UNIVERSAL COEFFICIENT THEOREMS FOR GENERALIZED HOMOLOGY AND STABLE COHOMOTOPY

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We show that if \( h \) is a nice (e.g. representable) homology functor and \( G \) is an Abelian group, then there is a cohomology functor \( k(X; G) \) which is a "quasi-functor" of \( G \) and a short exact sequence

\[
0 \rightarrow \text{Ext}(h(\Sigma X), G) \rightarrow k(X; G) \rightarrow \text{Hom}(h(X), G) \rightarrow 0
\]

which is natural in \( X \), "strongly quasi-natural" in \( G \), and split if two additional conditions are satisfied.

If, for example, \( h(X) = H_n(X) \), then \( k(X; G) = H^n(X; G) \), and we obtain a proof of the ordinary Universal Coefficient Theorem which does not descend to the chain level but which does make heavy use of Brown's Representability Theorem [2]. After setting up the machinery and proving some technical results in \( \S \ 1 \), we derive in \( \S \ 2 \) quasi-naturality and, with suitable restrictions, splitting of the sequence.

The construction of \( k(X; G) \) involves an injective resolution of \( G \). We show (2.8) that \( k(X; G) \) is independent (up to non-canonical isomorphism) of the resolution chosen and we remark (in 2.12) that there is a particular injective resolution \( I'(G) \) which is even functorial.

In \( \S \ 3 \) we prove a corresponding Universal Coefficient Theorem for stable cohomotopy. We construct (3.8) the following short exact sequence for finitely generated \( G \) and finite dimensional \( X \)

\[
0 \rightarrow \text{Ext}_Z(G, \pi_{S^{-1}}X) \rightarrow \{X, L(G, n)\} \rightarrow \text{Hom}_Z(G, \pi_S^X) \rightarrow 0
\]

which is natural in \( X \), strongly quasi-natural in \( G \), and split if \( \{X, L(G, n)\} \) is a functor of \( G \). \( L(G, n) \) denotes the co-Moore space of type \( (G, n) \), \( \{X, Y\} = \text{stable homotopy classes of maps} \), and \( \pi_S^X(X) = \{X, S^\infty\} \). In \( \S \ 4 \) we present some examples and a conjecture.

Let us recall from [5] the definition of a quasi-functor. Suppose \( \mathcal{A} \) and \( \mathcal{B} \) are categories and \( S: |\mathcal{B}| \rightarrow |\mathcal{A}| \) is a function from the objects of \( \mathcal{B} \) to the objects of \( \mathcal{A} \). We call \( S \) a quasi-functor if given any morphism \( \beta: B \rightarrow B' \) in \( \mathcal{B} \) there is a nonempty set \( S(\beta) \) of morphisms in \( \mathcal{A} \) satisfying

(a) \( S(\beta) \subset \mathcal{A}(SB, SB') \);
(b) \( \beta: B \rightarrow B' \) and \( \beta': B' \rightarrow B'' \) imply

\[
S(\beta' \beta) \supset \{\alpha' \alpha | \alpha' \in S(\beta'), \alpha \in S(\beta)\}
\]

(c) \( 1_{SB} \in S(1_B) \).
Now if $S, U : \mathcal{B} \to \mathcal{A}$ are quasi-functors, we say that $\nu$ is a strong quasi-natural transformation from $S$ to $U$ provided that $\nu$ associates to each $B \in \mathcal{B}$ a morphism $\nu_B : S(B) \to U(B)$ and if $\beta : B \to B'$ then the following diagram is commutative for all $s \in S(\beta)$ and all $u \in U(\beta)$:

\[
\begin{array}{ccc}
S(B) & \xrightarrow{\nu_B} & U(B) \\
\downarrow s & & \downarrow u \\
S(B') & \xrightarrow{\nu_{B'}} & U(B')
\end{array}
\]

We call $\nu$ quasi-natural if for every $s \in S(\beta)$ there exists $u \in U(\beta)$ such that the above diagram commutes, and symmetrically, if for every $u$ there exists $s$ making the diagram commute. Note that if $S$ is a quasi-functor which is not a functor and if $\nu : S \to S$ is the identity, then $\nu$ is quasi-natural but not strongly quasi-natural.

