ALGEBRAIC STRUCTURE FOR A SET OF NONLINEAR INTEGRAL OPERATIONS

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DAVID LOWELL LOVELADY

A generalized addition is introduced for a set of generators, and a generalized multiplication is introduced for a set of evolution systems. Then the mapping which takes a generator to the corresponding evolution system becomes an isomorphism. Necessary and sufficient conditions are found for the generalized addition to reduce to addition, and hence, under these conditions, we are able to write a formula for the evolution system generated by the sum of two generators.

Preliminaries. Let \( S = [0, \infty) \), and let \((G, +)\) be a complete normed abelian group with norm \( N \). Let \( H \) be the set to which \( A \) belongs only in case \( A \) is a function from \( G \) to \( G \), \( A[0] = 0 \), and there is a number \( b \) so that \( N[A[p] - A[q]] \leq bN[p - q] \) whenever \((p, q)\) is in \( G \times G \). If \( A \) is in \( H \), let \( N_1[A]\) be the least number \( b \) so that \( N[A[p] - A[q]] \leq bN[p - q] \) whenever \((p, q)\) is in \( G \times G \), and let \( N_1[A]\) be the least number \( b \) so that \( N_1[A[p]] \leq bN_1[p] \) whenever \( p \) is in \( G \).

Let \( OA^+, OM^+, \) and \( \mathcal{E}^+ \) be as in [8]. Let \( OA \) be the set to which \( V \) belongs only in case \( V \) is a function from \( S \times S \) to \( H \) so that

(i) \( V(x, y) + V(y, z) = V(x, z) \) whenever \((x, y, z)\) is in \( S \times S \times S \) and \( y \) is between \( x \) and \( z \), and

(ii) there is a member \( \alpha \) of \( OA^+ \) so that

\[
N_1[V(a, b)] \leq \alpha(a, b)
\]

whenever \((a, b)\) is in \( S \times S \).

If \( \alpha \) and \( V \) are related as in (ii), \( \alpha \) will be said to dominate \( V \).

Let \( OM \) be the set to which \( W \) belongs only in case \( W \) is a function from \( S \times S \) to \( H \) so that

(i) \( W(x, y)W(y, z) = W(x, z) \) whenever \((x, y, z)\) is in \( S \times S \times S \) and \( y \) is between \( x \) and \( z \), where the multiplication is composition, and

(ii) there is a member \( \mu \) of \( OM^+ \) so that

\[
N_1[W(a, b) - I] \leq \mu(a, b) - 1
\]

whenever \((a, b)\) is in \( S \times S \), where \( I \) in \( H \) is given by \( I[p] = p \).

The following theorem is due to Mac Nerney [9].

**Theorem 1.** There is a bijection \( \mathcal{E} \) from \( OA \) onto \( OM \) so that if \( V \) is in \( OA \) and \( W \) is in \( OM \), then (i), (ii), (iii), (iv), and (v) are
equivalent.

(i) \( W = \mathcal{E}[V] \).

(ii) \( W(a, b)[p] = \mathcal{N}^a[I + V][p] \) whenever \((a, b, p)\) is in \( S \times S \times G \).

(iii) \( V(a, b)[p] = \mathcal{N}^b[W - I][p] \) whenever \((a, b, p)\) is in \( S \times S \times G \).

(iv) There is \((\alpha, \mu)\) in \( \mathcal{E}^+ \) so that

\[
N_\mu[W(a, b) - I - V(a, b)] \leq \mu(a, b) - 1 - \alpha(a, b)
\]

whenever \((a, b)\) is in \( S \times S \).

(v) If \((a, p)\) is in \( S \times G \), and \( h \) is given by \( h(t) = W(t, a)[p] \), then \( h \) has bounded \( N_\mu \)-variation on each bounded interval of \( S \), and is the only such function such that

\[
h(t) = p + (R) \int_t^a V[h]
\]

whenever \( t \) is in \( S \).

Remark 1. The notions of \( \mathcal{N}, \Sigma, \) and \((R)\int\) are to be taken as in [9].

Let \( OAI \) be that subset of \( OA \) to which \( V \) belongs only in case each of \( I + V(t, t^+) \), \( I + V(t, t^-) \), \( I + V(t^+, t) \), and \( I + V(t^-, t) \) has inverse in \( H \) whenever \( t \) is in \( S \). The following theorem is due to Herod [6] (see also [4] and [5]).

