

Pacific Journal of Mathematics

**UNIVERSAL COEFFICIENT THEOREMS FOR GENERALIZED
HOMOLOGY AND STABLE COHOMOTOPY**

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We show that if h is a nice (e.g. representable) homology functor and G is an Abelian group, then there is a cohomology functor $k(X; G)$ which is a “quasi-functor” of G and a short exact sequence

$$0 \longrightarrow \text{Ext}(h(\Sigma X), G) \longrightarrow k(X; G) \longrightarrow \text{Hom}(h(X), G) \longrightarrow 0$$

which is natural in X , “strongly quasi-natural” in G , and split if two additional conditions are satisfied.

If, for example, $h(X) = H_n(X)$, then $k(X; G) = H^n(X; G)$, and we obtain a proof of the ordinary Universal Coefficient Theorem which does not descend to the chain level but which does make heavy use of Brown’s Representability Theorem [2]. After setting up the machinery and proving some technical results in § 1, we derive in § 2 quasi-naturality and, with suitable restrictions, splitting of the sequence.

The construction of $k(X; G)$ involves an injective resolution of G . We show (2.8) that $k(X; G)$ is independent (up to *non*-canonical isomorphism) of the resolution chosen and we remark (in 2.12) that there is a particular injective resolution $I^*(G)$ which is even functorial.

In § 3 we prove a corresponding Universal Coefficient Theorem for stable cohomotopy. We construct (3.8) the following short exact sequence for finitely generated G and finite dimensional X

$$0 \longrightarrow \text{Ext}_Z(G, \pi_S^{n-1}X) \longrightarrow \{X, L(G, n)\} \longrightarrow \text{Hom}_Z(G, \pi_S^n X) \longrightarrow 0$$

which is natural in X , strongly quasi-natural in G , and split if $\{X, L(G, n)\}$ is a functor of G . $L(G, n)$ denotes the co-Moore space of type (G, n) , $\{X, Y\}$ = stable homotopy classes of maps, and $\pi_S^n(X) = \{X, S^n\}$. In § 4 we present some examples and a conjecture.

Let us recall from [5] the definition of a quasi-functor. Suppose \mathcal{A} and \mathcal{B} are categories and $S: |\mathcal{B}| \rightarrow |\mathcal{A}|$ is a function from the objects of \mathcal{B} to the objects of \mathcal{A} . We call S a *quasi-functor* if given any morphism $\beta: B \rightarrow B'$ in \mathcal{B} there is a nonempty set $S(\beta)$ of morphisms in \mathcal{A} satisfying

- (a) $S(\beta) \subset \mathcal{A}(SB, SB')$;
- (b) $\beta: B \rightarrow B'$ and $\beta': B' \rightarrow B''$ imply

$$S(\beta'\beta) \supset \{\alpha'\alpha \mid \alpha' \in S(\beta'), \alpha \in S(\beta)\};$$

- (c) $1_{SB} \in S(1_B)$.

Now if $S, U: \mathcal{B} \rightarrow \mathcal{A}$ are quasi-functors, we say that ν is a strong quasi-natural transformation from S to U provided that ν associates to each $B \in |\mathcal{B}|$ a morphism $\nu_B: S(B) \rightarrow U(B)$ and if $\beta: B \rightarrow B'$ then the following diagram is commutative for all $s \in S(\beta)$ and all $u \in U(\beta)$

$$\begin{array}{ccc} S(B) & \xrightarrow{\nu_B} & U(B) \\ s \downarrow & & \downarrow u \\ S(B') & \xrightarrow{\nu_{B'}} & U(B') \end{array} .$$

We call ν quasi-natural if for every $s \in S(\beta)$ there exists $u \in U(\beta)$ such that the above diagram commutes, and symmetrically, if for every u there exists s making the diagram commute. Note that if S is a quasi-functor which is not a functor and if $\nu: S \rightarrow S$ is the identity, then ν is quasi-natural but not strongly quasi-natural.

