

# Pacific Journal of Mathematics

## **ALGEBRAIC STRUCTURE FOR A SET OF NONLINEAR INTEGRAL OPERATIONS**

DAVID LOWELL LOVELADY

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**A generalized addition is introduced for a set of generators, and a generalized multiplication is introduced for a set of evolution systems. Then the mapping which takes a generator to the corresponding evolution system becomes an isomorphism. Necessary and sufficient conditions are found for the generalized addition to reduce to addition, and hence, under these conditions, we are able to write a formula for the evolution system generated by the sum of two generators.**

**Preliminaries.** Let  $S = [0, \infty)$ , and let  $(G, +)$  be a complete normed abelian group with norm  $N_1$ . Let  $H$  be the set to which  $A$  belongs only in case  $A$  is a function from  $G$  to  $G$ ,  $A[0] = 0$ , and there is a number  $b$  so that  $N_1[A[p] - A[q]] \leq bN_1[p - q]$  whenever  $(p, q)$  is in  $G \times G$ . If  $A$  is in  $H$ , let  $N_2[A]$  be the least number  $b$  so that  $N_1[A[p] - A[q]] \leq bN_1[p - q]$  whenever  $(p, q)$  is in  $G \times G$ , and let  $N_3[A]$  be the least number  $b$  so that  $N_1[A[p]] \leq bN_1[p]$  whenever  $p$  is in  $G$ .

Let  $OA^+$ ,  $OM^+$ , and  $\mathcal{E}^+$  be as in [8]. Let  $OA$  be the set to which  $V$  belongs only in case  $V$  is a function from  $S \times S$  to  $H$  so that

- (i)  $V(x, y) + V(y, z) = V(x, z)$  whenever  $(x, y, z)$  is in  $S \times S \times S$  and  $y$  is between  $x$  and  $z$ , and
- (ii) there is a member  $\alpha$  of  $OA^+$  so that

$$N_2[V(a, b)] \leq \alpha(a, b)$$

whenever  $(a, b)$  is in  $S \times S$ .

If  $\alpha$  and  $V$  are related as in (ii),  $\alpha$  will be said to dominate  $V$ .

Let  $OM$  be the set to which  $W$  belongs only in case  $W$  is a function from  $S \times S$  to  $H$  so that

- (i)  $W(x, y)W(y, z) = W(x, z)$  whenever  $(x, y, z)$  is in  $S \times S \times S$  and  $y$  is between  $x$  and  $z$ , where the multiplication is composition, and
- (ii) there is a member  $\mu$  of  $OM^+$  so that

$$N_2[W(a, b) - I] \leq \mu(a, b) - 1$$

whenever  $(a, b)$  is in  $S \times S$ , where  $I$  in  $H$  is given by  $I[p] = p$ .

The following theorem is due to Mac Nerney [9].

**THEOREM 1.** *There is a bijection  $\mathcal{E}$  from  $OA$  onto  $OM$  so that if  $V$  is in  $OA$  and  $W$  is in  $OM$ , then (i), (ii), (iii), (iv), and (v) are*

equivalent.

- (i)  $W = \mathcal{E}[V]$ .
- (ii)  $W(a, b)[p] = {}_a\Pi^b[I + V][p]$  whenever  $(a, b, p)$  is in  $S \times S \times G$ .
- (iii)  $V(a, b)[p] = {}_a\Sigma^b[W - I][p]$  whenever  $(a, b, p)$  is in  $S \times S \times G$ .
- (iv) There is  $(\alpha, \mu)$  in  $\mathcal{E}^+$  so that

$$N_3[W(a, b) - I - V(a, b)] \leq \mu(a, b) - 1 - \alpha(a, b)$$

whenever  $(a, b)$  is in  $S \times S$ .

(v) If  $(a, p)$  is in  $S \times G$ , and  $h$  is given by  $h(t) = W(t, a)[p]$ , then  $h$  has bounded  $N_1$ -variation on each bounded interval of  $S$ , and is the only such function such that

$$h(t) = p + (R) \int_t^a V[h]$$

whenever  $t$  is in  $S$ .

