

# Pacific Journal of Mathematics

## **ON A PROBLEM OF DANZER**

RAM PRAKASH BAMBAH AND ALAN C. WOODS

## ON A PROBLEM OF DANZER

R. P. BAMBAH AND A. C. WOODS

By a Danzer set  $S$  we shall mean a subset of the  $n$ -dimensional Euclidean space  $R_n$  which has the property that every closed convex body of volume one in  $R_n$  contains a point of  $S$ . L. Danzer has asked if for  $n \geq 2$  there exist such sets  $S$  with a finite density. The answer to this question is still unknown. In this note our object is to prove two theorems about Danzer sets.

If  $A$  is a  $n$ -dimensional lattice, any translate  $\Gamma = A + p$  of  $A$  will be called a grid  $\Gamma$ ;  $A$  will be called the lattice of  $\Gamma$  and the determinant  $d(A)$  of  $A$  will be called the determinant of  $\Gamma$  and will be denoted by  $d(\Gamma)$ . In § 2 we prove

**THEOREM 1.** For  $n \geq 2$ , a Danzer set cannot be the union of a finite number of grids.

Let  $S$  be a Danzer set and  $X > 0$  a positive real number. Let  $N(S, X)$  be the number of points of  $S$  in the box  $\max_{1 \leq i \leq n} |x_i| \leq X$ . Let  $D(S, X) = N(S, X)/(2X)^n$ . In § 3 we prove

**THEOREM 2.** There exist Danzer sets  $S$  with  $D(S, X) = O((\log X)^{n-1})$  as  $X \rightarrow \infty$ .

The case  $n = 2$  of the theorem is known, although no proof seems to have been published. The referee has pointed out that a lower bound of 2 can easily be established for the density of a Danzer set in  $n = 2$ , but the authors are unaware of any further results in this direction.

**2. Proof of Theorem 1.** We shall assume throughout that  $n \geq 2$ . It is obvious that if  $S$  is a Danzer set and  $T$  is a volume preserving affine transformation of  $R_n$  onto itself, then  $T(S)$  is also a Danzer set.

Let  $S_1, S_2, \dots$  be a sequence of sets in  $R_n$ . Let  $S$  be the set of points  $X$  such that there exists a subsequence  $S_{i_1}, S_{i_2}, \dots$  of  $\{S_r\}$  and points  $X_{i_r} \in S_{i_r}$ , such that  $X_{i_r} \rightarrow X$  as  $r \rightarrow \infty$ . We write

$$S = \lim_{r \rightarrow \infty} S_r = \lim S_r .$$

**LEMMA 1.** Let  $\{S_r\}$  be a sequence of Danzer sets in  $R_n$ . Then  $S = \lim S_r$  is also a Danzer set.

*Proof.* Let  $K$  be a closed convex body of Volume 1. Then for

each  $r$ ,  $K \cap S_r \neq \phi$ , so that for each  $r$ , there exists  $X_r \in K \cap S_r$ . Since  $K$  is compact,  $\{X_r\}$  has a convergent subsequence  $\{X_{i_r}\}$  converging to a point  $X$  in  $K \cap S$ .

LEMMA 2. *Let  $S^{(j)} = \lim_{r \rightarrow \infty} S_r^{(j)}$ ,  $j = 1, \dots, k$ . Then*

$$\bigcup_{j=1}^k S^{(j)} = \lim_{r \rightarrow \infty} \left( \bigcup_{j=1}^k S_r^{(j)} \right).$$