Early versions of these results comprised a portion of the author's doctoral dissertation written at Cornell University under the direction of Professor Peter Hilton. I am grateful to Professor Hilton for pointing out a number of substantial improvements. I should also like to thank the referee for his very helpful suggestions.

One may view this paper as an alternative to Adams' approach (see [1]).

1. **The machinery.** Let us recall that a homology functor on the category \mathcal{W}_*^ω of based connected CW complexes is a covariant functor $h: \mathcal{W}_*^\omega \rightarrow Ab$, the category of abelian groups, satisfying the following two conditions:

(i) if $A \xrightarrow{f} X \xrightarrow{g} C$ is a cofiber sequence, then

$$h(A) \xrightarrow{h(f)} h(X) \xrightarrow{h(g)} h(C)$$

is exact;

(ii) the natural map

$$\coprod_{\alpha \in \Gamma} h(X_\alpha) \longrightarrow h\left(\bigvee_{\alpha \in \Gamma} X_\alpha\right)$$

is an isomorphism for any index set Γ , where \coprod and \bigvee denote coproducts in Ab and \mathcal{W}_*^ω , respectively.

A contravariant functor $k: \mathcal{W}_*^\omega \rightarrow Ab$ is a cohomology functor provided that it satisfies the duals of (i) and (ii).

DEFINITION 1.1. We say that a homology functor is *special* provided that for every pair (X, A) of spaces in $|\mathcal{W}_*^\omega|$

$$\zeta: \lim_{\substack{\longrightarrow \\ n}} h(X^n \cup A) \longrightarrow h(X)$$

is a monomorphism, where X^n is the n -skeleton of X and ζ is induced by the inclusions $\zeta_n: X^n \cup A \rightarrow X$. For example, h is special if it is representable in the sense of Whitehead [7]. We call a cohomology functor $k: \mathscr{W}_*^\omega \rightarrow Ab$ *special* if it satisfies the dual condition—that is, the natural map

$$\rho: k(X) \longrightarrow \lim_{\substack{\longleftarrow \\ n}} k(X^n \cup A)$$

is epic.

For the remainder of this section, let h be a fixed but arbitrary special homology functor on \mathscr{W}_*^ω .

LEMMA 1.2. *Let I be an injective Abelian group. Then there is a based CW complex $\hat{B}(I)$ and a natural equivalence*

$$(1.3) \quad \hat{\eta}_I: [-, \hat{B}(I)] \longrightarrow \text{Hom}(h(-), I)$$

of cohomology functors on \mathscr{W}_*^ω , where $[-, -]$ denotes homotopy classes of maps.

Proof. Since $\text{Hom}(-, I)$ is an exact functor, $\text{Hom}(h(-), I)$ is a special cohomology functor on \mathscr{W}_*^ω . Hence, by the Representability Theorem of E. H. Brown [2], the conclusion follows.

LEMMA 1.4. *\hat{B} is a functor on injective Abelian groups.*

Proof. Let I and J be injective and let $\psi: I \rightarrow J$. Let $\hat{B}(\psi): \hat{B}(I) \rightarrow \hat{B}(J)$ be the unique (up to homotopy) map which makes the diagram below commutative.

$$(1.5) \quad \begin{array}{ccc} [-, \hat{B}(I)] & \xrightarrow{\hat{\eta}_I} & \text{Hom}(h(-), I) \\ \downarrow \hat{B}(\psi)_\# & & \downarrow \psi_\# \\ [-, \hat{B}(J)] & \xrightarrow{\hat{\eta}_J} & \text{Hom}(h(-), J) \end{array}$$

where the vertical arrows are induced by $\hat{B}(\psi)$ and ψ , respectively. (The existence and uniqueness of a map $\hat{B}(\psi)$ inducing the natural transformation $\hat{\eta}_J^{-1}\psi_\#\hat{\eta}_I$ follows from the Yoneda Lemma of category theory.)

For brevity, we shall write $\hat{\psi}$ instead of $\hat{B}(\psi)$. Let $I': 0 \rightarrow G \xrightarrow{\varphi} I \xrightarrow{\psi} J \rightarrow 0$ be a short exact sequence in which I and J are injective.

