectangular.

Let $\alpha_1, \cdots, \alpha_n$ be positive real numbers. Let Γ_α be the grid generated by the parallelepiped $|x_i| \leqq \alpha_i$. Let B be a box $|x_i| \leqq \beta_i$, where $\beta_i \geqq \alpha_i$ for $i = 1, \cdots, n$. Then Γ_α is clearly a covering set for B.

Let K be a closed convex body of volume one. Let K_1 be the steiner symmetrical of K with respect to the plane $x_1 = 0$. Let K_2 be the steiner symmetrical of K_1 with respect to $x_2 = 0$ and so on. Then K_n is symmetrical about all the coordinate planes and has volume one. We next have

LEMMA 5. *If a rectangular lattice Λ is a covering set for K_n, then it is a covering set for K also.*

(The lemma and its proof are easy adaptions of Lemma 2 of Sawyer (3). For completeness, we give the proof below).

Proof. Let Λ be the rectangular lattice consisting of points $(\alpha_1 u_1, \cdots, \alpha_n u_n)$, $\alpha_i > 0$ fixed real numbers and u_i running over the set of integers. It is enough to prove that if Λ is a covering set for K_1, then it is a covering set for K also.

Let $\Lambda_1 =$ subset of Λ in the plane $x_1 = 0$. The sets $K_1 + \Lambda$ cover R_n. We assert each line $x_2 = a_2, \cdots, x_n = a_n$ meets $K_1 + P$ is a segment of length at least α_1 for some $P \in \Lambda_1$. Such a line meets only a finite number of translates $K_1 + P_s$, $P_s \in \Lambda_1$, each of them in a segment $|x_1| \leqq b_s$ and hence meets $K_1 + \Lambda_1$ in the segment $|x_1| \leqq b = \max b_s$. If $b < \frac{1}{2}\alpha_1$, then $K_1 + \Lambda$ meets the line in segments $|x_1 - m\alpha_1| \leqq b < \frac{1}{2}\alpha_1$, where m takes integral values. This leaves part of the line uncovered by sets $K_1 + \Lambda$, contrary to the fact that Λ is a covering set for K_1. Thus $b \geqq \frac{1}{2}\alpha_1$, i.e., $b_s \geqq \frac{1}{2}\alpha_1$ for some s. Therefore, the line meets $K_1 + P_s$ and hence $K + P_s$ in a segment of length at least α_1, and is therefore, covered by the sets $K + \Lambda$. Since this is true for all such lines, Λ is a covering set for K.

COROLLARY. *A rectangular grid Γ which is a covering set for K_n is also a covering set for K.*

Because of the corollary, in oder to prove that a given set S is a Danzer set, it is enough to prove that for every given closed convex body K of volume one, which is symmetrical about all the coordinate planes, S contains a rectangular grid Γ which is a covering set for K.

Let K be a closed convex body of volume one, which is symmetrical about the coordinate planes. Then K contains a point $(a_1, \cdots, a_n)$, $a_i > 0$, such that $2^n a_1 \cdots a_n \geqq n!/n^n$. (See, e.g., Sawyer [3]). Then K contains a box B_β: $|x_i| \leqq \beta_i$, $\beta_i \leqq a_i$ with volume $2^n\beta_1 \cdots \beta_n = n!/n^n$. A covering rectangular grid of B_β is automatically a covering set for K. Therefore, S is a Danzer set if for all closed boxes B_β of volume $n!/n^n$, S contains a rectangular grid Γ_α generated by $|x_i| \leqq \alpha_i$ with $\alpha_i \leqq \beta_i$.

We now construct a set A of points $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\alpha_i > 0$, such that for each set $(\beta_1, \cdots, \beta_n)$, $\beta_i > 0$, $\beta_1 \cdots \beta_n = n!/(2n)^n = k$, say, there exists an $\alpha \in A$, such that $\alpha_i \leqq \beta_i$. Then the grid Γ_α will provide a convering by B_β and the set $S = \bigcup_{\alpha \in A} \Gamma_\alpha$ will be a Danzer set.

Let H be the set of point x such that $x_1 \cdots x_n = k$, $x_i > 0$. Divide the part $x_1 > 0, \cdots, x_{n-1} > 0$ of the plane $x_n = 0$ into $n-1$ dimensional parallelepipeds $\pi_{k_1,\cdots,k_{n-1}}$ defined by

$$e^{k_i} \leqq x_i \leqq e^{k_i+1},\ i = 1, \cdots, n-1,\ (k_1, \cdots, k_{n-1}) \in Z^{n-1}\ ,$$

when Z is the set of rational integers. Let $H_{k_1,\cdots,k_{n-1}} = \{x\colon x \in H \text{ and } (x_1, \cdots, x_{n-1}) \in \pi_{k_1,\cdots,k_{n-1}}\}$. Then $H = \bigcup_{(k_1,\cdots,k_{n-1}) \in S^{n-1}} H_{k_1,\cdots,k_{n-1}}$. If $X \in H_{k_1,\cdots,k_{n-1}}$, then $x_i \geqq e^{k_i}$, $i = 1, \cdots, n-1$ and

$$x_n = \frac{k}{x_1 \cdots x_{n-1}} \geqq \frac{k}{e^{k_1+\cdots+k_{n-1}+n-1}}\ .$$