REMARK 1. The notions of  $\Pi, \Sigma$ , and  $(R) \int$  are to be taken as in [9].

Let  $OAI$  be that subset of  $OA$  to which  $V$  belongs only in case each of  $I + V(t, t^+), I + V(t, t^-), I + V(t^+, t)$ , and  $I + V(t^-, t)$  has inverse in  $H$  whenever  $t$  is in  $S$ . The following theorem is due to Herod [6] (see also [4] and [5]).

THEOREM 2. Let  $(V, W)$  be in  $\mathcal{E}$ . Then (i) and (ii) are equivalent.

- (i)  $V$  is in  $OAI$ .
- (ii) Each value of  $W$  has inverse in  $H$ .

Furthermore, there is a bijection  $\mathcal{E}$  from  $OAI$  onto  $OAI$  such that if  $V$  is in  $OAI$ , then each of (iii), (iv), (v), and (vi) is true.

- (iii)  $\mathcal{E}[\mathcal{E}[V]] = V$ .
- (iv)  $\mathcal{E}[V](a, b) = -V(b, a)$  for each  $(a, b)$  in  $S \times S$  only in case  ${}_a\Sigma^b N_3[V[I - V] - V] = 0$  whenever  $(a, b)$  is in  $S \times S$ .
- (v)  $\mathcal{E}[\mathcal{E}[V]](a, b) \cdot \mathcal{E}[V](b, a) = \mathcal{E}[V](b, a) \cdot \mathcal{E}[\mathcal{E}[V]](a, b) = I$  whenever  $(a, b)$  is in  $S \times S$ .
- (vi)  $\mathcal{E}[V](a, b)[p] = {}_b\Sigma^a V[I + V]^{-1}[p]$  whenever  $(a, b, p)$  is in  $S \times S \times G$ .

### The $\oplus$ Operation.

LEMMA 1. If each of  $\alpha$  and  $\beta$  is in  $OA^+$ , and  $(a, b)$  is in  $S \times S$ , then  ${}_a\Sigma^b\alpha[1 + \beta]$  exists and is the greatest lower bound of the set to which  $r$  belongs only in case there is a chain  $(t_k)_{k=0}^n$  from  $a$  to  $b$  so that  $r = \Sigma_{k=1}^n \alpha(t_{k-1}, t_k)[1 + \beta(t_{k-1}, t_k)]$ .

*Proof.* It suffices to show that if  $(a, b, c)$  is in  $S \times S \times S$ , and  $b$  is between  $a$  and  $c$ , then

$$\alpha(a, c)[1 + \beta(a, c)] \geq \alpha(a, b)[1 + \beta(a, b)] + \alpha(b, c)[1 + \beta(b, c)].$$

But  $\alpha(a, c) \geq \alpha(a, b)$  and  $\alpha(a, c) \geq \alpha(b, c)$ , so

$$\begin{aligned} \alpha(a, c)\beta(a, c) &= \alpha(a, c)\beta(a, b) + \alpha(a, c)\beta(b, c) \\ &\geq \alpha(a, b)\beta(a, b) + \alpha(b, c)\beta(b, c), \end{aligned}$$

and the proof is complete.

**THEOREM 3.** *If each of  $V_1$  and  $V_2$  is in  $OA$ , and  $(a, b, p)$  is in  $S \times S \times G$ , then  ${}_a\Sigma^b V_i[I + V_2][p]$  exists. If, for  $i = 1, 2$ ,  $\alpha_i$  in  $OA^+$  dominates  $V_i$ , then*

$$\begin{aligned} N_3[V_1(a, b)[I + V_2(a, b)] - {}_a\Sigma^b V_1[I + V_2]] \\ \leq \alpha_1(a, b)[1 + \alpha_2(a, b)] - {}_a\Sigma^b \alpha_1[1 + \alpha_2] \end{aligned}$$

whenever  $(a, b)$  is in  $S \times S$ . Furthermore, if  $U$  is given by  $U(a, b)[p] = {}_a\Sigma^b V_1[I + V_2][p]$ , then  $U$  is in  $OA$ .