Early versions of these results comprised a portion of the author’s doctoral dissertation written at Cornell University under the direction of Professor Peter Hilton. I am grateful to Professor Hilton for pointing out a number of substantial improvements. I should also like to thank the referee for his very helpful suggestions.

One may view this paper as an alternative to Adams’ approach (see [1]).

1. The machinery. Let us recall that a homology functor on the category $\mathcal{W}_*^\omega$ of based connected CW complexes is a covariant functor $h : \mathcal{W}_*^\omega \to Ab$, the category of abelian groups, satisfying the following two conditions:

   (i) if $A \overset{f}{\longrightarrow} X \overset{g}{\longrightarrow} C$ is a cofiber sequence, then

   \[
   h(A) \xrightarrow{h(f)} h(X) \xrightarrow{h(g)} h(C)
   \]

   is exact;

   (ii) the natural map

   \[
   \prod_{\alpha \in \Gamma} h(X_\alpha) \longrightarrow h(\bigvee_{\alpha \in \Gamma} X_\alpha)
   \]

   is an isomorphism for any index set $\Gamma$, where $\prod$ and $\bigvee$ denote coproducts in $Ab$ and $\mathcal{W}_*^\omega$, respectively.

A contravariant functor $k : \mathcal{W}_*^\omega \to Ab$ is a cohomology functor provided that it satisfies the duals of (i) and (ii).

**Definition 1.1.** We say that a homology functor is special provided that for every pair $(X, A)$ of spaces in $|\mathcal{W}_*^\omega|$:

\[
\zeta : \lim_{\rightarrow} h(X^n \cup A) \longrightarrow h(X)
\]
is a monomorphism, where $X^n$ is the $n$-skeleton of $X$ and $\zeta$ is induced by the inclusions $\zeta_n: X^n \cup A \to X$. For example, $h$ is special if it is representable in the sense of Whitehead [7]. We call a cohomology functor $k: \mathcal{W}_\ast \to \text{Ab}$ special if it satisfies the dual condition—that is, the natural map
\[
\rho: k(X) \longrightarrow \lim_{\rightarrow n} k(X^n \cup A)
\]
is epic.

For the remainder of this section, let $h$ be a fixed but arbitrary special homology functor on $\mathcal{W}_\ast$.

**Lemma 1.2.** Let $I$ be an injective Abelian group. Then there is a based CW complex $B(I)$ and a natural equivalence
\[
\tilde{\gamma}_I: [-, B(I)] \longrightarrow \text{Hom}(h(-), I)
\]
of cohomology functors on $\mathcal{W}_\ast$, where $[-, -]$ denotes homotopy classes of maps.

**Proof.** Since $\text{Hom}(-, I)$ is an exact functor, $\text{Hom}(h(-), I)$ is a special cohomology functor on $\mathcal{W}_\ast$. Hence, by the Representability Theorem of E. H. Brown [2], the conclusion follows.

**Lemma 1.4.** $B$ is a functor on injective Abelian groups.

**Proof.** Let $I$ and $J$ be injective and let $\psi: I \to J$. Let $\hat{B}(\psi): \hat{B}(I) \to \hat{B}(J)$ be the unique (up to homotopy) map which makes the diagram below commutative.

\[
\begin{aligned}
[-, \hat{B}(I)] &\xrightarrow{\tilde{\gamma}_I} \text{Hom}(h(-), I) \\
\downarrow_{\hat{B}(\psi)_*} &\downarrow_{\psi_*}
\end{aligned}
\]
where the vertical arrows are induced by $\hat{B}(\psi)$ and $\psi$, respectively. (The existence and uniqueness of a map $\hat{B}(\psi)$ inducing the natural transformation $\tilde{\gamma}_I^{-1}\psi_\ast\tilde{\gamma}_I$ follows from the Yoneda Lemma of category theory.)