DEFINITION 1.6. We define $B(I')$ to be the mapping kernel of $\hat{\psi}$, so $B(I')$ fits into the following pull-back square

$$(1.7) \quad \begin{array}{ccc} B(\Gamma) & \longrightarrow & E\hat{B}(J) \\ j \downarrow & & \downarrow p \\ \hat{B}(I) & \xrightarrow{\hat{\psi}} & \hat{B}(J) \end{array}$$

where $E\hat{B}(J)$ is the (contractible) space of paths in $\hat{B}(J)$ starting at the base point, $p(\omega) = \omega(1)$, and the fibre of the fibration p is $\Omega\hat{B}(J)$. Note that $\hat{B}(I)$ and $\hat{B}(J)$ are homotopy associative and homotopy commutative H -spaces, and $\hat{\psi}$ is an H -map, so that $B(\Gamma)$ is also a homotopy associative and commutative H -space.

By Eckmann-Hilton duality, the map $\hat{\psi}$ fits into a co-Puppe sequence $P(\Gamma)$:

$$(1.8) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & \Omega B(\Gamma) & \xrightarrow{\Omega j} & \Omega \hat{B}(I) & \xrightarrow{\Omega \hat{\psi}} & \Omega \hat{B}(J) \\ & & & & & & \\ & & \longrightarrow & B(\Gamma) & \xrightarrow{j} & \hat{B}(I) & \xrightarrow{\hat{\psi}} \hat{B}(J) . \end{array}$$

LEMMA 1.9. *B and P are quasi-functors on injective resolutions Γ and morphisms of short exact sequences.*

Proof. Let $\Gamma: 0 \longrightarrow G \xrightarrow{\varphi} I \xrightarrow{\psi} J \longrightarrow 0$ and $\Gamma': 0 \longrightarrow G' \xrightarrow{\varphi'} I' \xrightarrow{\psi'} J' \longrightarrow 0$ be injective resolutions, and let μ be a morphism from Γ to Γ'

$$\mu = (e, f, g): \quad \begin{array}{ccccccc} 0 & \longrightarrow & G & \xrightarrow{\varphi} & I & \xrightarrow{\psi} & J \longrightarrow 0 \\ & & e \downarrow & & f \downarrow & & g \downarrow \\ 0 & \longrightarrow & G' & \xrightarrow{\varphi'} & I' & \xrightarrow{\psi'} & J' \longrightarrow 0 . \end{array}$$

Now we may choose a map $m: B(\Gamma) \rightarrow B(\Gamma')$ so that the diagram of homotopy classes of maps

$$(1.10) \quad \begin{array}{ccccccccc} \cdots & \longrightarrow & \Omega \hat{B}(I) & \longrightarrow & \Omega \hat{B}(J) & \longrightarrow & B(\Gamma) & \longrightarrow & \hat{B}(I) & \longrightarrow & \hat{B}(J) \\ & & \downarrow \Omega \hat{f} & & \downarrow \Omega \hat{g} & & \downarrow m & & \downarrow \hat{f} & & \downarrow \hat{g} \\ \cdots & \longrightarrow & \Omega \hat{B}(I') & \longrightarrow & \Omega \hat{B}(J') & \longrightarrow & B(\Gamma') & \longrightarrow & \hat{B}(I') & \longrightarrow & \hat{B}(J') \end{array}$$

is commutative. Thus, m induces a morphism \bar{m} from $P(\Gamma)$ to $P(\Gamma')$. However, the homotopy class of m is not uniquely determined. We now define $B(\mu)$ to be the set of all such homotopy classes m and $P(\mu)$ to be the set of all corresponding morphisms \bar{m} from $P(\Gamma)$ to $P(\Gamma')$. B and P are quasi-functors because the composite of commutative diagrams is a commutative diagram.