Let

$$\alpha = \alpha_{k_1,\cdots,k_{n-1}} = \left(e^{k_1}, \cdots, e^{k_{n-1}}, \frac{k}{e^{k_1+\cdots+k_{n-1}+n-1}}\right)$$

Then Γ_α is a grid of determinant $2^n k/e^{n-1}$. Let

$$A = \{\alpha_{k_1,\cdots,k_{n-1}} : (k_1, \cdots, k_{n-1}) \in Z^{n-1}\} .$$

For each $\beta = (\beta_1, \cdots, \beta_n) \in H_{k_1,\cdots,k_{n-1}}$, $\alpha_{k_1,\cdots,k_{n-1}} \in A$ has the property that Γ_α is a covering set for B_β. Therefore $S = \bigcup_{\alpha \in A} \Gamma_\alpha$ is a Danzer set. To prove Theorem 2, it will be enough to prove $D(S, X) = O((\log X)^{n-1})$, as $X \to \infty$.

Let $B(X)$ be the box $|x_i| \leqq X$. Since $N(S, X)$, $N(\Gamma_\alpha, X)$ denote the number of points of S and Γ_α, respectively, in $B(X)$, it follows that

$$(*) \qquad N(S, X) \leqq \sum_{\alpha \in A} N(\Gamma_\alpha, X) .$$

If $\alpha = \alpha_{k_1,\cdots,k_{n-1}} \in A$, then the points of Γ_α have coordinates

$$\left(e^{k_1}u_1, e^{k_2}u_2, \cdots, e^{k_{n-1}}u_{n-1}, \frac{k}{e^{k_1+\cdots+k_{n-1}+n-1}}u_n\right)$$
$$= (e^{k_1}u_1, e^{k_2}u_2, \cdots, e^{k_{n-1}}u_{n-1}, ke^l u_n) ,$$

say, where u_i are odd integers. If $\Gamma_\alpha \cap B(X) \neq \phi$, then

$$e^{k_1} \leqq X, \cdots, e^{k_{n-1}} \leqq X, ke^l \leqq X ,$$

so that for

$$i = 1, 2, \cdots, n-1,\ e^{k_i} \geqq \frac{k}{e^{n-1}} \cdot \frac{e^{k_i}}{e^{k_1+\cdots+k_{n-1}}} \cdot \frac{1}{X}$$
$$\geqq \frac{k}{e^{n-1}} \cdot \frac{1}{X^{n-1}} .$$

Therefore,

$$\Gamma_\alpha \cap B(X) \neq \phi \Rightarrow \frac{k}{(eX)^{n-1}} \leqq e^{k_i} \leqq X, \text{ for } i = 1, \cdots, n-1$$
$$\Rightarrow \log k - (n-1)(1 + \log X) \leqq k_i \leqq \log X \quad \text{for } i = 1, \cdots, n-1 .$$

Therefore, the number $\nu(X)$ of α for which $\Gamma_\alpha \cap B(X) \neq \phi$, satisfies

$$(**) \qquad \nu(X) \leqq (n(1 + \log X) - \log k)^{n-1} = O(\log X)^{n-1} .$$

If $\Gamma_\alpha \cap B(X) \neq \phi$, then the number $N(\Gamma_\alpha, X)$ of points of Γ_α in $B(X)$ is the number of points $(u_1, \cdots, u_n) \in Z^n$, u_i odd, with

$$-X \leqq u_i e^{k_i} \leqq X,\ i = 1, \cdots, n-1$$

and

$$-X \leqq u_n \frac{k}{e^{k_1+\cdots+k_{n-1}+n-1}} \leqq X .$$

Writing $[\xi]$ for the largest integer $\leqq \xi$, we have

$$
\begin{aligned}
N(\Gamma_\alpha, X) &= \Big(\prod_{i=1}^{n-1} 2\Big[\frac{1}{2}\Big(\frac{X}{e^{k_i}} + 1\Big)\Big]\Big) 2\Big[\frac{1}{2}\Big(\frac{Xe^{k_1+\cdots+k_{n-1}+n-1}}{k} + 1\Big)\Big] \\
(***) \qquad &\leqq 2^n \Big(\prod_{i=1}^{n-1} \frac{X}{e^{k_i}}\Big) \frac{Xe^{k_1+\cdots+k_{n-1}+n-1}}{k} \\
&= (2X)^n e^{n-1}/k \,.
\end{aligned}
$$

Combining $(*)$, $(**)$ and $(***)$, we get

$$D(S, X) = N(S, X)/(2X)^n = O((\log X)^{n-1}) \,.$$

Thus S is a Danzer set which provides an example for Theorem 2.

REFERENCES

1. L. Danzer, *Convexity*, Proc. Coll., Copenhagen, (1965), 310.
2. J. W. S. Cassels, *Introduction to the Geometry of Numbers*, Springer-Verlag, 1959.
3. D. B, Sawyer, J. London Math. Soc., **41** (1966), 466-468.

Received July 29, 1970. Research partially supported by NSF Grant GP-9588.

THE OHIO STATE UNIVERSITY
AND
PANJAB UNIVERSITY, CHANDIGRAH, INDIA

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Pacific Journal of Mathematics

Vol. 37, No. 2 February, 1971