For brevity, we shall write $\hat{\psi}$ instead of $\hat{B}(\psi)$. Let $0 \longrightarrow G \xrightarrow{\varphi} I \xrightarrow{\psi} J \longrightarrow 0$ be a short exact sequence in which $I$ and $J$ are injective.

**Definition 1.6.** We define $B(I')$ to be the mapping kernel of $\hat{\psi}$, so $B(I')$ fits into the following pull-back square
where $E\hat{B}(J)$ is the (contractible) space of paths in $\hat{B}(J)$ starting at the base point, $p(\omega) = \omega(1)$, and the fibre of the fibration $p$ is $\Omega\hat{B}(J)$. Note that $\hat{B}(I)$ and $\hat{B}(J)$ are homotopy associative and homotopy commutative $H$-spaces, and $\hat{\varphi}$ is an $H$-map, so that $B(\Gamma')$ is also a homotopy associative and commutative $H$-space.

By Eckmann-Hilton duality, the map $\hat{\varphi}$ fits into a co-Puppe sequence $P(\Gamma)$:

\begin{equation}
\cdots \to \Omega B(\Gamma) \xrightarrow{\Omega j} \Omega \hat{B}(I) \xrightarrow{\Omega \hat{\varphi}} \Omega \hat{B}(J) \to B(\Gamma) \xrightarrow{j} \hat{B}(I) \xrightarrow{\hat{\varphi}} \hat{B}(J).
\end{equation}

**Lemma 1.9.** $B$ and $P$ are quasi-functors on injective resolutions $\Gamma$ and morphisms of short exact sequences.

**Proof.** Let $\Gamma': 0 \to G \xrightarrow{\varphi} I \xrightarrow{\varphi} J \to 0$ and $\Gamma'': 0 \to G' \xrightarrow{\varphi'} I' \xrightarrow{\varphi'} J' \to 0$ be injective resolutions, and let $\mu$ be a morphism from $\Gamma'$ to $\Gamma''$

\[
0 \to G \xrightarrow{\varphi} I \xrightarrow{\varphi} J \to 0.
\]

\[
\mu = (e, f, g):
\]

\[
0 \to G' \xrightarrow{\varphi'} I' \xrightarrow{\varphi'} J' \to 0.
\]

Now we may choose a map $m: B(\Gamma) \to B(\Gamma'')$ so that the diagram of homotopy classes of maps

\begin{equation}
\cdots \to \Omega \hat{B}(I) \xrightarrow{\Omega \hat{f}} \Omega \hat{B}(J) \to B(\Gamma') \to \hat{B}(I) \to \hat{B}(J)
\end{equation}

is commutative. Thus, $m$ induces a morphism $\bar{m}$ from $P(\Gamma)$ to $P(\Gamma'')$. However, the homotopy class of $m$ is not uniquely determined. We now define $B(\mu)$ to be the set of all such homotopy classes $m$ and $P(\mu)$ to be the set of all corresponding morphisms $\bar{m}$ from $P(\Gamma)$ to $P(\Gamma'')$. $B$ and $P$ are quasi-functors because the composite of commutative diagrams is a commutative diagram.

**Definition 1.11.** We define for any injective resolution $\Gamma$ the
cohomology functor \( k(-; \Gamma) = [-, B(\Gamma)] \). By the preceding lemma, \( k(-; \Gamma) \) is a quasi-functor of \( \Gamma \).

2. The sequence. Now we are ready to state and prove our main result.

**Theorem 2.1.** Let \( h \) be any special homology functor, let \( X \in |\mathcal{W}| \), and let \( \Gamma: 0 \to G \to I \to J \to 0 \) be an injective resolution. Then there is a short exact sequence

\[
\sigma(X; \Gamma): 0 \to \text{Ext}(h(\Sigma X), G) \to k(X; \Gamma) \to \text{Hom}(h(X), G) \to 0
\]

in which the arrows are natural in \( X \) and strongly quasi-natural in \( \Gamma \).

