SOME RESULTS ON COMPLETABILITY IN COMMUTATIVE RINGS

MARION EDWARD MOORE AND ARTHUR STEGER
SOME RESULTS ON COMPLETABILITY 
IN COMMUTATIVE RINGS

MARION MOORE AND ARTHUR STEGER

In this paper, \( R \) always denotes a commutative ring with identity. The ideal of nilpotents and the Jacobson radical of the ring \( R \) are denoted by \( N(R) \) and \( J(R) \), respectively. The vector \([a_1, \ldots, a_n]\) is called a primitive row vector provided \( 1 \in (a_1, \ldots, a_n) \); a primitive row vector \([a_1, \ldots, a_n]\) is called completable provided there exists an \( n \times n \) unimodular matrix over \( R \) with first row \( a_1, \ldots, a_n \). A ring \( R \) is called a \( B \)-ring if given a primitive row vector \([a_1, \ldots, a_n]\), \( n \geq 3 \), and

\[
(a_1, \ldots, a_{n-2}) \not\in J(R),
\]

there exists \( b \in R \) such that \( 1 \in (a_1, \ldots, a_{n-2}, a_{n-1} + ba_n) \). Similarly, \( R \) is defined to be a Strongly \( B \)-ring (SB-ring), if \( d \in (a_1, \ldots, a_n) \), \( n \geq 3 \), and \( (a_1, \ldots, a_{n-2}) \not\in J(R) \) implies that there exists \( b \in R \) such that \( d \in (a_1, \ldots, a_{n-2}, a_{n-1} + ba_n) \).

In this paper it is proved that every primitive vector over a \( B \)-ring is completable. It is shown that the following are \( B \)-rings: \( \pi \)-regular rings, quasi-semi-local rings, Noetherian rings in which every (proper) prime ideal is maximal, and adequate rings. In addition it is proved that \( R[X] \) is a \( B \)-ring if and only if \( R \) is a completely primary ring. It is then shown that the following are SB-rings: quasi-local rings, any ring which is both an Hermite ring and a \( B \)-ring, and Dedekind domains. Finally, it is shown that \( R[X] \) is an SB-ring if and only if \( R \) is a field.

2. \( B \)-rings.

**Lemma 2.1.** Let \( R \) be a ring with \( A \subseteq J(R) \), \( A \) an ideal of \( R \). Then \( R \) is a \( B \)-ring if and only if \( R/A \) is a \( B \)-ring.

**Proof.** Necessity: Let \( R \) be a \( B \)-ring and let

\[
(1 + A) \in (a_1 + A, \ldots, a_n + A), \quad n \geq 3
\]

and

\[
(a_1 + A, \ldots, a_{n-2} + A) \not\subseteq J(R/A) = J(R)/A,
\]

where \( a_i \in R, i = 1, \ldots, n \). Then \( 1 + A = \sum_{i=1}^{n} a_i b_i + A, b_i \in R \); hence \([a_1, \ldots, a_n]\) is primitive. Since \( (a_1, \ldots, a_{n-2}) \not\subseteq J(R) \), it follows that \([a_1 + A, \ldots, a_{n-2} + A, a_{n-1} + ba_n + A]\) is primitive for some \( b \in R \). Therefore, \( R/A \) is a \( B \)-ring.

Sufficiency: Suppose \( R/A \) is a \( B \)-ring and suppose \([a_1, \ldots, a_n]\) is a
primitive vector with \((a_1, \cdots, a_{n-2}) \subseteq J(R)\). Hence \([a_i + A, \cdots, a_n + A]\) is a primitive vector; and, since \((a_1, \cdots, a_{n-2}) \subseteq J(R)\), we have \((a_1 + A, \cdots, a_{n-2} + A) \not\subseteq J(R/A)\). Since \(R/A\) is a \(B\)-ring, there exists \(b + A \in R/A\) such that \([a_i + A, \cdots, a_{n-2} + A, (a_{n-1} + ba_n) + A]\) is primitive. It follows that \((1 - u) \in A \subseteq J(R)\), where

\[
u = \sum_{i=1}^{n-2} a_i b_i + (a_{n-1} + b a_n) b_{n-1}, b_i \in R, i = 1, \cdots, n - 1.
\]

Therefore, \(\nu\) is a unit of \(R\); i.e., \([a_1, \cdots, a_{n-2}, a_{n-1} + ba_n]\) is primitive.