DEFINITION 1.11. We define for any injective resolution Γ the

cohomology functor $k(-; \Gamma) = [-, B(\Gamma)]$. By the preceding lemma, $k(-; \Gamma)$ is a quasi-functor of Γ .

2. The sequence. Now we are ready to state and prove our main result.

THEOREM 2.1. *Let h be any special homology functor, let $X \in |\mathcal{W}_*^w|$, and let $\Gamma: 0 \longrightarrow G \xrightarrow{\varphi} I \xrightarrow{\psi} J \longrightarrow 0$ be an injective resolution. Then there is a short exact sequence*

$$\sigma(X; \Gamma): 0 \longrightarrow \text{Ext}(h(\Sigma X), G) \longrightarrow k(X; \Gamma) \longrightarrow \text{Hom}(h(X), G) \longrightarrow 0$$

in which the arrows are natural in X and strongly quasi-natural in Γ .

REMARK 2.2. A word is necessary here to describe the second and fourth terms of $\sigma(X; \Gamma)$ as functors of Γ . If Γ is an injective resolution of G , Γ' is an injective resolution of G' , and $\mu = (e, f, g): \Gamma \rightarrow \Gamma'$, then the corresponding morphisms from $\text{Ext}(h(\Sigma X), G)$ to $\text{Ext}(h(\Sigma X), G')$ and from $\text{Hom}(h(X), G)$ to $\text{Hom}(h(X), G')$ are, respectively, $\text{Ext}(1, e)$ and $\text{Hom}(1, e)$.

Proof of 2.1. Applying the functor $[X, -]$ to 1.8 and using the adjointness of Ω and Σ , we obtain the exact sequence

$$(2.3) \quad \begin{aligned} [\Sigma X, \hat{B}(I)] &\xrightarrow{\hat{\phi}_\#(\Sigma X)} [\Sigma X, \hat{B}(J)] \longrightarrow [X, B(\Gamma)] \\ &\longrightarrow [X, \hat{B}(I)] \xrightarrow{\hat{\phi}_\#(X)} [X, \hat{B}(J)] \end{aligned}$$

and so, by homological algebra, a short exact sequence

$$(2.4) \quad 0 \longrightarrow \text{cok}(\hat{\psi}_\#(\Sigma X)) \longrightarrow k(X; \Gamma) \longrightarrow \ker(\hat{\psi}_\#(X)) \longrightarrow 0$$

which is natural in X and strongly quasi-natural in Γ .

But by 1.5 there are isomorphisms

$$(2.5) \quad \begin{aligned} s: \text{cok}(\hat{\psi}_\#(\Sigma X)) &\cong \text{cok}(\psi_\#(h(\Sigma X))) , \\ t: \ker(\hat{\psi}_\#(X)) &\cong \ker(\psi_\#(h(X))) ; \end{aligned}$$

and these isomorphisms are natural in X and Γ . (Note that the above groups are *functor* of Γ .) Moreover, there are also isomorphisms, well-known from homological algebra,

$$(2.6) \quad \begin{aligned} u: \text{cok}(\psi_\#(h(\Sigma X))) &\cong \text{Ext}(h(\Sigma X), G) , \\ v: \ker(\psi_\#(h(X))) &\cong \text{Hom}(h(X), G) , \end{aligned}$$

which are natural in X and Γ . These isomorphisms simply express the independence of Hom and Ext of the resolution of G . Now the

composite isomorphisms us and vt transform 2.4 into $\sigma(X; G)$ and preserve naturality in X and strong quasi-naturality in Γ .

The following lemma is well-known.

LEMMA 2.7. *Let $e: G \rightarrow G'$ be any homomorphism and let Γ and Γ' be injective resolutions of G and G' , respectively. Then e extends (non-uniquely) to a morphism $(e, f, g): \Gamma \rightarrow \Gamma'$ of resolutions.*

Now we can state a corollary to Theorem 2.1.