**Remark 2.2.** A word is necessary here to describe the second and fourth terms of \( \sigma(X; \Gamma) \) as functors of \( \Gamma \). If \( \Gamma \) is an injective resolution of \( G \), \( \Gamma' \) is an injective resolution of \( G' \), and \( \mu = (e, f, g): \Gamma \to \Gamma' \), then the corresponding morphisms from \( \text{Ext}(h(\Sigma X), G) \) to \( \text{Ext}(h(\Sigma X), G') \) and from \( \text{Hom}(h(X), G) \) to \( \text{Hom}(h(X), G') \) are, respectively, \( \text{Ext}(1, e) \) and \( \text{Hom}(1, e) \).

**Proof of 2.1.** Applying the functor \([X, -]\) to 1.8 and using the adjointness of \( \Omega \) and \( \Sigma \), we obtain the exact sequence

\[
(2.3) \quad [\Sigma X, \hat{B}(I)] \xrightarrow{\hat{\psi}_z(\Sigma X)} [\Sigma X, \hat{B}(J)] \longrightarrow [X, \hat{B}(I)] \xrightarrow{\hat{\psi}_z(X)} [X, \hat{B}(J)]
\]

and so, by homological algebra, a short exact sequence

\[
(2.4) \quad 0 \longrightarrow \cok(\hat{\psi}_z(X)) \longrightarrow k(X; \Gamma) \longrightarrow \ker(\hat{\psi}_z(X)) \longrightarrow 0
\]

which is natural in \( X \) and strongly quasi-natural in \( \Gamma \).

But by 1.5 there are isomorphisms

\[
(2.5) \quad s: \cok(\hat{\psi}_z(X)) \cong \cok(\psi_z(h(\Sigma X))),
\]

and these isomorphisms are natural in \( X \) and \( \Gamma \). (Note that the above groups are *functor* of \( \Gamma \).) Moreover, there are also isomorphisms, well-known from homological algebra,

\[
(2.6) \quad u: \cok(\psi_z(h(\Sigma X))) \cong \text{Ext}(h(\Sigma X), G),
\]

which are natural in \( X \) and \( \Gamma \). There isomorphisms simply express the independence of \( \text{Hom} \) and \( \text{Ext} \) of the resolution of \( G \). Now the
composite isomorphisms $us$ and $vt$ transform 2.4 into $\sigma(X; G)$ and preserve naturality in $X$ and strong quasi-naturality in $\Gamma$.

The following lemma is well-known.

**Lemma 2.7.** Let $e: G \to G'$ be any homomorphism and let $\Gamma$ and $\Gamma'$ be injective resolutions of $G$ and $G'$, respectively. Then $e$ extends (non-uniquely) to a morphism $(e, f, g): \Gamma \to \Gamma'$ of resolutions.

Now we can state a corollary to Theorem 2.1.

**Corollary 2.8.** Let $\Gamma$ and $\Gamma'$ be two injective resolutions of the same group $G$, let $h$ be a special homology theory, and let $X \in | \mathcal{W}^* |$. Then there is a (non-unique) isomorphism $\sigma(X; \Gamma) \cong \sigma(X; \Gamma')$.

**Proof.** By 2.7, $1: G \to G$ extends to $(1, f, g): \Gamma \to \Gamma'$ which yields a morphism $M: \sigma(X; \Gamma) \to \sigma(X; \Gamma')$. Neither process is unique. But $M$ induces the identity on the second and fourth terms, and therefore $M$ must be an isomorphism by the 5-lemma.