**Theorem 2.1.** If \(R\) is a \(B\)-ring then every primitive row vector over \(R\) is completable.

*Proof.* Let \(R\) be a \(B\)-ring and let \(1 \in (a_1, \cdots, a_n)\). The theorem clearly holds for \(n = 1\). If \(n = 2\), then \(1 = a_1 x + a_2 y, x, y \in R\) and the matrix \(\begin{pmatrix} a_1 & a_2 \\ -y & x \end{pmatrix}\) is unimodular; hence the result holds for \(n = 2\).

Let \(n \geq 3\), and suppose the result is established for \(k < n\).

**Case 1.** If \((a_1, \cdots, a_{n-2}) \subseteq J(R)\) and \(1 = \sum_{i=1}^{n-2} a_i b_i, b_i \in R\), then \(1 - \sum_{i=1}^{n-2} a_i b_i = a_{n-1} b_{n-1} + a_n b_n\) is a unit \(\nu \in R\). Let

\[
V = \begin{pmatrix} a_{n-1} & a_n & a_1 & a_2 & \cdots & a_{n-2} \\ -b_n & b_{n-1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.
\]

Then \(V\) has determinant \(\nu\), and it follows that \([a_1, \cdots, a_n]\) is completable.

**Case 2.** If \((a_1, \cdots, a_{n-2}) \not\subseteq J(R)\), then \(1 \in (a_1, \cdots, a_{n-2}, a_{n-1} + ba_n)\), for some \(b \in R\). By the induction hypothesis, \([a_1, \cdots, a_{n-2}, a_{n-1} + ba_n]\) is completable to an \((n - 1) \times (n - 1)\) unimodular matrix \(D\). Let

\[
U = \begin{pmatrix} I_{n-2} & 0 & 0 \\ 0 & \cdots & 1 \\ 0 & \cdots & -b \end{pmatrix} \quad \text{and let} \quad B = \begin{pmatrix} a_n \\ D \\ \vdots \\ 0 \cdots 0 \end{pmatrix}.
\]

Then \(BU\) is an \(n \times n\) unimodular matrix whose first row is \([a_1, \cdots, a_n]\). For convenience, we introduce the notation \(Z(A)\) to mean the set of maximal ideals containing the ideal \(A\); \(Z(a)\) will denote the set of maximal ideals containing the element \(a\).

**Theorem 2.2** If \(R\) is a ring such that for every ideal \(A \not\subseteq J(R)\), \(Z(A)\) is finite, then \(R\) is a \(B\)-ring.
Proof. The essentials of the proof are due to Reiner [4]. Let $1 \in (a_1, \ldots, a_n)$, $n \geq 3$, and $(a_1, \ldots, a_{n-1}) \not\subseteq J(R)$. By the hypothesis on $R$, $Z(A)$ is finite where $A = (a_1, \ldots, a_{n-2})$. Let $Z(A) = \{M_1, \ldots, M_r\}$, and note that if $b \in R$ and $a_{n-1} + ba_n \in M_i, i = 1, \ldots, r, \text{ then } [a_1, \ldots, a_{n-2}, a_{n-1} + ba_n]$ is primitive.

For any $M_i \in Z(A)$ such that $a_n \in M_i$, we have $a_{n-1} + ba_n \in M_i$, for all $b \in R$; otherwise, $a_{n-1} \in M_i$, and $(a_1, \ldots, a_n) \subseteq M_i$ which contradicts the hypothesis that $[a_1, \ldots, a_n]$ is primitive.

For those $M_i \in Z(A)$ for which $a_n \in M_i$, we have $(a_n, M_i) = (1)$. Hence there exists an $x_i$ such that $a_n x_i \equiv a_{n-1} \pmod{M_i}$. For these $M_i$, we can find (by the Chinese Remainder Theorem) an element $b \in R$ such that $b \equiv 1 - x_i \pmod{M_i}$. It follows that $a_{n-1} + ba_n \in M_i, i = 1, \ldots, r$. Hence $[a_1, \ldots a_{n-2}, a_{n-1} + ba_n]$ is primitive.