COROLLARY 2.8. *Let Γ and Γ' be two injective resolutions of the same group G , let h be a special homology theory, and let $X \in |\mathscr{W}_*^\omega|$. Then there is a (non-unique) isomorphism $\sigma(X; \Gamma) \cong \sigma(X; \Gamma')$.*

Proof. By 2.7, $1: G \rightarrow G$ extends to $(1, f, g): \Gamma \rightarrow \Gamma'$ which yields a morphism $M: \sigma(X; \Gamma) \rightarrow \sigma(X; \Gamma')$. Neither process is unique. But M induces the identity on the second and fourth terms, and therefore M must be an isomorphism by the 5-lemma.

Select for every Abelian group G an injective resolution $\Gamma(G)$ and define $\sigma(X; G) = \sigma(X; \Gamma(G))$. By 2.7, $\Gamma(G)$ is a quasi-functor of G and so $\sigma(X; G)$ is strongly quasi-natural in G . By 2.8, $\sigma(X; G)$ is independent, up to noncanonical isomorphism, of the resolution chosen. We shall fix, for definiteness, a particular $\Gamma(G)$ in 2.12.

Now we need a lemma.

LEMMA 2.9. *Let $G = G_1 \oplus G_2$ and let $\ell_j: G_j \rightarrow G$ denote the canonical injection ($j = 1, 2$). Let $X \in |\mathscr{W}_*^\omega|$ be fixed but arbitrary. Choose $m_j \in k(X; \ell_j)$ so that by strong quasi-naturality we have the commutative diagram*

$$(2.10) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \text{Ext}(h(\Sigma X), G_j) & \longrightarrow & k(X; G_j) & \longrightarrow & \text{Hom}(h(X), G_j) \longrightarrow 0 \\ & & \text{Ext}(1, \ell_j) \downarrow & & m_j \downarrow & & \text{Hom}(1, \ell_j) \downarrow \\ 0 & \longrightarrow & \text{Ext}(h(\Sigma X), G) & \longrightarrow & k(X; G) & \longrightarrow & \text{Hom}(h(X), G) \longrightarrow 0 . \end{array}$$

Then

$$m_1 \oplus m_2: k(X; G_1) \oplus k(X; G_2) \longrightarrow k(X; G)$$

is an isomorphism.

Proof. Ext and Hom are additive and, therefore, by the 5-lemma, $m_1 \oplus m_2$ is an isomorphism.

This lemma permits us to apply an elegant theorem of Hilton [3] to the sequence $\sigma(X; G)$.

THEOREM 2.11. (Universal Coefficient Theorem). *Let h be any special homology theory, let $X \in |\mathscr{W}_*^\omega|$, and let G be an Abelian group.*

(a) *Then there is a representable cohomology functor $k(X; G)$ which is a quasi-functor of G and a short exact sequence*

$$\sigma(X; G): 0 \longrightarrow \text{Ext}(h(\Sigma X), G) \xrightarrow{\tau_{XG}} k(X; G) \xrightarrow{\eta_{XG}} \text{Hom}(h(X), G) \longrightarrow 0$$

in which τ_{XG} and η_{XG} are natural in X and strongly quasi-natural in G .

(b) *Moreover, if for some fixed $X \in |\mathscr{W}_*^\omega|$ we have*

(i) *$k(X; G)$ is a functor of G and*

(ii) *$\text{Hom}(h(X), G)$ is a direct sum of cyclic groups, then $\sigma(X; G)$ splits for that X and every G .*

Proof. Part (a) is simply 2.1 with $\Gamma = \Gamma(G)$. Part (b) follows from [3] since Hom is a left-exact functor and, by (i) and 2.9, $k(X; G)$ is an additive functor of G so that $\sigma(X; G)$ is pure. Condition (ii) yields splitting.