*Proof.*  $X \in \cup S^{(j)} \Rightarrow X \in S^{(j)}$  for some  $j$ , say  $j = j_0 \Rightarrow$  there exist a subsequence  $\{S_{i_r}^{(j_0)}\}$  of  $\{S_r^{(j_0)}\}$  and points  $X_{i_r} \in S_{i_r}^{(j_0)}$  such that  $X_{i_r} \rightarrow X \Rightarrow X_{i_r} \in \cup S_{i_r}$  and  $X_{i_r} \rightarrow X \Rightarrow X \in \lim_{r \rightarrow \infty} (\bigcup_{j=1}^k S_r^{(j)})$ . Thus  $\cup S^{(j)} \subset \lim (\bigcup_{j=1}^k S_r^{(j)})$ . Let  $X \in \lim (U_{j=1}^k S_r^{(j)})$ . Then there exists a sequence  $\{i_r\}$  of natural numbers and  $X_{i_r} \in \bigcup_{j=1}^k S_{i_r}^{(j)}$  such that  $X_{i_r} \rightarrow X$ . Since  $k$  is finite, there exists a  $j = j_0$  say, and an infinite subsequence  $k_r$  of  $i_r$  such that  $X_{k_r} \in S_{k_r}^{(j_0)}$ . Then  $X_{k_r} \rightarrow X$  and  $X \in S^{(j_0)}$ , so that  $X \in \cup S^{(j)}$  and  $\lim_{r \rightarrow \infty} (\bigcup_{j=1}^k S_r^{(j)}) \subset \cup S^{(j)}$ .

This completes the proof of the lemma.

LEMMA 3. *Let  $\Gamma_1, \Gamma_2, \dots$  be a sequence of grids in  $R_n$  with equal determinants  $d(\Gamma_r) = \Delta$ . Then  $\{\Gamma_r\}$  has a subsequence  $\{\Gamma_{i_r}\}$ , such that  $\lim_{r \rightarrow \infty} \Gamma_{i_r}$  is either a grid or is contained in a hyperplane.*

*Proof.* If  $\lim_{r \rightarrow \infty} \Gamma_r = \phi$ , there is nothing to prove. Assume, therefore, that  $\Gamma = \lim_{r \rightarrow \infty} \Gamma_r \neq \phi$ . Let  $X \in \Gamma$ . Then there exists a subsequence  $\{i_r\}$  of natural numbers and points  $X_{i_r} \in \Gamma_{i_r}$ , such that  $X_{i_r} \rightarrow X$ . Then  $A_{i_r} = \Gamma_{i_r} - X_{i_r}$  is a sequence of homogeneous lattices and  $\lim \Gamma_{i_r} = X + \lim A_{i_r}$ . Therefore, it is enough to prove the theorem for lattices.

Let  $\{A_r\}$  be a sequence of lattices with determinants  $d(A_r) = \Delta$ , independent of  $r$ . Let  $\mu_1(A_r), \dots, \mu_n(A_r)$  be the successive minima of the Euclidean distance with respect to  $A_r$ , i.e.,  $\mu_i(A_r) = \inf \mu$ : such that  $|X| < \mu$  has  $i$  linearly independent points of  $A_r$ .

Suppose, first, that there exists  $\delta > 0$ , such that  $\mu_1(A_r) \geq \delta$  for infinitely many  $r$ . Then a subsequence satisfies the conditions of Mahler's compactness theorem and has a subsequence convergent in the sense of Mahler (see, e.g., Cassels [2]). The last subsequence converges to the limiting lattice in our sense also.

We may, therefore, assume  $\mu_1(A_r) \rightarrow 0$  as  $r \rightarrow \infty$ . Since

$$\mu_1(A_r) \cdots \mu_n(A_r) \geq \frac{2^n}{n!} \cdot \frac{1}{J_n},$$

where  $J_n$  is the volume of the sphere  $|X| < 1$ , (see, e.g., Cassels [2]),

and since  $n \geq 2$ , it follows that  $\mu_n(A_r) \rightarrow \infty$  as  $r \rightarrow \infty$ . For each  $r$ , let  $P_{r_1}, \dots, P_{r_n}$  be points such that  $|P_{r_i}| = \mu_i(A_r)$ . Let  $\pi_r$  be the plane through  $0, p_{r_1}, \dots, p_{r_{n-1}}$ . It is easily seen that there exists a subsequence  $\{A_{i_r}\}$  of  $\{A_r\}$  such that the sequence  $\{\pi_{i_r}\}$  converges to a plane  $\pi$ . We assert that  $\lim_{r \rightarrow \infty} \{A_{i_r}\} \subset \pi$ . For, let  $X \in \lim_{r \rightarrow \infty} A_{i_r}$ . Then  $X = \lim X_{k_r}$ , where  $k_r$  is a subsequence of  $i_r$  and  $X_{k_r} \in A_{k_r}$ . There exists  $M$  independent of  $k_r$ , such that  $|X_{k_r}| \leq M$  for all  $k_r$ . Also