It follows from this theorem that quasi-semi-local rings and Noetherian rings in which every proper prime ideal is maximal (in particular, Dedekind domains) are $B$-rings.

**Lemma 2.2.** Let $R$ be an $F$-ring (i.e., a ring in which every finitely generated ideal is principal) which satisfies the condition that if $1 \in (a_1, a_2, a_3)$ with $a_i \in J(R)$ then $1 \in (a_1, a_2 + ba_3)$ for some $b \in R$. Then $R$ is a $B$-ring.

**Proof.** Let $1 \in (a_1, \ldots, a_n)$, $n \geq 3$, and let $(a) = (a_1, \ldots, a_{n-1}) \not\subseteq J(R)$. Hence $1 \in (a, a_{n-1}, a_n)$. By the hypothesis on $R$, $1 \in (a, a_{n-1} + ba_n)$; hence, $R$ is a $B$-ring.

**Theorem 2.3.** If $R$ is an $F$-ring which satisfies the condition that for every $a, c \in R$ with $a \in J(R)$, there is an $r \in R$ such that $Z(r) = Z(a) \setminus Z(c)$, then $R$ is a $B$-ring.

**Proof.** The proof is essentially the same as the proof of Theorem 5 of [2]. Let $1 \in (a, b, c), a \in J(R)$. By the hypothesis on $R$ there exists $r \in R$ such that $Z(r) = Z(a) \setminus Z(c)$. Hence $(c, r) = (1)$, so there exists $q \in R$ such that $1 \in (r, b + qc)$. We claim $(a, b + qc) = (1)$. Otherwise, there exists a maximal ideal $M$ of $R$ such that $(a, b + qc) \subseteq M$. Hence $M \in Z(a)$ and $M \in Z(b + qc)$. Since $1 \in (r, b + qc)$ it follows that $M \in Z(r)$, so $M \in Z(c)$. But we now have $M \in Z(b)$, contrary to $(a, b, c) = (1)$. Therefore $(a, b + qc) = (1)$. Lemma 2.2 completes the proof.

**Theorem 2.4.** Every adequate ring is a $B$-ring.

**Proof.** In the proof of Theorem 5.3 of [3], Kaplansky shows that
if $R$ is an adequate ring and if $1 \in (a, b, c)$, $a \neq 0$, then there exists $q \in R$ such that $1 \in (a, b + qc)$. Since an adequate ring is an $F$-ring, the result follows from Lemma 2.2.

**Theorem 2.5.** Every $\pi$-regular ring is a B-ring.

**Proof.** If $R$ is a $\pi$-regular ring, and if $a \in R/N(R)$, then by Lemma 2.2 of [5], $a$ is an associate of $e + \beta$, $e$ an idempotent and $\beta$ a nilpotent of the $\pi$-regular ring $R/N(R)$. Since $\beta = 0$, $a = we$, $w$ a unit of $R/N(R)$. Therefore, $e^2 = we$ and $w^{-1}e^2 = we = a$. Hence, $R/N(R)$ is a regular ring and therefore an adequate ring ([1, Th. 11]). Theorem 2.4 and Lemma 2.1 complete the proof.

**Theorem 2.6.** Let $D$ be an integral domain, $K$ its quotient field. Let $R = \{(a, \cdots, a_k, a, a, \cdots): a_k \in K, a \in D\}$, where $k$ is a nonnegative integer ($k$ may be different for distinct elements of $R$). The operations in $R$ are component-wise addition and multiplication. If $R$ is a B-ring then $D$ is a B-domain.

