

Pacific Journal of Mathematics

**THE FUNCTIONS OF BOUNDED INDEX AS A SUBSPACE OF A
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KEITH A. EKBLAW

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Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be entire functions. Define $d(f, g) = \text{Sup} \{|a_0 - b_0|, (|a_n - b_n|)^{1/n} \mid n = 1, 2, \dots\}$. It is the purpose of this note to show that, in the topology generated by d , the entire functions of bounded index, B , are of the first category.

Further, for Γ , the corresponding space of all entire functions, and $B_n = \{f \in B \mid \text{the index of } f \text{ is } \leq n\}$ is shown that $B - B_n$ is dense in Γ for any nonnegative integer n . It is also shown that $\Gamma - B$ is dense in Γ . (For definition and main results see [2], [3].)

LEMMA 1. *For any $f \in \Gamma$, $N \geq 0$, and $\varepsilon > 0$ there exists a $\delta > 0$ such that if $g \in \Gamma$ and $d(f, g) < \delta$ then $d(f^{(k)}, g^{(k)}) < \varepsilon$ for $k = 0, 1, \dots, N$.*

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \Gamma$, $N \geq 0$, and $\varepsilon > 0$ be given. Let

$$T > \text{Sup} \left\{ \left(\frac{(n+k)!}{n!} \right)^{1/n} \mid n = 1, 2, \dots \text{ and } k = 0, 1, \dots, N \right\}.$$

It is straightforward to verify that if $g(z) = \sum_{n=0}^{\infty} b_n z^n \in \Gamma$ and $d(f, g) < \frac{\varepsilon}{T + \varepsilon}$ then

$$\begin{aligned} d(f^{(k)}, g^{(k)}) &= \text{Sup} \left\{ k! |a_k - b_k|, \left(\frac{(n+k)!}{n!} |a_{n+k} - b_{n+k}| \right)^{1/n} \mid n = 1, 2, \dots \right\} \\ &< T \cdot \frac{\varepsilon}{T + \varepsilon} < \varepsilon \text{ for } k = 0, 1, \dots, N. \end{aligned}$$

REMARK. If $f \in \Gamma - B$ then f is said to be of unbounded index and the index of $f = \infty$.

LEMMA 2. *If n is a nonnegative integer and f is of index $> n$ (bounded or unbounded) then there exists a $\delta > 0$ such that if $g \in \Gamma$ and $d(f, g) < \delta$ then $g \in \Gamma - B_n$.*

Proof. Let n be given such that $n \geq 0$. Let $f \in \Gamma$ be given such that the index of f (bounded or unbounded) is $> n$. Let k be

a positive integer $> n$ and z_1 a complex number such that f is of index k at the point z_1 . Let $\delta_1 > 0$ be such that for

$$j < k, \frac{|f^{(k)}(z_1)|}{k!} - \delta_1 > \frac{|f^{(j)}(z_1)|}{j!}.$$

Let $R \geq |z_1|$. It is known that for every $j \leq k$ there exists an ε_j such that if $g_j \in \Gamma$ and $d(f^{(j)}, g_j) < \varepsilon_j$ then $|f^{(j)}(z) - g_j(z)| < \delta_1/2$ for $|z| \leq R$, and in particular at z_1 [1; p. 220]. In Lemma 1 we let $N = k$ and $\varepsilon = \text{Min} \{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_k\}$. Hence there exists a δ such that for $g \in \Gamma$ and $d(f, g) < \delta$ we have

$$\frac{|g^{(k)}(z_1)|}{k!} > \frac{|g^{(j)}(z_1)|}{j!} \text{ for } j = 0, 1, \dots, k - 1.$$

Thus g is of index $\geq k > n$.

LEMMA 3. *If $p(z)$ is a polynomial of degree n then $h(z) = e^z + p(z)$ is of index $\leq n + 1$.*

Proof. Let $k > n + 1$. Thus,

$$\frac{|h^{(k)}(z)|}{k!} = \frac{|e^z|}{k!} < \frac{|e^z|}{(n + 1)!} = \frac{|h^{(n+1)}(z)|}{(n + 1)!}$$

and hence h is of index $\leq n + 1$.

THEOREM 1. *For any n , B_n is nowhere dense in B and thus $B = \bigcup_{k=0}^{\infty} B_k$ is of the first category.*

Proof. Let n be given. Lemma 2 shows that B_n is closed. Thus let $f \in B_n$ and $\varepsilon > 0$. Let

$$e^{z^2} = \sum_{k=0}^{\infty} b_k z^k, f(z) = \sum_{k=0}^{\infty} a_k z^k, \text{ and } f_j(z) = \sum_{k=0}^j a_k z^k + \sum_{k=j+1}^{\infty} b_k z^k.$$

Since the order of f_j is two, for every j , we have that $f_j \in \Gamma - B$ [3]. Let i be such that $d(f, f_i) < \varepsilon/2$ and let $f_i = \sum_{k=0}^{\infty} c_k z^k$. For every $j > 0$ let $g_j(z) = \sum_{k=0}^j c_k z^k + \sum_{k=j+1}^{\infty} z^k/k!$. By the previous lemma the index of g_j is $\leq j + 1$. Thus, for every j , $g_j \in B$. In Lemma 2 we let $\delta < \varepsilon/2$ be such that if $g \in \Gamma$ and $d(f_i, g) < \delta$ then the index of g is $\geq n + 1$. Let m be such that $d(f_i, g_m) < \delta$. Thus $d(f, g_m) < \varepsilon$ and $g_m \in B - B_n$. Hence, for every integer n , B_n is nowhere dense in B and $B = \bigcup_{k=0}^{\infty} B_k$ is of the first category.

THEOREM 2. *The following are dense in Γ :*

- (a) $\Gamma - B$; and
- (b) $B - B_n$, for any integer n .

Proof. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \Gamma$.

(a) Let

$$e^{z^2} = \sum_{k=0}^{\infty} b_k z^k, \text{ and } f_j = \sum_{k=0}^j a_k z^k + \sum_{k=j+1}^{\infty} b_k z^k.$$

As in the proof of Theorem 1, $f_j \in \Gamma - B$ for every j and $\lim_{j \rightarrow \infty} d(f, f_j) = 0$.

(b) Now let

$$f_j(z) = \sum_{k=0}^j a_k z^k + \sum_{k=j+1}^{\infty} \frac{z^k}{k!}.$$

By Lemma 3, f_j is of bounded index for every j . For each j , if the index of f_j is $> n$ let $g_j = f_j$. If the index of f_j is $\leq n$ then by Theorem 1 there exists a function $g \in B - B_n$ such that $d(f_j, g) < 1/j$. Let $g_j = g$. Hence the $\lim_{j \rightarrow \infty} d(f, g_j) = 0$ and for every j , $g_j \in B - B_n$.

In conclusion it should be noted that the polynomials could be excluded from the class of entire functions, Γ , and the proofs of the preceding Lemmas and Theorems would remain valid.

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