$$X_{k_r} = g_{r,1}P_{k_r,1} + \dots + g_{r,n}P_{k_r,n}, g_{r,i} \text{ real,}$$

and if  $g_{r,n} \neq 0$  then  $|X_{k_r}| \geq \mu_n(A_{k_r})$ . Since  $\mu_n(A_{k_r}) \rightarrow \infty$  as  $r \rightarrow \infty$ ,  $g_{r,n} = 0$  for all large  $r$  and  $X \in \pi$ . This proves the lemma

LEMMA 4. *Let  $\{\pi_i\}$  be a sequence of hyperplanes. Then  $\{\pi_i\}$  has a subsequence  $\{\pi_{i_\mu}\}$  whose limit lies in a hyperplane.*

*Proof.* If  $\pi = \lim_{i \rightarrow \infty} \pi_i = \phi$  then there is nothing to prove. Assume, therefore,  $X \in \pi$ . Then there is a subsequence  $\{k_r\}$  of natural numbers and points  $X_{k_r} \in \pi_{k_r}$  such that  $X_{k_r} \rightarrow X$ . The planes  $\hat{\pi}_{k_r} = \pi_{k_r} - X_{k_r}$  pass through  $0$  and have a subsequence  $\hat{\pi}_{i_r}$  which converges to a plane  $\hat{\pi}$  say. Then  $\lim_{r \rightarrow \infty} \pi_{i_r} = \hat{\pi} + X$ . This proves the lemma.

*Proof of Theorem 1.* We shall prove more, namely, a Danzer set cannot be the union of a finite number of hyperplanes and a finite number of grids.

Let  $S = \bigcup_{i=1}^j \pi_i \bigcup_{j=1}^l \Gamma_j$  be a Danzer set, such that  $\pi_i$  are hyperplanes and  $\Gamma_j$  are grids. Let  $t \geq 1$ . Let  $X \neq Y, X, Y \in \Gamma_1$ . For each positive integer  $k$ , let  $T_k$  be a volume preserving affine transformation such that  $T_k(X) = X$  and  $|T_k(Y) - X| = k^{-1}|Y - X|$ . Since  $n \geq 2$ , such transformations exist. For each  $k$ ,  $T_k(S)$  is a Danzer set, and by Lemma 1, so is the limit of every subsequence of  $\{T_k(S)\}$ . By Lemmas 3 and 4 we can choose a subsequence  $\{T_{k_r}\}$  of  $\{T_k\}$  such that each  $\lim_{t \rightarrow \infty} T_{k_r}(\pi_i)$  lies in a hyperplane, while each  $\lim_{t \rightarrow \infty} T_{k_r}(\Gamma_j)$  is either a grid or lies in a hyperplane. Since

$$\lim_{r \rightarrow \infty} T_{k_r}(S) = \bigcup_{i=1}^t \lim T_{k_r}(\pi_i) \bigcup_{j=1}^t \lim T_{k_r}(\Gamma_j)$$

and  $\lim T_{k_r}(\Gamma_1)$  is in a hyperplane, the Danzer set  $\lim T_{k_r}(S)$  lies in the union of a finite number of hyperplanes and  $t_1 < t$  grids, so that we have (by increasing  $T_{k_r}(S)$  if necessary) a Danzer set consisting of a finite number of hyperplanes and  $t_1 < t$  grids. Repeating this process a number of times we obtain a Danzer set that is the union of a finite number of hyperplanes. This can easily be seen to lead to a contradiction which proves the theorem.