Select for every Abelian group $G$ an injective resolution $\Gamma(G)$ and define $\sigma(X; G) = \sigma(X; \Gamma(G))$. By 2.7, $\Gamma(G)$ is a quasi-functor of $G$ and so $\sigma(X; G)$ is strongly quasi-natural in $G$. By 2.8, $\sigma(X; G)$ is independent, up to noncanonical isomorphism, of the resolution chosen. We shall fix, for definiteness, a particular $\Gamma(G)$ in 2.12.

Now we need a lemma.

**Lemma 2.9.** Let $G = G_1 \oplus G_2$ and let $\iota_j: G_j \to G$ denote the canonical injection ($j = 1, 2$). Let $X \in | \mathcal{W}^* |$ be fixed but arbitrary. Choose $m_j \in k(X; \iota_j)$ so that by strong quasi-naturality we have the commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & \text{Ext} \ (h(\Sigma X), G_j) & \longrightarrow & k(X; G_j) & \longrightarrow & \text{Hom} \ (h(X), G_j) & \longrightarrow & 0 \\
(2.10) & & \downarrow \text{Ext} \ (1, \iota_j) & & \downarrow m_j & & \downarrow \text{Hom} \ (1, \iota_j) & & \\
0 & \longrightarrow & \text{Ext} \ (h(\Sigma X), G) & \longrightarrow & k(X; G) & \longrightarrow & \text{Hom} \ (h(X), G) & \longrightarrow & 0 .
\end{array}
$$

Then

$$m_1 \oplus m_2: k(X; G_1) \oplus k(X; G_2) \longrightarrow k(X; G)
$$

is an isomorphism.

**Proof.** Ext and Hom are additive and, therefore, by the 5-lemma, $m_1 \oplus m_2$ is an isomorphism.

This lemma permits us to apply an elegant theorem of Hilton [3] to the sequence $\sigma(X; G)$. 

**Theorem 2.11.** (Universal Coefficient Theorem). Let \( h \) be any special homology theory, let \( X \in \mathcal{W}_{n}^{*} \), and let \( G \) be an Abelian group.

(a) Then there is a representable cohomology functor \( k(X; G) \) which is a quasi-functor of \( G \) and a short exact sequence

\[
\sigma(X; G) : 0 \rightarrow \text{Ext}(h(\Sigma X), G) \xrightarrow{\tau_{XG}} k(X; G) \xrightarrow{\eta_{XG}} \text{Hom}(h(X), G) \rightarrow 0
\]

in which \( \tau_{XG} \) and \( \eta_{XG} \) are natural in \( X \) and strongly quasi-natural in \( G \).

(b) Moreover, if for some fixed \( X \in \mathcal{W}_{n}^{*} \) we have

(i) \( k(X; G) \) is a functor of \( G \) and

(ii) \( \text{Hom}(h(X, G) \) is a direct sum of cyclic groups, then \( \sigma(X; G) \) splits for that \( X \) and every \( G \).

**Proof.** Part (a) is simply 2.1 with \( \Gamma_{*} = \Gamma_{*}(G) \). Part (b) follows from [3] since \( \text{Hom} \) is a left-exact functor and, by (i) and 2.9, \( k(X; G) \) is an additive functor of \( G \) so that \( \sigma(X; G) \) is pure. Condition (ii) yields splitting.

2.12 Construction of \( \Gamma(G) \)

The following construction of \( \Gamma(G) \) was related to me by Peter Hilton. Let \( G \) be any Abelian group. Then \( G \) has a canonical free resolution \( 0 \rightarrow RG \xrightarrow{\lambda} FG \xrightarrow{\rho} G \rightarrow 0 \), where \( FG = \text{free Abelian group on underlying set of } G \) and \( RG = \text{kernel } (FG \rightarrow G) \). Let \( QG = \prod_{g \in G} Q_{g} \) where \( Q_{g} = \mathbb{Q} \), the rationals, for every \( g \in G \), and define \( \pi : FG \rightarrow QG \) by \( \pi(\hat{g}) = 1 \in Q_{g} \) where \( \hat{g} \) is the generator of \( FG \) corresponding to \( g \). Then setting \( \pi_{\lambda} = \bar{\lambda} : RG \rightarrow QG \), we have the following commutative exact diagram