**Theorem 2.** Let \((V, W)\) be in \( \mathcal{E} \). Then (i) and (ii) are equivalent.

(i) \( V \) is in \( OAI \).

(ii) Each value of \( W \) has inverse in \( H \).

Furthermore, there is a bijection \( \mathcal{E} \) from \( OAI \) onto \( OAI \) such that if \( V \) is in \( OAI \), then each of (iii), (iv), (v), and (vi) is true.

(iii) \( \mathcal{E}[\mathcal{E}[V]] = V \).

(iv) \( \mathcal{E}[V](a, b) = -V(b, a) \) for each \((a, b)\) in \( S \times S \) only in case

\[
\mathcal{N}^a \mathcal{N}^b[V[I - V] - V] = 0 \quad \text{whenever} \quad (a, b) \quad \text{is in} \quad S \times S.
\]

(v) \( \mathcal{E}[\mathcal{E}[V]](a, b) \cdot \mathcal{E}[V](b, a) = \mathcal{E}[V](b, a) \cdot \mathcal{E}[\mathcal{E}[V]](a, b) = I \)

whenever \((a, b)\) is in \( S \times S \).

(vi) \( \mathcal{E}[V](a, b)[p] = -\mathcal{N}^aV[I + V]^{-1}[p] \) whenever \((a, b, p)\) is in \( S \times S \times G \).

The \( \oplus \) Operation.

**Lemma 1.** If each of \( \alpha \) and \( \beta \) is in \( OA^+ \), and \((a, b)\) is in \( S \times S \), then \( \mathcal{N}^a \alpha[1 + \beta] \) exists and is the greatest lower bound of the set to which \( r \) belongs only in case there is a chain \((t_k)_{k=0}^n \) from \( a \) to \( b \) so that \( r = \mathcal{N}^a \alpha(t_{k-1}, t_k)[1 + \beta(t_{k-1}, t_k)] \).
Proof. It suffices to show that if \((a, b, c)\) is in \(S \times S \times S\), and \(b\) is between \(a\) and \(c\), then
\[
\alpha(a, c)[1 + \beta(a, c)] \geq \alpha(a, b)[1 + \beta(a, b)] + \alpha(b, c)[1 + \beta(b, c)] .
\]
But \(\alpha(a, c) \geq \alpha(a, b)\) and \(\alpha(a, c) \geq \alpha(b, c)\), so
\[
\alpha(a, c)\beta(a, c) = \alpha(a, c)\beta(a, b) + \alpha(a, c)\beta(b, c) \\
\geq \alpha(a, b)\beta(a, b) + \alpha(b, c)\beta(b, c) ,
\]
and the proof is complete.

**Theorem 3.** If each of \(V_i\) and \(V_2\) is in OA, and \((a, b, p)\) is in \(S \times S \times G\), then \(Z\Sigma^b V_i[I + V_2][p]\) exists. If, for \(i = 1, 2\), \(\alpha_i\) in \(OA^+\) dominates \(V_i\), then
\[
N[a^i(a, b)][I + V_2(a, b)] - \Sigma^b V_i[I + V_2] \\
\leq \alpha_i(a, b)[1 + \alpha_i(a, b)] - \Sigma^b \alpha_i[1 + \alpha_i]
\]
whenever \((a, b)\) is in \(S \times S\). Furthermore, if \(U\) is given by \(U(a, b)[p] = Z\Sigma^b V_i[I + V_2][p]\), then \(U\) is in OA.

**Proof.** Let \((a, b, c, p)\) be in \(S \times S \times S \times G\), with \(b\) between \(a\) and \(c\). Now
\[
N_i[V_i(a, c)[I + V_2(a, c)][p] - V_i(a, b)[I + V_2(a, b)][p] \\
- V_i(b, c)[I + V_2(b, c)][p]] \\
= N_i[V_i(a, b)[I + V_2(a, c)][p] - V_i(a, b)[I + V_2(a, b)][p] \\
+ V_i(b, c)[I + V_2(a, c)][p] - V_i(b, c)[I + V_2(b, c)][p]] \\
\leq [\alpha_i(a, b)\alpha_i(b, c) + \alpha_i(b, c)\alpha_i(a, b)]N_i[p] \\
= N_i[p](\alpha_i(a, c)[1 + \alpha_i(a, c)] - \alpha_i(a, b)[1 + \alpha_i(a, b)] \\
- \alpha_i(b, c)[1 + \alpha_i(b, c)]) .
\]
The theorem is now clear.