*Proof.* Let  $(a, b, c, p)$  be in  $S \times S \times S \times G$ , with  $b$  between  $a$  and  $c$ . Now

$$\begin{aligned} N_1[V_1(a, c)[I + V_2(a, c)][p] - V_1(a, b)[I + V_2(a, b)][p] \\ - V_1(b, c)[I + V_2(b, c)][p]] \\ = N_1[V_1(a, b)[I + V_2(a, c)][p] - V_1(a, b)[I + V_2(a, b)][p] \\ + V_1(b, c)[I + V_2(a, c)][p] - V_1(b, c)[I + V_2(b, c)][p]] \\ \leq [\alpha_1(a, b)\alpha_2(b, c) + \alpha_1(b, c)\alpha_2(a, b)]N_1[p] \\ = N_1[p](\alpha_1(a, c)[1 + \alpha_2(a, c)] - \alpha_1(a, b)[1 + \alpha_2(a, b)] \\ - \alpha_1(b, c)[1 + \alpha_2(b, c)]). \end{aligned}$$

The theorem is now clear.

**DEFINITION 1.** If each of  $V_1$  and  $V_2$  is in  $OA$ , then  $V_1 \oplus V_2$  is that member  $U$  of  $OA$  given by

$$U(a, b)[p] = V_2(a, b)[p] + {}_a\Sigma^b V_1[I + V_2][p].$$

**DEFINITION 2.** If  $V$  is in  $OA$ ,  $V^*$  will be that member of  $OA$  given by  $V^*(a, b) = V(b, a)$ .

**THEOREM 4.** *If each of  $V_1, V_2$ , and  $V_3$  is in  $OA$ , then*

$$V_1 \oplus (V_2 \oplus V_3) = (V_1 \oplus V_2) \oplus V_3,$$

and consequently  $(OA, \oplus)$  is a semigroup.  $(OAI, \oplus)$  is a subgroup of  $(OA, \oplus)$ , each subgroup of  $(OA, \oplus)$  is contained in  $OAI$ , and if  $V$  is in  $OAI$ , then

$$V \oplus \mathcal{S}[V]^* = \mathcal{S}[V]^* \oplus V = 0.$$

*Proof.* Let  $U$  be given by

$$\begin{aligned} U(a, b)[p] &= V_3(a, b)[p] + {}_a\Sigma^b V_2[I + V_3][p] \\ &\quad + {}_a\Sigma^b V_1[I + V_2][I + V_3][p]. \end{aligned}$$

A moment's reflection shows

$$V_1 \oplus (V_2 \oplus V_3) = U = (V_1 \oplus V_2) \oplus V_3,$$

so the first part of the theorem is clear.

Now if  $A$  is in  $H$ , and  $I + A$  has inverse in  $H$ , then

$$\begin{aligned} -A[I + A]^{-1} + A[I - A[I + A]^{-1}] \\ = -A[I + A]^{-1} + A[[I + A] - A][I + A]^{-1} = 0. \end{aligned}$$

This, with (vi) of Theorem 2, says that if  $V$  is in  $OAI$ , then  $V \oplus \mathcal{S}[V]^* = 0$ . Similarly,  $\mathcal{S}[V]^* \oplus V = 0$ , so  $(OAI, \oplus)$  is a group.