3. **Proof of Theorem 2.** Let  $K$  be a closed convex body in  $R_n$ . The set  $S \subset R_n$  is said to be a covering set for  $K$  if  $R_n \subset \bigcup_{A \in S} (K + A)$ . The set  $S$  contains a point of each translate of  $K$  if and only if  $S$  is a covering set for  $-K$ . Clearly a set  $S$  is a Danzer set if and only if it is a covering set for each closed convex body of volume one. Therefore, in order to prove a given set  $S$  is a Danzer set, it is enough to prove that for every closed convex body  $K$  of volume one,  $S$  contains a covering set for  $K$ .

If  $\Gamma$  is a grid with lattice  $A$ , then it is easy to see that  $\Gamma$  is a covering set for  $K$  if and only if  $A$  is.

Let  $\pi$  be a parallelepiped. Let  $A_0$  be one of its vertices and  $A_1, \dots, A_n$  be the  $n$  vertices joined to  $A_0$  by edges of  $\pi$ . Let  $A$  be the lattice generated by  $A_1 - A_0, \dots, A_n - A_0$ . By the grid generated by  $\pi$  we shall mean the grid  $A + A_0$ . It is easily seen that if a closed convex body  $K$  contains a parallelepiped which generates a grid  $\Gamma$ , then  $\Gamma$  is a covering set for  $K$ .

A lattice  $A$  will be called rectangular if it consists of points  $(\alpha_1 u_1, \dots, \alpha_n u_n)$ , where  $\alpha_i$  are fixed positive real numbers and  $u_i$  take integral values. A grid  $\Gamma$  will be called rectangular if its lattice is rectangular.

Let  $\alpha_1, \dots, \alpha_n$  be positive real numbers. Let  $\Gamma_\alpha$  be the grid generated by the parallelepiped  $|x_i| \leq \alpha_i$ . Let  $B$  be a box  $|x_i| \leq \beta_i$ , where  $\beta_i \geq \alpha_i$  for  $i = 1, \dots, n$ . Then  $\Gamma_\alpha$  is clearly a covering set for  $B$ .

Let  $K$  be a closed convex body of volume one. Let  $K_1$  be the steiner symmetrical of  $K$  with respect to the plane  $x_1 = 0$ . Let  $K_2$  be the steiner symmetrical of  $K_1$  with respect to  $x_2 = 0$  and so on. Then  $K_n$  is symmetrical about all the coordinate planes and has volume one. We next have

LEMMA 5. *If a rectangular lattice  $A$  is a covering set for  $K_n$ , then it is a covering set for  $K$  also.*

(The lemma and its proof are easy adaptations of Lemma 2 of Sawyer (3). For completeness, we give the proof below).

*Proof.* Let  $A$  be the rectangular lattice consisting of points  $(\alpha_1 u_1, \dots, \alpha_n u_n)$ ,  $\alpha_i > 0$  fixed real numbers and  $u_i$  running over the set of integers. It is enough to prove that if  $A$  is a covering set for  $K_1$ , then it is a covering set for  $K$  also.

Let  $A_1 =$  subset of  $A$  in the plane  $x_1 = 0$ . The sets  $K_1 + A$  cover  $R_n$ . We assert each line  $x_2 = a_2, \dots, x_n = a_n$  meets  $K_1 + P$  is a segment of length at least  $\alpha_1$  for some  $P \in A_1$ . Such a line meets only a finite number of translates  $K_1 + P_s, P_s \in A_1$ , each of them in a seg-

ment  $|x_1| \leq b_s$  and hence meets  $K_1 + A_1$  in the segment  $|x_1| \leq b = \max b_s$ . If  $b < \frac{1}{2}\alpha_1$ , then  $K_1 + A$  meets the line in segments  $|x_1 - m\alpha_1| \leq b < \frac{1}{2}\alpha_1$ , where  $m$  takes integral values. This leaves part of the line uncovered by sets  $K_1 + A$ , contrary to the fact that  $A$  is a covering set for  $K_1$ . Thus  $b \geq \frac{1}{2}\alpha_1$ , i.e.,  $b_s \geq \frac{1}{2}\alpha_1$  for some  $s$ . Therefore, the line meets  $K_1 + P_s$  and hence  $K + P_s$  in a segment of length at least  $\alpha_1$ , and is therefore, covered by the sets  $K + A$ . Since this is true for all such lines,  $A$  is a covering set for  $K$ .