**Definition 1.** If each of \(V_i\) and \(V_2\) is in OA, then \(V_i \oplus V_2\) is that member \(U\) of OA given by
\[
U(a, b)[p] = V_2(a, b)[p] + \Sigma^b V_i[I + V_2][p] .
\]

**Definition 2.** If \(V\) is in OA, \(V^\ast\) will be that member of OA given by \(V^\ast(a, b) = V(b, a)\).

**Theorem 4.** If each of \(V_i\), \(V_2\), and \(V_3\) is in OA, then
\[
V_1 \oplus (V_2 \oplus V_3) = (V_1 \oplus V_2) \oplus V_3 ,
\]
and consequently $(OA, \oplus)$ is a semigroup. $(OAI, \oplus)$ is a subgroup of $(OA, \oplus)$, each subgroup of $(OA, \oplus)$ is contained in $OAI$, and if $V$ is in $OAI$, then

$$V \oplus \mathcal{C}[V]^* = \mathcal{C}[V]^* \oplus V = 0.$$  

**Proof.** Let $U$ be given by

$$U(a, b)[p] = V_3(a, b)[p] + \sum V_2[I + V_3][p]$$

$$+ \sum V_1[I + V_2][I + V_3][p].$$

A moment's reflection shows

$$V_1 \oplus (V_2 \oplus V_3) = U = (V_1 \oplus V_2) \oplus V_3,$$

so the first part of the theorem is clear.

Now if $A$ is in $H$, and $I + A$ has inverse in $H$, then


This, with (vi) of Theorem 2, says that if $V$ is in $OAI$, then $V \oplus \mathcal{C}[V]^* = 0$. Similarly, $\mathcal{C}[V]^* \oplus V = 0$, so $(OAI, \oplus)$ is a group.

To complete the proof it suffices to show that if $U$ and $V$ are in $OA$, and $U \oplus V = V \oplus U = 0$, then $U$ is in $OAI$ and $V = \mathcal{C}[U]^*$. If $t$ is in $S$, then $[U \oplus V](t, t^+) = 0$, so

$$U(t, t^+)[I + V(t, t^+)] + V(t, t^+) = 0,$$

$$U(t, t^+)[I + V(t, t^+)] + V(t, t^+) = I,$$

$$[I + U(t, t^+)]I + V(t, t^+) = I.$$  

Similarly, since $[V \oplus U](t, t^+) = 0$, we have

$$[I + V(t, t^+)]I + U(t, t^+) = I.$$  

Similar computations for $(t, t^-), (t^+, t)$, and $(t^-, t)$ show that each of $U$ and $V$ is in $OAI$. Also, it is clear that $V$ is given by

$$V(a, b)[p] = -\sum U[I + U]^{-1}[p] = \mathcal{C}[U]^*(a, b)[p],$$

so the proof is complete.

**Lemma 2.** Let each of $\alpha_1$ and $\alpha_2$ be in $OA^+$, and let $\beta$ be a continuous member of $OA^+$. Suppose $\beta(a, b) \leq \sum \alpha_1 \alpha_2$ whenever $(a, b)$ is in $S \times S$. Then $\beta = 0$.

**Remark 2.** Lemma 2 is immediate, and we shall not prove it here.
THEOREM 5. Let each of $V_1$ and $V_2$ be in OA. Then (i) and (ii) are equivalent, and (iii) and (iv) are equivalent.

(i) $V_1 \oplus V_2 = V_1 + V_2$.

(ii) $V_1[I + V_2] - V_1 = 0$ at all "pairs" of the forms $(t, t^+), (t, t^-), (t^+, t)$, and $(t^-, t)$ for $t$ in $S$.

(iii) $V_1 \oplus V_2 = V_2 \oplus V_1$.

(iv) $V_1 - V_2 = V_1[I + V_2] - V_2[I + V_1]$ at all "pairs" of the forms $(t, t^+), (t, t^-), (t^+, t)$, and $(t^-, t)$ for $t$ in $S$.