\[
\begin{array}{ccc}
0 & 0 & 0 \\
& \downarrow & \downarrow & \downarrow \\
0 & RG & FG & G & \rightarrow 0 \\
& \downarrow 1 & \downarrow \pi & \downarrow \bar{\lambda} \\
0 & RG & QG & I'G & \rightarrow 0 \\
& \downarrow & \downarrow \phi_{G} & \\
0 & J'G & \\
& \downarrow & \\
0 & \\
\end{array}
\]

(2.13)

where \( I'G = \text{cok } (\bar{\lambda}), \phi_{G} \) is induced by \((1, \pi)\), and \( J'G = \text{cok } (\phi_{G}) \) with \( \psi_{G} : I'G \rightarrow J'G \) the canonical map. Put \( \Gamma(G) = \text{right-hand column in} \).
2.13. Then $\Gamma(G)$ is an injective resolution of $G$ since injective Abelian groups are closed under coproducts and quotients. Moreover, $\Gamma(G)$ is even functorial in $G$.

Remark 2.14. The epimorphism $\eta_{\mathcal{X}G}$ of 2.11(a) can be interpreted as providing a weak adjunction from $h$ to $B(-)$, where $B(G)$ is the space which represents $k(-; G)$. Thus, $B(-): Ab \to \mathcal{W}^\omega_*$ is a weak right adjoint (in the sense of [5]) to $h: \mathcal{W}^\omega_* \to Ab$, just as $K(-, n): Ab \to \mathcal{W}^\omega_*$, which associates to a group $G$ the Eilenberg-MacLane space $K(G, n)$, is a weak right adjoint to $H_n: \mathcal{W}^\omega_* \to Ab$, the ordinary homology functor.

Remark 2.15. The results of this section hold for theories as well as functors. Moreover, they can also be modified to hold for other categories than $\mathcal{W}^\omega_*$. Finally, there is nothing special about using $Ab$ as a target; we could just as well do everything for $R$-module-valued homology and cohomology functors where $R$ is a (commutative) ring of cohomological dimension 1.

3. The universal coefficient theorem for stable cohomotopy

Let $G$ be a finitely generated Abelian group. Then there is a standard projective resolution $\rho(G)$ of $G$

\begin{equation}
0 \to RG \xrightarrow{\sigma_G} FG \xrightarrow{\tau_G} G \to 0
\end{equation}

where $FG$ is the free Abelian group on a set $SG$ of generators of $G$, $\tau_G$ is the canonical projection, $RG$ is the kernel of $\tau_G$, and $\sigma_G$ is the canonical injection of $RG$ into $FG$. As in Lemma 2.7 $\rho(G)$ is a quasi-functor of $G$. Define

\begin{equation}
\tilde{F}_n G = \bigvee_{t \in SG} S^n_t, S^n_t = S^n, t \in SG, n \geq 0,
\end{equation}

and, similarly, define

\begin{equation}
\tilde{R}_n G = \bigvee_{q \in \Gamma} S^n_{(q)}, S^n_{(q)} = S^n, n \geq 0, q \in \Gamma = \text{set of generators of } RG.
\end{equation}

Lemma 3.4. Let $n \geq 1$. Then there exists a map $\tilde{\sigma}_G: \tilde{F}_n G \to \tilde{R}_n G$ (unique up to homotopy) which induces $\sigma_G$ upon applying $H^*(-; Z)$.

Proof. If $\varphi: Z \to Z$, then $\varphi$ is just multiplication by some integer $m$ ($m = 0$ is not excluded), and we write $\varphi = m$. Then any map $f$ of degree $m$ from $S^n$ to $S^n$ induces $\varphi$ in $n$th cohomology, and we can write $\varphi^n = m$.