COROLLARY. *A rectangular grid  $\Gamma$  which is a covering set for  $K_n$  is also a covering set for  $K$ .*

Because of the corollary, in order to prove that a given set  $S$  is a Danzer set, it is enough to prove that for every given closed convex body  $K$  of volume one, which is symmetrical about all the coordinate planes,  $S$  contains a rectangular grid  $\Gamma$  which is a covering set for  $K$ .

Let  $K$  be a closed convex body of volume one, which is symmetrical about the coordinate planes. Then  $K$  contains a point  $(a_1, \dots, a_n)$ ,  $a_i > 0$ , such that  $2^n a_1 \dots a_n \geq n!/n^n$ . (See, e.g., Sawyer [3]). Then  $K$  contains a box  $B_\beta: |x_i| \leq \beta_i, \beta_i \leq a_i$  with volume  $2^n \beta_1 \dots \beta_n = n!/n^n$ . A covering rectangular grid of  $B_\beta$  is automatically a covering set for  $K$ . Therefore,  $S$  is a Danzer set if for all closed boxes  $B_\beta$  of volume  $n!/n^n$ ,  $S$  contains a rectangular grid  $\Gamma_\alpha$  generated by  $|x_i| \leq \alpha_i$  with  $\alpha_i \leq \beta_i$ .

We now construct a set  $A$  of points  $\alpha = (\alpha_1, \dots, \alpha_n)$ ,  $\alpha_i > 0$ , such that for each set  $(\beta_1, \dots, \beta_n)$ ,  $\beta_i > 0, \beta_1 \dots \beta_n = n!/(2n)^n = k$ , say, there exists an  $\alpha \in A$ , such that  $\alpha_i \leq \beta_i$ . Then the grid  $\Gamma_\alpha$  will provide a covering by  $B_\beta$  and the set  $S = \bigcup_{\alpha \in A} \Gamma_\alpha$  will be a Danzer set.

Let  $H$  be the set of point  $x$  such that  $x_1 \dots x_n = k, x_i > 0$ . Divide the part  $x_1 > 0, \dots, x_{n-1} > 0$  of the plane  $x_n = 0$  into  $n - 1$  dimensional parallelepipeds  $\pi_{k_1, \dots, k_{n-1}}$  defined by

$$e^{k_i} \leq x_i \leq e^{k_i+1}, i = 1, \dots, n - 1, (k_1, \dots, k_{n-1}) \in Z^{n-1},$$

when  $Z$  is the set of rational integers. Let  $H_{k_1, \dots, k_{n-1}} = \{x: x \in H \text{ and } (x_1, \dots, x_{n-1}) \in \pi_{k_1, \dots, k_{n-1}}\}$ . Then  $H = \bigcup_{(k_1, \dots, k_{n-1}) \in S^{n-1}} H_{k_1, \dots, k_{n-1}}$ . If  $X \in H_{k_1, \dots, k_{n-1}}$ , then  $x_i \geq e^{k_i}, i = 1, \dots, n - 1$  and

$$x_n = \frac{k}{x_1 \dots x_{n-1}} \geq \frac{k}{e^{k_1 + \dots + k_{n-1} + n-1}}.$$

Let

$$\alpha = \alpha_{k_1, \dots, k_{n-1}} = \left( e^{k_1}, \dots, e^{k_{n-1}}, \frac{k}{e^{k_1 + \dots + k_{n-1} + n-1}} \right)$$

Then  $\Gamma_\alpha$  is a grid of determinant  $2^n k/e^{n-1}$ . Let

$$A = \{\alpha_{k_1, \dots, k_{n-1}} : (k_1, \dots, k_{n-1}) \in Z^{n-1}\}.$$

For each  $\beta = (\beta_1, \dots, \beta_n) \in H_{k_1, \dots, k_{n-1}}$ ,  $\alpha_{k_1, \dots, k_{n-1}} \in A$  has the property that  $\Gamma_\alpha$  is a covering set for  $B_\beta$ . Therefore  $S = \bigcup_{\alpha \in A} \Gamma_\alpha$  is a Danzer set. To prove Theorem 2, it will be enough to prove  $D(S, X) = O((\log X)^{n-1})$ , as  $X \rightarrow \infty$ .

