

Pacific Journal of Mathematics

THE DIOPHANTINE EQUATION

$$Y(Y + 1)(Y + 2)(Y + 3) = 2X(X + 1)(X + 2)(X + 3)$$

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It is shown that the only solution in positive integers of the equation of the title is $X = 4, Y = 5$.

Substituting $y = 2Y + 3, x = 2X + 3$ gives with a little manipulation

$$\left\{ \frac{y^2 - 5}{4} \right\}^2 - 2 \left\{ \frac{x^2 - 5}{4} \right\}^2 = -1,$$

and since the fundamental solution of $v^2 - 2u^2 = -1$ is $\alpha = 1 + \sqrt{2}$, we find that if $\beta = 1 - \sqrt{2}$ and

$$(1) \quad u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}; \quad v_n = \frac{\alpha^n + \beta^n}{2}$$

we must have simultaneously

$$(2) \quad y^2 = 5 + 4v_N,$$

and

$$(3) \quad x^2 = 5 + 4u_N,$$

where N is odd and $N \geq 3$.

We find easily from (1) since $\alpha\beta = -1$ and $\alpha + \beta = 2$, that

$$(4) \quad u_{-n} = (-1)^{n-1}u_n$$

$$(5) \quad v_{-n} = (-1)^n v_n$$

$$(6) \quad u_{m+n} = u_m v_n + u_n v_m$$

$$(7) \quad v_{m+n} = v_m v_n + 2u_m u_n.$$

Throughout k denotes an even integer, and then we find using (4)–(7) that

$$(8) \quad v_{2k} = 2v_k^2 - 1 = 4u_k^2 + 1$$

$$(9) \quad u_{2k} = 2u_k v_k$$

$$(10) \quad v_{3k} = v_k(8u_k^2 + 1) = v_k(2v_{2k} - 1)$$

$$(11) \quad u_{3k} = u_k(8u_k^2 + 3).$$

We then have, using (6)–(9) that

$$(12) \quad u_{n+2k} \equiv -u_n \pmod{v_k}$$

and

$$(13) \quad v_{n+2k} \equiv -v_n \pmod{v_k}.$$

We have also the following table of values

| n | u_n | v_n |
|-----|-------|-------|
| 0 | 0 | 1 |
| 1 | 1 | 1 |
| 2 | 2 | 3 |
| 3 | 5 | 7 |
| 4 | 12 | 17 |
| 5 | 29 | 41 |
| 6 | 70 | 99 |
| 7 | 169 | 239 |
| 8 | 408 | 577 |
| 9 | 985 | 1393 |
| 10 | 2378 | 3363 |
| 11 | 5741 | 8119 |
| 12 | 13860 | 19601 |

The proof is now accomplished in eight stages:-

(a). (2) is impossible if $N \equiv 3 \pmod{6}$.

For,

$$\begin{aligned} v_{n+6} &= v_n v_6 + 2u_n u_6 && \text{by (7)} \\ &= 99v_n + 140u_n \\ &\equiv -v_n \pmod{5}, \end{aligned}$$

and so if $N \equiv 3 \pmod{6}$, $v_N \equiv \pm v_3 \equiv \pm 2 \pmod{5}$, whence $y^2 = 5 + 4v_N$ is impossible modulo 5.

(b). (2) is impossible if $N \equiv -3$ or $-5 \pmod{16}$.

For, using (13) we find that for such N ,

$$\begin{aligned} v_N &\equiv v_{-3} \quad \text{or} \quad v_{-5} && \pmod{v_4} \\ &\equiv -v_3 \quad \text{or} \quad -v_5 && \pmod{17}, \text{ using (5)} \\ &\equiv -7 && \pmod{17}. \end{aligned}$$

But then $5 + 4v_N \equiv -6 \pmod{17}$, and since the Jacobi-Legendre symbol $(-6|17) = -1$, (2) is impossible.

(c). (3) is impossible if $N \equiv \pm 7 \pmod{16}$.

For, using, (12) we find that in this case

$$\begin{aligned} u_n &\equiv \pm u_{\pm 7} && \pmod{v_8} \\ &\equiv \pm 169 && \pmod{577}. \end{aligned}$$

Thus we find that

$$5 + 4u_N \equiv 681 \text{ or } -671 \pmod{577}, \text{ and since}$$

$$(681 | 577) = (-671 | 577) = -1,$$

(3) is impossible.