Proof. We shall indicate the first equivalence, and leave the second to the reader. Since $[V_1 \oplus V_2] - [V_1 + V_2] = \Sigma V_3[I + V_2] - V_1$, it is clear that (i) implies (ii). Now suppose (ii). For $i = 1, 2$, let $\alpha_i$ in $OA^+$ dominate $V_i$. Let $\beta$ in $OA^+$ be given by $\beta(a, b) = \Sigma^b N_\alpha[V_1[I + V_2] - V_i]$. Now, by (ii), $\beta$ is continuous, and clearly $\beta(a, b) \leq \Sigma^b \alpha \alpha$, whenever $(a, b)$ is in $S \times S$. Thus $\beta = 0$, (i) follows, and the proof is complete.

The $\otimes$ Operation and the Exponential Identity.

THEOREM 6. Let each of $(V_1, W_1)$ and $(V_2, W_2)$ be in $\mathcal{S}$, and let $(a, b, p)$ be in $S \times S \times G$. Then each of

\[ \varepsilon \Pi^b [I + V_1][I + V_2][p] \quad \text{and} \quad \varepsilon \Pi^b W_1 W_2[p] \]

exists, and they are equal. Furthermore, if $M$ is given by

\[ M(a, b)[p] = \varepsilon \Pi^b W_1 W_2[p], \]

then $M$ is in $OM$.

Proof. Let $U = V_1 \oplus V_2$. Let $\alpha$ be a member of $OA^+$ which dominates each of $U, V_1$, and $V_2$, and let $\mu = \mathcal{S}^+ [\alpha]$. Let $(a, b, p)$ be in $S \times S \times G$, and let $(t_k)_{k=0}^\infty$ be a chain from $a$ to $b$. Now, by [7, Lemma 4],

\[ N_i[I^\infty_k = 1[I + U(t_{k-1}, t_k)][p] - I^\infty_k[I + V_1(t_{k-1}, t_k)][I + V_2(t_{k-1}, t_k)][p] \]

\[ \leq N_i[p] \mu(a, b)^2 \Sigma_{k=1}^\infty N_\alpha[V_1(t_{k-1}, t_k)[I + V_2(t_{k-1}, t_k)] \]

\[ - \tau_{k-1} \Sigma^t V_1[I + V_2] \]

\[ \leq N_i[p] \mu(a, b)^2 \Sigma_{k=1}^\infty \alpha(t_{k-1}, t_k)[1 + \alpha(t_{k-1}, t_k)] - \Sigma^b \alpha[1 + \alpha]. \]

It is now clear that $\varepsilon \Pi^b [I + V_1][I + V_2][p]$ exists and equals $\varepsilon \Pi^b [I + U][p]$ whenever $(a, b, p)$ is in $S \times S \times G$. Now [9, Lemma 1.2] tells us that $\varepsilon \Pi^b W_1 W_2[p] = \varepsilon \Pi^b [I + V_1][I + V_2][p]$ whenever $(a, b, p)$ is in $S \times S \times G$. Since these products describe $\mathcal{S}[U]$, it is clear that $M$ is in $OM$ and the proof is complete.
DEFINITION 3. If each of $W_1$ and $W_2$ is in $OM$, $W_1 \otimes W_2$ is that member $M$ of $OM$ given by $M(a, b)[p] = \mathcal{H}^b W_1 W_2[p]$.

There emerges from the proof of Theorem 6 a fact which we now record.

**Theorem 7.** If each of $V_1$ and $V_2$ is in $OA$, then

$$\mathcal{E}[V_1 \ominus V_2] = \mathcal{E}[V_1] \otimes \mathcal{E}[V_2].$$

**Remark 3.** Theorem 7, together with the first equivalence of Theorem 5, includes and extends Theorem 6 of [7].

**Theorem 8.** Let $V_1$ be in $OA$, $V_2$ in $OAI$. Let $U$ in $OA$ be given by

$$U(a, b)[p] = \mathcal{H}^b V_1[I + V_2]^{-i}[p].$$

Then

$$\mathcal{E}[V_1 + V_2] = \mathcal{E}[U] \otimes \mathcal{E}[V_2].$$

**Proof.** Let $(a, b, p)$ be in $S \times S \times G$. Now

$$[\mathcal{E}[U] \otimes \mathcal{E}[V_2]](a, b)[p] = \mathcal{H}^b \mathcal{E}[U] \mathcal{E}[V_2][p]$$

$$= \mathcal{H}^b[I + U][I + V_2][p]$$

$$= \mathcal{H}^b[I + V_1[I + V_2]^{-i}][I + V_2][p]$$

$$= \mathcal{H}^b[I + V_1 + V_2][p]$$

$$= \mathcal{E}[V_1 + V_2](a, b)[p].$$

This completes the proof.