2.12 Construction of $\Gamma(G)$

The following construction of $\Gamma(G)$ was related to me by Peter Hilton. Let G be any Abelian group. Then G has a canonical free resolution $0 \longrightarrow RG \xrightarrow{\lambda} FG \xrightarrow{\rho} G \longrightarrow 0$, where FG = free Abelian group on underlying set of G and RG = kernel ($FG \rightarrow G$). Let $QG = \coprod_{g \in G} Q_g$ where $Q_g = Q$, the rationals, for every $g \in G$, and define $\pi: FG \rightarrow QG$ by $\pi(\hat{g}) = 1 \in Q_g$ where \hat{g} is the generator of FG corresponding to g . Then setting $\pi\lambda = \bar{\lambda}: RG \rightarrow QG$, we have the following commutative exact diagram

$$(2.13) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & RG & \xrightarrow{\lambda} & FG & \xrightarrow{\rho} & G \longrightarrow 0 \\ & & \downarrow 1 & & \downarrow \pi & & \downarrow \varphi'_G \\ 0 & \longrightarrow & RG & \xrightarrow{\bar{\lambda}} & QG & \longrightarrow & I'G \longrightarrow 0 \\ & & \downarrow & & & & \downarrow \phi_G \\ & & 0 & & & & J'G \\ & & & & & & \downarrow \\ & & & & & & 0 \end{array}$$

where $I'G = \text{cok}(\bar{\lambda})$, φ'_G is induced by $(1, \pi)$, and $J'G = \text{cok}(\varphi'_G)$ with $\psi'_G: I'G \rightarrow J'G$ the canonical map. Put $\Gamma(G)$ = right-hand column in

2.13. Then $\Gamma(G)$ is an injective resolution of G since injective Abelian groups are closed under coproducts and quotients. Moreover, $\Gamma(G)$ is even functorial in G .

REMARK 2.14. The epimorphism η_{xG} of 2.11(a) can be interpreted as providing a weak adjunction from h to $B(-)$, where $B(G)$ is the space which represents $k(-; G)$. Thus, $B(-): Ab \rightarrow \mathscr{W}_*^w$ is a weak right adjoint (in the sense of [5]) to $h: \mathscr{W}_*^w \rightarrow Ab$, just as $K(-, n): Ab \rightarrow \mathscr{W}_*^w$, which associates to a group G the Eilenberg-MacLane space $K(G, n)$, is a weak right adjoint to $H_n: \mathscr{W}_*^w \rightarrow Ab$, the ordinary homology functor.

REMARK 2.15. The results of this section hold for theories as well as functors. Moreover, they can also be modified to hold for other categories than \mathscr{W}_*^w . Finally, there is nothing special about using Ab as a target; we could just as well do everything for R -module-valued homology and cohomology functors where R is a (commutative) ring of cohomological dimension 1.

3. The universal coefficient theorem for stable cohomotopy. Let G be a finitely generated Abelian group. Then there is a standard projective resolution $\rho(G)$ of G

$$(3.1) \quad 0 \longrightarrow RG \xrightarrow{\sigma_G} FG \xrightarrow{\tau_G} G \longrightarrow 0$$

where FG is the free Abelian group on a set SG of generators of G , τ_G is the canonical projection, RG is the kernel of τ_G , and σ_G is the canonical injection of RG into FG . As in Lemma 2.7 $\rho(G)$ is a quasi-functor of G . Define

$$(3.2) \quad \tilde{F}_n G = \bigvee_{t \in SG} S_{(t)}^n, S_{(t)}^n = S^n, t \in SG, n \geq 0,$$

and, similarly, define

$$(3.3) \quad \tilde{R}_n G = \bigvee_{q \in \Gamma} S_{(q)}^n, S_{(q)}^n = S^n, n \geq 0, q \in \Gamma = \text{set of generators of } RG.$$

LEMMA 3.4. Let $n \geq 1$. Then there exists a map $\tilde{\sigma}_G^n: \tilde{F}_n G \rightarrow \tilde{R}_n G$ (unique up to homotopy) which induces σ_G upon applying $H^n(-; Z)$.

Proof. If $\varphi: Z \rightarrow Z$, then φ is just multiplication by some integer m ($m = 0$ is not excluded), and we write $\varphi = m$. Then any map f of degree m from S^n to S^n induces φ in n th cohomology, and we can write $\tilde{\varphi}^n = m$.