(d). (3) is impossible if $N \equiv \pm 7 \pmod{24}$.

For then

$$\begin{aligned} u_N &\equiv u_{\pm 7} \pmod{v_6} \\ &\equiv 169 \pmod{99}, \end{aligned}$$

whence $u_N \equiv -2 \pmod{9}$, and then $5 + 4u_N \equiv -3 \pmod{9}$, and so (3) is impossible.

(e). (2) and (3) together are impossible if $N \equiv 3 \pmod{16}$.

If $N = 3$, then $5 + 4v_N = 33 \neq y^2$. If $N \neq 3$, then we may write

$$N - 3 = 2lk,$$

where l is odd and $k = 2^r$ with $r \geq 3$. Then by using (13) l times we obtain

$$\begin{aligned} 5 + 4u_N &= 5 + 4u_{3+2lk} \\ &\equiv 5 + (-1)^l 4u_3 \pmod{v_k} \\ &\equiv -15 \pmod{v_k}, \text{ since } l \text{ is odd.} \end{aligned}$$

But from (8) we find easily by induction upon r , that if $k = 2^r$ with $r \geq 3$, that $v_k \equiv 1 \pmod{4}$, $v_k \equiv 1 \pmod{3}$ and $v_k \equiv 2 \pmod{5}$, whence $(-15 | v_k) = -1$ and (3) is impossible.

Combining the results of (a)—(e) we find that we can only have

$$(14) \quad N \equiv 1, 5, -1, 37 \pmod{48},$$

and we deal with each of these in turn.

(f). (3) is impossible if $N \equiv 37 \pmod{48}$.

For then $u_N \equiv u_{-11} \equiv 5741 \pmod{v_{12}}$ and then $5 + 4u_N \equiv 22969 \pmod{19601}$.

But

$$\begin{aligned} (22969 | 19601) &= (3368 | 19601) \\ &= (2^3 | 19601)(421 | 19601) \\ &= (19601 | 421) \\ &= (235 | 421) \\ &= (421 | 5)(421 | 47) \\ &= (-2 | 47) = -1, \end{aligned}$$

and so (3) is impossible.

(g). (3) is impossible if $N \equiv 1 \pmod{48}$, $N \neq 1$ or if $N \equiv -1 \pmod{48}$ and $N \neq -1$.

Since if N is odd, $u_{-N} = u_N$ by (4) it suffices to consider $N \equiv 1 \pmod{48}$, $N \neq 1$. Then we may write $N = 1 + 3k(2l + 1)$, where $k = 2^r$ and $r \geq 4$, and so using (12) we find that

$$\begin{aligned} u_N &= u_{1+3k+2l \cdot 3k} \\ &\equiv (-1)^l u_{1+3k} && \pmod{v_{3k}} \\ &\equiv \pm(u_{3k} + v_{3k}) && \pmod{v_{3k}} \text{ using (6)} \\ &\equiv \pm u_{3k} && \pmod{v_{3k}} \\ &\equiv \pm u_k(8u_k^2 + 3) && \pmod{v_k(8u_k^2 + 1)}, \end{aligned}$$

using (10) and (11). Thus

$$u_N \equiv \pm 2u_k \pmod{8u_k^2 + 1}.$$

But now, writing $u = u_k$, we find

$$\begin{aligned} (15) \quad (5 + 4u_N | 8u^2 + 1) &= (5 \pm 8u | 8u^2 + 1) \\ &= (8u \pm 5 | 8u^2 + 1) \\ &= (8u^2 + 1 | 8u \pm 5) \\ &= (8 | 8u \pm 5)(8^2u^2 + 8 | 8u \pm 5) \\ &= -(33 | 8u \pm 5) \\ &= -(8u \pm 5 | 33). \end{aligned}$$

But $u = u_k$ with $k = 2^r$ and $r \geq 4$, and we find that $3 | u_8$, whence $3 | u_k$ in view of (9). Also $v_8 \equiv 5 \pmod{11}$ whence by induction, using (8), $v_n \equiv 5 \pmod{11}$ for $n = 2^r$ and $r \geq 3$. Thus $u_{2n} \equiv -u_n \pmod{11}$ in view of (9), and so since $u_8 \equiv 1 \pmod{11}$, $u \equiv \pm 1 \pmod{11}$. Thus we have $u \equiv \pm 12 \pmod{33}$ whence $8u \equiv \mp 3 \pmod{33}$. Considering therefore the right hand side of (15), we observe that $8u \pm 5 \equiv \pm 2$ or $\pm 8 \pmod{33}$ and in any one of the four cases the right hand side of (15) equals -1 , and accordingly (3) is impossible.