We illustrate the proof. Suppose $R$ is a B-ring and let $1 \in (a, b, c)$, $a, b, c \in D$, $1 = aa' + bb' + cc'$. Let $\hat{a} = (1, a, a, \cdots)$, $\hat{b} = (0, b, b, \cdots)$, $\hat{c} = (0, c, \cdots)$, $\hat{a}' = (1, a', a', \cdots)$, $\hat{b}' = (0, b', b', \cdots)$, $\hat{c}' = 0$, $c'$, $c'$, $\cdots$). Then $\hat{1} = \hat{a}\hat{a}' + \hat{b}\hat{b}' + \hat{c}\hat{c}'$. If $\hat{a} \in J(R)$, then $\hat{1} - \hat{a} = (0, 1 - a, 1 - a, \cdots)$ is a unit of $R$. Since this is false, $\hat{a} \notin J(R)$, hence $\hat{1} \in (\hat{a}, \hat{b} + \hat{y}\hat{c})$ for some $\hat{y} \in R$. Therefore $\hat{1} = \hat{a}\hat{a}' + (\hat{b} + \hat{y}\hat{c})\hat{c}'$, where $\hat{a}, \hat{c}, \hat{e} \in R$. Let $\hat{d} = (d_1, \cdots, d_p, d, d, \cdots)$, $\hat{e} = (e_1, \cdots, e, e, e, \cdots)$, $\hat{y} = (y, \cdots, y, y, y, \cdots)$ and let $\lambda = \max(1, p, q, r)$. In the $(\lambda + 1)$st entry of $\hat{a}\hat{d} + (\hat{b} + \hat{y}\hat{c})\hat{e}$, we have $ad + (b + yc)e$; i.e., $1 \in (a, b + ye)$. Hence, $D$ is a B-domain.

**Theorem 2.7.** $R[X]$ is a B-ring if and only if $R$ is a completely primary ring.

**Proof.** Sufficiency: Let $R$ be a completely primary ring. Since $R/N(R)$ is a field and since $(R/N(R))[X] \cong R[X]/N(R)[X]$, it follows from Theorem 2.2 that $R[X]/N(R)[X]$ is a B-ring. Since $N(R)[X] = N(R[X])$, the result follows from Lemma 1.2.1.

Necessity: Assume that $R$ is not completely primary and that $R[X]$ is a B-ring. Let $r$ be a nonunit, nonnilpotent element of $R$. Then $1 \in (r, 1 + X, X^2)$ and $r \in J(R[X])$. By the assumption that $R[X]$ is a B-ring, we have $1 \in (r, 1 + X + X^2f(X))$ for some $f(X) \in R[X]$. Let $\bar{a}$ denote the image of $a \in R$ under the natural homomorphism of $R[X]$ onto $(R/rR)[X]$. Then $\bar{1} \in (0, 1 + X + X^2\bar{f}(X))$ and $1 + X + X^2\bar{f}(X)$ is a unit of $(R/rR)[X]$. This is a contradiction since the coefficient of $X$ is not nilpotent.
Since $R[X]$ cannot be completely primary, (clearly, $X$ is neither a unit nor a nilpotent) it follows that for every ring $R$, $R[X, Y] = R[X][Y]$ is not a $B$-ring.

3. **Strongly $B$-rings.** We now turn our attention to the study of $SB$-rings. Our main objective here is to compare the theory of this particular subclass of $B$-rings with that of $B$-rings given in the last section.

**Lemma 3.1.** $R$ is an $SB$-ring if and only if for every $s, c, c_3 \in R$ with $s \in (c, c_2, c_3)$ and $c_1 \in J(R)$, it follows that $s \in (c, c_2 + bc_3)$ for some $b \in R$.

**Proof.** The necessity clearly follows from the definition of an $SB$-ring.

Sufficiency: Let $r = (a_1, \ldots, a_n)$, $n \geq 3$, with $(a_1, \ldots, a_{n-1}) \not\subseteq J(R)$. Without loss of generality, we may assume that $a_{n-1} \in J(R)$. Suppose $r = \sum_{i=1}^{n} a_i x_i$ and let $s = a_{n-2} x_{n-2} + a_{n-1} x_n + a_n x_n$. Then $r \in (a_1, \ldots, a_{n-2}, a_{n-3}, a_3)$ and $s \in (a_{n-2}, a_{n-1}, a_n)$. Since $a_{n-2} \in J(R), s \in (a_{n-2}, a_{n-1} + ba_n)$ for some $b \in R$. Therefore $r \in (a_1, \ldots, a_{n-2}, s) \subseteq (a_1, \ldots, a_{n-1} + ba_n)$, and the proof is complete.

In view of Lemma 3.1, we need only consider triples instead of arbitrary $n$-tuples in our study of $SB$-rings.