Thus, by stable additivity, $[\tilde{F}_n G, \tilde{R}_n G]$ is in one-to-one correspond-
ence with integer matrices \((m_{qt})\), and the set \(\text{Hom}(RG, FG)\) of homomorphisms is in one-to-one correspondence with integer matrices \((m_{qt})\). Moreover, \((m_{qt})\) is induced by its transpose \((m_{qt})\) so we let

\[
(3.5) \quad \tilde{\sigma}_G^n = (m_{tb})
\]

where \((m_{qt})\) is the matrix corresponding to \(\sigma_G\).

Since \(\sum \tilde{F}_n G = \tilde{F}_{n+1} G\), \(\sum \tilde{R}_n G = \tilde{R}_{n+1} G\), and \(\sum \tilde{\sigma}_G^n = \tilde{\sigma}_G^{n+1}\), we have the following Puppe sequence \(\tilde{\sigma}_G^n\) for \(\tilde{\sigma}_G^n, n \geq 1\)

\[
(3.6) \quad \tilde{F}_n G \xrightarrow{\tilde{\sigma}_G^n} \tilde{R}_n G \longrightarrow L(G, n+1) \longrightarrow \tilde{F}_{n+1} G \xrightarrow{\tilde{\sigma}_G^{n+1}} \tilde{R}_{n+1} G
\]

where \(L(G, n+1)\) is (reduced) mapping cone of \(\tilde{\sigma}_G^n\). Thus, \(L(G, n+1)\) is just the co-Moore space of type \((G, n + 1)\); i.e. \(H^q(L(G, n+1); Z) = 0\) for \(q \neq n + 1\), \(H^{n+1}(L(G, n+1); Z) = G\), and \(\pi_*(L(G, n+1)) = 0\) by Van Kampen when \(n \geq 2\). Since \(\rho(G)\) is a quasi-functor of \(G\), so is \(\tilde{\rho}_*(G)\) and, hence, \(L(G, n + 1)\).

Let \(\mathcal{W}_\infty^w\) denote the category of based connected finite-dimensional \(CW\) complexes. If \(X \in |\mathcal{W}_\infty^w|\) and \(Y \in |\mathcal{W}_\infty^w|\), then we define

\[
\{X, Y\} = \lim_{\longrightarrow} [\Sigma^k X, \Sigma^k Y],
\]

and we recall that \(\{X, -\}\) is a special homology functor on \(\mathcal{W}_\infty^w\).

Therefore, applying \(\{X, -\}\) to 3.6, we obtain an exact sequence

\[
(3.7) \quad \{X, \tilde{F}_n G\} \xrightarrow{\tilde{\sigma}_G^n} \{X, \tilde{R}_n G\} \longrightarrow \{X, L(G, n+1)\} \longrightarrow \{X, \tilde{F}_{n+1} G\} \xrightarrow{\tilde{\sigma}_G^{n+1}} \{X, \tilde{R}_{n+1} G\}.
\]

But clearly \(\{X, \tilde{F}_n G\} \cong \text{Hom}(FG, \pi_S^n(X))\) by an isomorphism which is natural in \(X\) and also natural in \(G(\pi_S^n(X) = \{X, S^*\})\). Therefore, as in § 2 we obtain the following theorem.

**Theorem 3.8.** Let \(G\) be a finitely generated Abelian group. Let \(n \geq 2\) and let \(X \in |\mathcal{W}_\infty^w|\). Then there is a short exact sequence

\[
(3.9) \quad 0 \longrightarrow \text{Ext} (G, \pi_S^{n-1}(X)) \longrightarrow \{X, L(G, n)\} \longrightarrow \text{Hom} (G, \pi_S^n(X)) \longrightarrow 0
\]

which is natural in \(X\) and strongly quasi-natural in \(G\). The sequence splits if, for some fixed \(X, \{X, L(G, n)\}\) is a functor of \(G\).

As a corollary of this theorem, we have the following result of Hilton-Olum-see [4].