**Remark 4.** Note that by using Theorems 5, 7, and 8 we can compute, under two different sets of hypotheses, $\mathcal{E}[V_1 + V_2]$ in terms of the $\otimes$ operation.

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Received June 4, 1970.

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<table>
<thead>
<tr>
<th>Authors</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Charles Compton Alexander</td>
<td>Semi-developable spaces and quotient images of metric spaces</td>
<td>277</td>
</tr>
<tr>
<td>Ram Prakash Bambah and Alan C. Woods</td>
<td>On a problem of Danzer</td>
<td>295</td>
</tr>
<tr>
<td>John A. Beekman and Ralph A. Kallman</td>
<td>Gaussian Markov expectations and related integral equations</td>
<td>303</td>
</tr>
<tr>
<td>Frank Michael Cholewinski and Deborah Tepper Haimo</td>
<td>Inversion of the Hankel potential transform</td>
<td>319</td>
</tr>
<tr>
<td>John H. E. Cohn</td>
<td>The diophantine equation $Y(Y + 1)(Y + 2)(Y + 3) = 2X(X + 1)(X + 2)(X + 3)$</td>
<td>331</td>
</tr>
<tr>
<td>Philip C. Curtis, Jr. and Henrik Stetkaer</td>
<td>A factorization theorem for analytic functions operating in a Banach algebra</td>
<td>337</td>
</tr>
<tr>
<td>Doyle Otis Cutler and Paul F. Dubois</td>
<td>Generalized final rank for arbitrary limit ordinals</td>
<td>345</td>
</tr>
<tr>
<td>Keith A. Ekblaw</td>
<td>The functions of bounded index as a subspace of a space of entire functions</td>
<td>353</td>
</tr>
<tr>
<td>Dennis Michael Girard</td>
<td>The asymptotic behavior of norms of powers of absolutely convergent Fourier series</td>
<td>357</td>
</tr>
<tr>
<td>John Gregory</td>
<td>An approximation theory for elliptic quadratic forms on Hilbert spaces: Application to the eigenvalue problem for compact quadratic forms</td>
<td>383</td>
</tr>
<tr>
<td>Paul C. Kainen</td>
<td>Universal coefficient theorems for generalized homology and stable cohomotopy</td>
<td>397</td>
</tr>
<tr>
<td>Aldo Joram Lazar and James Ronald Retherford</td>
<td>Nuclear spaces, Schauder bases, and Choquet simplexes</td>
<td>409</td>
</tr>
<tr>
<td>David Lowell Lovelady</td>
<td>Algebraic structure for a set of nonlinear integral operations</td>
<td>421</td>
</tr>
<tr>
<td>John McDonald</td>
<td>Compact convex sets with the equal support property</td>
<td>429</td>
</tr>
<tr>
<td>Forrest Miller</td>
<td>Quasivector topologies</td>
<td>445</td>
</tr>
<tr>
<td>Marion Edward Moore and Arthur Steger</td>
<td>Some results on completability in commutative rings</td>
<td>453</td>
</tr>
<tr>
<td>A. P. Morse</td>
<td>Taylor’s theorem</td>
<td>461</td>
</tr>
<tr>
<td>Richard E. Phillips, Derek J. S. Robinson and James Edward Roseblade</td>
<td>Maximal subgroups and chief factors of certain generalized soluble groups</td>
<td>475</td>
</tr>
<tr>
<td>Doron Ravdin</td>
<td>On extensions of homeomorphisms to homeomorphisms</td>
<td>481</td>
</tr>
<tr>
<td>John William Rosenthal</td>
<td>Relations not determining the structure of $\mathcal{L}$</td>
<td>497</td>
</tr>
<tr>
<td>Prem Lal Sharma</td>
<td>Proximity bases and subbases</td>
<td>515</td>
</tr>
<tr>
<td>Larry Smith</td>
<td>On ideals in $\Omega^g$</td>
<td>527</td>
</tr>
<tr>
<td>Warren R. Wogen</td>
<td>von Neumann algebras generated by operators similar to normal operators</td>
<td>539</td>
</tr>
<tr>
<td>R. Grant Woods</td>
<td>Co-absolutes of remainders of Stone-Čech compactifications</td>
<td>545</td>
</tr>
</tbody>
</table>