Thus, by stable additivity, $[\tilde{F}_n G, \tilde{R}_n G]$ is in one-to-one correspond-

ence with integer matrices (m_{iq}) , and the set $\text{Hom}(RG, FG)$ of homomorphisms is in one-to-one correspondence with integer matrices (m_{qt}) . Moreover, (m_{qt}) is induced by its transpose (m_{tiq}) so we let

$$(3.5) \quad \tilde{\sigma}_G^n = (m_{tiq}) ,$$

where (m_{qt}) is the matrix corresponding to σ_G .

Since $\sum \tilde{F}_n G = \tilde{F}_{n+1} G$, $\sum \tilde{R}_n G = \tilde{R}_{n+1} G$, and $\sum \tilde{\sigma}_G^n = \tilde{\sigma}_G^{n+1}$, we have the following Puppe sequence $\tilde{\rho}_G^n$ for $\tilde{\sigma}_G^n$, $n \geq 1$

$$(3.6) \quad \tilde{F}_n G \xrightarrow{\tilde{\sigma}_G^n} \tilde{R}_n G \longrightarrow L(G, n+1) \longrightarrow \tilde{F}_{n+1} G \xrightarrow{\tilde{\sigma}_G^{n+1}} \tilde{R}_{n+1} G$$

where $L(G, n+1) = (\text{reduced})$ mapping cone of $\tilde{\sigma}_G^n$. Thus, $L(G, n+1)$ is just the co-Moore space of type $(G, n+1)$; i.e. $H^q(L(G, n+1); Z) = 0$ $q \neq n+1$, $H^{n+1}(L(G, n+1); Z) = G$, and $\pi_1(L(G, n+1)) = 0$ by Van Kampen when $n \geq 2$. Since $\rho(G)$ is a quasi-functor of G , so is $\tilde{\rho}^n(G)$ and, hence, $L(G, n+1)$.

Let \mathcal{W}_*^∞ denote the category of based connected finite-dimensional CW complexes. If $X \in |\mathcal{W}_*^\infty|$ and $Y \in |\mathcal{W}_*^\omega|$, then we define

$$\{X, Y\} = \lim_{\vec{k}} [\Sigma^k X, \Sigma^k Y] ,$$

and we recall that $\{X, -\}$ is a special homology functor on \mathcal{W}_*^ω .

Therefore, applying $\{X, -\}$ to 3.6, we obtain an exact sequence

$$(3.7) \quad \begin{aligned} \{X, \tilde{F}_n G\} &\xrightarrow{\tilde{\sigma}_G^n} \{X, \tilde{R}_n G\} \longrightarrow \{X, L(G, n+1)\} \\ &\longrightarrow \{X, \tilde{F}_{n+1} G\} \xrightarrow{\tilde{\sigma}_G^{n+1}} \{X, \tilde{R}_{n+1} G\} . \end{aligned}$$

But clearly $\{X, \tilde{F}_n G\} \cong \text{Hom}(FG, \pi_S^n(X))$ by an isomorphism which is natural in X and also natural in $G(\pi_S^n(X) = \{X, S^n\})$. Therefore, as in § 2 we obtain the following theorem.

THEOREM 3.8. *Let G be a finitely generated Abelian group. Let $n \geq 2$ and let $X \in |\mathcal{W}_*^\infty|$. Then there is a short exact sequence*

$$(3.9) \quad \begin{aligned} 0 &\longrightarrow \text{Ext}(G, \pi_S^{n-1}(X)) \longrightarrow \{X, L(G, n)\} \\ &\longrightarrow \text{Hom}(G, \pi_S^n(X)) \longrightarrow 0 \end{aligned}$$

which is natural in X and strongly quasi-natural in G . The sequence splits if, for some fixed X , $\{X, L(G, n)\}$ is a functor of G .

As a corollary of this theorem, we have the following result of Hilton-Olun-see [4].