To complete the proof it suffices to show that if  $U$  and  $V$  are in  $OA$ , and  $U \oplus V = V \oplus U = 0$ , then  $U$  is in  $OAI$  and  $V = \mathcal{S}[U]^*$ . If  $t$  is in  $S$ , then  $[U \oplus V](t, t^+) = 0$ , so

$$\begin{aligned} U(t, t^+)[I + V(t, t^+)] + V(t, t^+) &= 0, \\ U(t, t^+)[I + V(t, t^+)] + [I + V(t, t^+)] &= I, \\ [I + U(t, t^+)] [I + V(t, t^+)] &= I. \end{aligned}$$

Similarly, since  $[V \oplus U](t, t^+) = 0$ , we have

$$[I + V(t, t^+)] [I + U(t, t^+)] = I.$$

Similar computations for  $(t, t^-)$ ,  $(t^+, t)$ , and  $(t^-, t)$  show that each of  $U$  and  $V$  is in  $OAI$ . Also, it is clear that  $V$  is given by

$$V(a, b)[p] = -{}_a\Sigma^b U[I + U]^{-1}[p] = \mathcal{S}[U]^*(a, b)[p],$$

so the proof is complete.

**LEMMA 2.** *Let each of  $\alpha_1$  and  $\alpha_2$  be in  $OA^+$ , and let  $\beta$  be a continuous member of  $OA^+$ . Suppose  $\beta(a, b) \leqslant {}_a\Sigma^b \alpha_1 \alpha_2$  whenever  $(a, b)$  is in  $S \times S$ . Then  $\beta = 0$ .*

**REMARK 2.** Lemma 2 is immediate, and we shall not prove it here.

**THEOREM 5.** *Let each of  $V_1$  and  $V_2$  be in  $OA$ . Then (i) and (ii) are equivalent, and (iii) and (iv) are equivalent.*

$$(i) \quad V_1 \oplus V_2 = V_1 + V_2.$$

(ii)  $V_1[I + V_2] - V_1 = 0$  at all "pairs" of the forms  $(t, t^+)$ ,  $(t, t^-)$ ,  $(t^+, t)$ , and  $(t^-, t)$  for  $t$  in  $S$ .

$$(iii) \quad V_1 \oplus V_2 = V_2 \oplus V_1.$$

(iv)  $V_1 - V_2 = V_1[I + V_2] - V_2[I + V_1]$  at all "pairs" of the forms  $(t, t^+)$ ,  $(t, t^-)$ ,  $(t^+, t)$ , and  $(t^-, t)$  for  $t$  in  $S$ .

*Proof.* We shall indicate the first equivalence, and leave the second to the reader. Since  $[V_1 \oplus V_2] - [V_1 + V_2] = \Sigma V_i[I + V_j] - V_i$ , it is clear that (i) implies (ii). Now suppose (ii). For  $i = 1, 2$ , let  $\alpha_i$  in  $OA^+$  dominate  $V_i$ . Let  $\beta$  in  $OA^+$  be given by  $\beta(a, b) = {}_a\Sigma^b N_3[V_1[I + V_2] - V_1]$ . Now, by (ii),  $\beta$  is continuous, and clearly  $\beta(a, b) \leq {}_a\Sigma^b \alpha_1 \alpha_2$  whenever  $(a, b)$  is in  $S \times S$ . Thus  $\beta = 0$ , (i) follows, and the proof is complete.

### The $\otimes$ Operation and the Exponential Identity.

**THEOREM 6.** *Let each of  $(V_1, W_1)$  and  $(V_2, W_2)$  be in  $\mathcal{E}$ , and let  $(a, b, p)$  be in  $S \times S \times G$ . Then each of*

$${}_aH^b[I + V_1][I + V_2][p] \quad \text{and} \quad {}_aH^b W_1 W_2 [p]$$

*exists, and they are equal. Furthermore, if  $M$  is given by*

$$M(a, b)[p] = {}_aH^b W_1 W_2 [p],$$

*then  $M$  is in  $OM$ .*

*Proof.* Let  $U = V_1 \oplus V_2$ . Let  $\alpha$  be a member of  $OA^+$  which dominates each of  $U, V_1$ , and  $V_2$ , and let  $\mu = \mathcal{E}^+[\alpha]$ . Let  $(a, b, p)$  be in  $S \times S \times G$ , and let  $(t_k)_{k=0}^n$  be a chain from  $a$  to  $b$ . Now, by [7, Lemma 4],