(h). (2) and (3) together are impossible if $N \equiv 5 \pmod{48}$, $N \neq 5$.

Suppose if possible that (2), (3) hold with $N \equiv 5 \pmod{48}$, $N \neq 5$. Let $N = 5 + 2l \cdot 3k$ where $k = 2^r$, $r \geq 3$ and l is odd. Then we have using (12) and (13)

$$(16) \quad x^2 = 5 + 4u_N \equiv 5 - 4u_5 \equiv -111 \pmod{v_{3k}}$$

$$(17) \quad y^2 = 5 + 4v_N \equiv 5 - 4v_5 \equiv -159 \pmod{v_{3k}}.$$

Now we have from (10) $v_{3k} = v_k(2v_{2k} - 1)$, and as before $v_k \equiv 1 \pmod{12}$ whence also $2v_{2k} - 1 \equiv 1 \pmod{12}$. Thus $(-3 | v_k) = (-3 | 2v_{2k} - 1) = 1$, and so (16) and (17) imply (since as we shall see presently neither v_k nor $2v_{2k} - 1$ is ever divisible by either 37 or 53) that

$$(18) \quad (v_k | 37) = (2v_{2k} - 1 | 37) = (v_k | 53) = (2v_{2k} - 1 | 53) = 1,$$

for some $k = 2^r$, $r \geq 3$. We shall demonstrate that (18) occurs for no such k .

In view of (8) it is clear that the residues modulo p for any prime p , of v_k with $k = 2^r$ are eventually periodic with respect to r . It transpires that if $p = 37$ or if $p = 53$, the length of the period is 9, and that the sequence of residues has already become periodic by the time $r = 3$. It is fortunately the case that in no one of the nine cases that arise are all the four conditions of (18) satisfied, and this concludes the proof. A table showing these calculations follows:-

| $k = 2^r$ | $r = 3$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-------------------------|---------|-----|----|-----|-----|-----|-----|-----|-----|-----|
| $v_k \pmod{37}$ | -15 | 5 | 12 | -9 | 13 | 4 | -6 | -3 | 17 | -15 |
| $2v_{2k} - 1 \pmod{37}$ | 9 | -14 | 18 | -12 | 7 | -13 | -7 | -4 | 6 | |
| $v_k \pmod{53}$ | -6 | 18 | 11 | -24 | -15 | 25 | -23 | -3 | 17 | -6 |
| $2v_{2k} - 1 \pmod{53}$ | -18 | 21 | 4 | 22 | -4 | 6 | -7 | -20 | -13 | |
| $(v_k 37)$ | -1 | -1 | +1 | +1 | -1 | +1 | -1 | +1 | -1 | |
| $(2v_{2k} - 1 37)$ | +1 | -1 | -1 | +1 | +1 | -1 | +1 | +1 | -1 | |
| $(v_k 53)$ | +1 | -1 | +1 | +1 | +1 | +1 | -1 | -1 | +1 | |
| $(2v_{2k} - 1 53)$ | -1 | -1 | +1 | -1 | +1 | +1 | +1 | -1 | +1 | |

Summarising the results, we see that (2) and (3) can hold simultaneously for N odd, $N \geq 3$ only for $N = 5$, and this value does indeed satisfy (2) and (3) with $x = 11$, $y = 13$. Thus $X = 4$, $Y = 5$ is the only solution of the given equation in positive integers. The complete solution in integers can now be written down; it consists of the sixteen "trivial" pairs of solutions obtained by equating both sides of the given equation to zero, and the four pairs $X = 4$ or -7 , $Y = 5$ or -8 .

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