COROLLARY 3.10. *Let G_1 and G_2 be finitely generated Abelian*

groups and $n \geq 4$. Then there is a short exact sequence

$$(3.11) \quad \begin{aligned} 0 \longrightarrow T(G_1)^* \otimes G_2 \otimes Z_2 &\longrightarrow [L(G_2, n), L(G_1, n)] \\ &\longrightarrow \text{Hom}(G_1, G_2) \longrightarrow 0 \end{aligned}$$

which is strongly quasi-natural in G_1 and G_2 , where $T(G)$ = torsion subgroup of G and $G^* = \text{Hom}(G, Q/Z)$ ($\cong G$ if G is finite).

Proof. Applying 3.9 to $G = G_1$ and $X = L(G_2, n)$, we get

$$(3.12) \quad \begin{aligned} 0 \longrightarrow \text{Ext}(G_1, \pi_S^{n-1}(L(G_2, n))) &\longrightarrow \{L(G_2, n), L(G_1, n)\} \\ &\longrightarrow \text{Hom}(G_1, \pi_S^n(L(G_2, n))) \longrightarrow 0. \end{aligned}$$

But for $n \geq 4$

$$\begin{aligned} \pi_S^{n-1}(L(G_2, n)) &\cong G_2 \otimes Z_2 \\ \{L(G_2, n), L(G_1, n)\} &\cong [L(G_2, n), L(G_1, n)], \end{aligned}$$

and

$$(3.13) \quad \begin{aligned} \pi_S^n(L(G_2, n)) &\cong G_2, \quad \text{so we have for } n \geq 4 \\ 0 \longrightarrow \text{Ext}(G_1, G_2 \otimes Z_2) &\longrightarrow [L(G_2, n), L(G_1, n)] \\ &\longrightarrow \text{Hom}(G_1, G_2) \longrightarrow 0. \end{aligned}$$

Now we are done since $\text{Ext}(G_1, -) \cong T(G_1)^* \otimes -$ as functors on the category of finitely generated Abelian groups.

4. Some examples and a conjecture. The general problem of computing $k^*(X; G)$, for a given homology theory h_* and group G , is very difficult, even when the group is injective. For example, if $h_q = \pi_q^S = \{S^q, -\}$ and $G = Q$, then

$$(4.1) \quad k^q(X; Q) \cong H^q(X; Q)$$

by an easy argument based on Serre's result [6] that $\pi_q^S(S^r)$ is finite for $r \neq q$. With h_* as above and $G = Q/Z$ it is easy to establish

$$(4.2) \quad k^q(S^r; Q/Z) \cong \begin{cases} \pi_q^S(S^r), & r \neq q \\ Q/Z, & r = q. \end{cases}$$

Thus computing $k^*(X; Q/Z)$ in this case amounts to knowing the stable homotopy groups of spheres!

If the homology theory h_* is represented by a spectrum B , then the spectrum $B(G)$ which represents $k^*(-; G)$ can be thought of as obtained from B by introducing G coefficients. The spectrum B also represents a cohomology theory, and we have the following

CONJECTURE 4.3. If $\pi_* B$ is a ring of cohomological dimension 1,

then there is a homotopy equivalence of spectra $B \simeq B(Z)$.

This conjecture simply says that our method and Adams' [1] coincide over rings of cohomological dimension 1-where his spectral sequence collapses to a Universal Coefficient Sequence.

REMARK 4.4. It is *not* true in general that $k^*(-; Z)$ is the cohomology theory associated to the spectrum B which represents h_* . For example, if, as above, $B =$ sphere spectrum and $h_* =$ stable homotopy is the homology theory represented by B , then

$$(4.5) \quad k^n(S^q; Z) = 0 \quad \text{for all } q > n.$$

But the cohomology functor associated to the sphere spectrum is stable cohomotopy, and certainly

$$(4.6) \quad \pi_S^n(S^q) \neq 0 \quad \text{for all } q > n.$$

In particular, $k^n(S^{n+1}; Z) = 0 \not\cong Z_2 = \pi_S^n(S^{n+1})$.

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Received October 23, 1970 and in revised form February 24, 1971.

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PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

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