**Corollary 3.10.** Let \(G_1\) and \(G_2\) be finitely generated Abelian
groups and \( n \geq 4 \). Then there is a short exact sequence

\[
0 \longrightarrow T(G_1) \otimes G_2 \otimes Z_2 \longrightarrow [L(G_2, n), L(G_1, n)] \longrightarrow \text{Hom}(G_1, G_2) \longrightarrow 0
\]

which is strongly quasi-natural in \( G_1 \) and \( G_2 \), where \( T(G) \) = torsion subgroup of \( G \) and \( G^* = \text{Hom}(G, Q/Z)(\equiv G \) if \( G \) is finite).

**Proof.** Applying 3.9 to \( G = G_1 \) and \( X = L(G_2, n) \), we get

\[
0 \longrightarrow \text{Ext}(G_1, \pi_3^{-1}(L(G_2, n))) \longrightarrow [L(G_2, n), L(G_1, n)] \longrightarrow \text{Hom}(G_1, \pi_3(L(G_2, n))) \longrightarrow 0.
\]

But for \( n \geq 4 \)

\[
\pi_3^{-1}(L(G_2, n)) \cong G_2 \otimes Z_2
\]

\[
[L(G_2, n), L(G_1, n)] \cong [L(G_2, n), L(G_1, n)]
\]

and

\[
\pi_3(L(G_2, n)) \cong G_2 , \text{ so we have for } n \geq 4
\]

\[
0 \longrightarrow \text{Ext}(G_1, G_2 \otimes Z_2) \longrightarrow [L(G_2, n), L(G_1, n)] \longrightarrow \text{Hom}(G_1, G_2) \longrightarrow 0.
\]

Now we are done since \( \text{Ext}(G_1, -) \cong T(G_1) \otimes - \) as functors on the category of finitely generated Abelian groups.

4. Some examples and a conjecture. The general problem of computing \( k^*(X; G) \), for a given homology theory \( h_* \) and group \( G \), is very difficult, even when the group is injective. For example, if \( h_q = \pi_q^S = \{S^q, -\} \) and \( G = Q \), then

\[
k^q(X; Q) \cong H^q(X; Q)
\]

by an easy argument based on Serre’s result [6] that \( \pi_q^S(S^r) \) is finite for \( r \neq q \). With \( h_* \) as above and \( G = Q/Z \) it is easy to establish

\[
k^q(S^r; Q/Z) \cong \begin{cases} 
\pi_q^S(S^r), & r \neq q \\
Q/Z, & r = q
\end{cases}
\]

Thus computing \( k^*(X; Q/Z) \) in this case amounts to knowing the stable homotopy groups of spheres!

If the homology theory \( h_* \) is represented by a spectrum \( B \), then the spectrum \( B(G) \) which represents \( k^*(-; G) \) can be thought of as obtained from \( B \) by introducing \( G \) coefficients. The spectrum \( B \) also represents a cohomology theory, and we have the following

**Conjecture 4.3.** If \( \pi_* B \) is a ring of cohomological dimension 1,
then there is a homotopy equivalence of spectra $B \simeq B(Z)$.

This conjecture simply says that our method and Adams' [1] coincide over rings of cohomological dimension 1—where his spectral sequence collapses to a Universal Coefficient Sequence.

**Remark 4.4.** It is *not* true in general that $k^*(-; Z)$ is the cohomology theory associated to the spectrum $B$ which represents $h_*$. For example, if, as above, $B = \text{sphere spectrum}$ and $h_* = \text{stable homotopy}$ is the homology theory represented by $B$, then

$$k^*(S^n; Z) = 0$$

for all $q > n$.

But the cohomology functor associated to the sphere spectrum is stable cohomotopy, and certainly

$$\pi^n(S^n) \neq 0$$

for all $q > n$.

In particular, $k^*(S^{n+1}; Z) = 0 \neq Z_2 = \pi^n(S^{n+1})$.

**References**


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