Let  $B(X)$  be the box  $|x_i| \leq X$ . Since  $N(S, X)$ ,  $N(\Gamma_\alpha, X)$  denote the number of points of  $S$  and  $\Gamma_\alpha$ , respectively, in  $B(X)$ , it follows that

$$(*) \quad N(S, X) \leq \sum_{\alpha \in A} N(\Gamma_\alpha, X).$$

If  $\alpha = \alpha_{k_1, \dots, k_{n-1}} \in A$ , then the points of  $\Gamma_\alpha$  have coordinates

$$\begin{aligned} & \left( e^{k_1 u_1}, e^{k_2 u_2}, \dots, e^{k_{n-1} u_{n-1}}, \frac{k}{e^{k_1 + \dots + k_{n-1} + n-1}} u_n \right) \\ & = (e^{k_1 u_1}, e^{k_2 u_2}, \dots, e^{k_{n-1} u_{n-1}}, k e^l u_n), \end{aligned}$$

say, where  $u_i$  are odd integers. If  $\Gamma_\alpha \cap B(X) \neq \phi$ , then

$$e^{k_1} \leq X, \dots, e^{k_{n-1}} \leq X, k e^l \leq X,$$

so that for

$$\begin{aligned} i = 1, 2, \dots, n-1, e^{k_i} & \geq \frac{k}{e^{n-1}} \cdot \frac{e^{k_i}}{e^{k_1 + \dots + k_{n-1}}} \cdot \frac{1}{X} \\ & \geq \frac{k}{e^{n-1}} \cdot \frac{1}{X^{n-1}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \Gamma_\alpha \cap B(X) \neq \phi & \Rightarrow \frac{k}{(eX)^{n-1}} \leq e^{k_i} \leq X, \text{ for } i = 1, \dots, n-1 \\ & \Rightarrow \log k - (n-1)(1 + \log X) \leq k_i \leq \log X \\ & \qquad \qquad \qquad \text{for } i = 1, \dots, n-1. \end{aligned}$$

Therefore, the number  $\nu(X)$  of  $\alpha$  for which  $\Gamma_\alpha \cap B(X) \neq \phi$ , satisfies

$$\begin{aligned} (**) \quad \nu(X) & \leq (n(1 + \log X) - \log k)^{n-1} \\ & = O(\log X)^{n-1}. \end{aligned}$$

If  $\Gamma_\alpha \cap B(X) \neq \phi$ , then the number  $N(\Gamma_\alpha, X)$  of points of  $\Gamma_\alpha$  in  $B(X)$  is the number of points  $(u_1, \dots, u_n) \in Z^n$ ,  $u_i$  odd, with

$$-X \leq u_i e^{k_i} \leq X, i = 1, \dots, n-1$$

and

$$-X \leq u_n \frac{k}{e^{k_1 + \dots + k_{n-1} + n-1}} \leq X.$$

Writing  $[\xi]$  for the largest integer  $\leq \xi$ , we have

$$\begin{aligned}
 N(\Gamma_\alpha, X) &= \left( \prod_{i=1}^{n-1} 2 \left[ \frac{1}{2} \left( \frac{X}{e^{k_i}} + 1 \right) \right] \right) 2 \left[ \frac{1}{2} \left( \frac{X e^{k_1 + \dots + k_{n-1} + n-1}}{k} + 1 \right) \right] \\
 (***) \quad &\leq 2^n \left( \prod_{i=1}^{n-1} \frac{X}{e^{k_i}} \right) \frac{X e^{k_1 + \dots + k_{n-1} + n-1}}{k} \\
 &= (2X)^n e^{n-1} / k .
 \end{aligned}$$

Combining (\*), (\*\*) and (\*\*\*), we get

$$D(S, X) = N(S, X) / (2X)^n = O((\log X)^{n-1}) .$$

Thus  $S$  is a Danzer set which provides an example for Theorem 2.

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