$$\begin{aligned} N_1[{}_aH^b_{k=1}^n [I + U(t_{k-1}, t_k)][p] - {}_aH^b_{k=1}^n [I + V_1(t_{k-1}, t_k)][I + V_2(t_{k-1}, t_k)][p]] \\ \leq N_1[p] \mu(a, b) {}_a\Sigma^b_{k=1}^n N_3[V_1(t_{k-1}, t_k)[I + V_2(t_{k-1}, t_k)] \\ - {}_{t_{k-1}}\Sigma^{t_k} V_1[I + V_2]] \\ \leq N_1[p] \mu(a, b) {}_a\Sigma^b_{k=1}^n \alpha(t_{k-1}, t_k) [1 + \alpha(t_{k-1}, t_k)] - {}_a\Sigma^b \alpha [1 + \alpha]. \end{aligned}$$

It is now clear that  ${}_aH^b[I + V_1][I + V_2][p]$  exists and equals  ${}_aH^b[I + U][p]$  whenever  $(a, b, p)$  is in  $S \times S \times G$ . Now [9, Lemma 1.2] tells us that  ${}_aH^b W_1 W_2 [p] = {}_aH^b [I + V_1][I + V_2][p]$  whenever  $(a, b, p)$  is in  $S \times S \times G$ . Since these products describe  $\mathcal{E}[U]$ , it is clear that  $M$  is in  $OM$  and the proof is complete.

DEFINITION 3. If each of  $W_1$  and  $W_2$  is in  $OM$ ,  $W_1 \otimes W_2$  is that member  $M$  of  $OM$  given by  $M(a, b)[p] = {}_aH^b W_1 W_2[p]$ .

There emerges from the proof of Theorem 6 a fact which we now record.

THEOREM 7. *If each of  $V_1$  and  $V_2$  is in  $OA$ , then*

$$\mathcal{E}[V_1 \oplus V_2] = \mathcal{E}[V_1] \otimes \mathcal{E}[V_2].$$

REMARK 3. Theorem 7, together with the first equivalence of Theorem 5, includes and extends Theorem 6 of [7].

THEOREM 8. *Let  $V_1$  be in  $OA$ ,  $V_2$  in  $OAI$ . Let  $U$  in  $OA$  be given by*

$$U(a, b)[p] = {}_a\Sigma^b V_1[I + V_2]^{-1}[p].$$

*Then*

$$\mathcal{E}[V_1 + V_2] = \mathcal{E}[U] \otimes \mathcal{E}[V_2].$$

*Proof.* Let  $(a, b, p)$  be in  $S \times S \times G$ . Now

$$\begin{aligned} [\mathcal{E}[U] \otimes \mathcal{E}[V_2]](a, b)[p] &= {}_aH^b \mathcal{E}[U] \mathcal{E}[V_2][p] \\ &= {}_aH^b [I + U][I + V_2][p] \\ &= {}_aH^b [I + V_1[I + V_2]^{-1}][I + V_2][p] \\ &= {}_aH^b [I + V_1 + V_2][p] \\ &= \mathcal{E}[V_1 + V_2](a, b)[p]. \end{aligned}$$

This completes the proof.

REMARK 4. Note that by using Theorems 5, 7, and 8 we can compute, under two different sets of hypotheses,  $\mathcal{E}[V_1 + V_2]$  in terms of the  $\otimes$  operation.

REMARK 5. The notion of continuously multiplying solutions for generators in order to construct the solution for a sum of generators has been used by Trotter [11] and Chernoff [1], [2] for the case of autonomous linear differential equations with discontinuous linear operators, by Helton [3] for the case of linear Stieltjes integral equations, and by Mermin [10] for the case of autonomous nonlinear differential equations with accretive operators.

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