**Lemma 3.2.** The homomorphic image of an $SB$-ring is an $SB$-ring.

**Proof.** Let $\bar{R}$ be the image of $R$ under the homomorphism $\phi$, and let $\bar{d} = (\bar{a}, \bar{a}_z, \bar{a}_3)$ with $\bar{a}_i \in J(\bar{R}), \bar{a}, \bar{a}_z, \bar{a}_3 \in \bar{R}$. Suppose $\bar{d} = \sum_{i=1}^{n} \bar{a}_i \bar{x}_i$, $\bar{x}_i \in \bar{R}$ and let $a_i \phi = \bar{a}_i, x_i \phi = \bar{x}_i, i = 1, 2, 3$. Let $d = \sum_{i=1}^{n} a_i x_i$. Since $(J(R))\phi \subseteq J(\bar{R})$, we have $a_i \in J(R)$; hence, $d \in (a, a_2 + ba_n)$ for some $b \in R$. Since $d \phi = \bar{d}$, we have $\bar{d} \in (\bar{a}, \bar{a}_z + b \bar{a}_n)$, where $b \phi = \bar{b}$. Hence $\bar{R}$ is an $SB$-ring.

**Theorem 3.1.** Every quasi-local ring is an $SB$-ring.

**Proof.** Let $d \in (a, a_2, a_3)$, with $a_i \in J(R), R$ a quasi-local ring. Since $a_i \in J(R), a_i$ is a unit of $R$; hence, $d \in (a, a_2 + ba_n) = (1)$ for every $b \in R$.

**Lemma 3.3.** Let $A = (a_1, \ldots, a_n), n \geq 3,$ be an ideal in a Dedekind domain $R$. If $B = (a_1, \ldots, a_{n-1}) \neq (0)$, then $A = (a_1, \ldots, a_{n-2}, a_{n-1} + ba_n)$ for some $b \in R$. 

Proof. Let \( A = \prod_{i=1}^{r} M_i^{\alpha_i} \) and let \( B = \prod_{i=1}^{t} M_i^{\beta_i} \) be the representations of the ideals \( A \) and \( B \) as a product of powers of distinct maximal ideals. Since \( B \subseteq A \), we may order the \( M_i \) so that \( 0 \leq \alpha_i < \beta_i \) for \( 1 \leq i \leq r \), and \( \alpha_i = \beta_i \) for \( r+1 \leq i \leq t \). Let \( 1 \leq k \leq r \). We claim that either \( a_{n-1} \) or \( a_n \) does not belong to \( M_i^{\alpha_i+1} \). For suppose both \( a_{n-1} \) and \( a_n \) belong to \( M_i^{\alpha_i+1} \). Then \( A \not\subseteq M_i^{\alpha_i+1} \), a contradiction. Since the \( M_i^{\alpha_i+1} \) are relative prime, the Chinese Remainder Theorem guarantees the existence of a \( b \in R \) satisfying:

\[
\begin{align*}
 b &\equiv 0 \pmod{M_i^{\alpha_i+1}} \quad \text{if} \quad a_{n-1} \in M_i^{\alpha_i+1}, \\
 b &\equiv 1 \pmod{M_i^{\alpha_i+1}} \quad \text{if} \quad a_n \in M_i^{\alpha_i+1},
\end{align*}
\]

for \( k = 1, 2, \ldots, r \). It follows that \( a_{n-1} + ba_n \in M_i^{\alpha_i+1} \) for \( k = 1, 2, \ldots, r \). Let \( (a_1, \ldots, a_n, a_{n-1} + ba_n) = \prod_{i=1}^{r} M_i^{\alpha_i} \). Since \( (a_1, \ldots, a_n, a_{n-1} + ba_n) \not\in A = \prod_{i=1}^{r} M_i^{\alpha_i} \), it follows that \( \mu_i \geq \alpha_i \), \( i = 1, 2, \ldots, t \). Since \( B = \prod_{i=1}^{t} M_i^{\beta_i} \subseteq \prod_{i=1}^{r} M_i^{\alpha_i} \subseteq \prod_{i=1}^{r} M_i^{\alpha_i} = A \), and since \( \beta_i = \alpha_i \), \( r+1 \leq i \leq t \), it follows that \( \mu_i = \beta_i = \alpha_i \), \( r+1 \leq i \leq t \). If \( \mu_i > \alpha_i \) for some \( i \) with \( 1 \leq i \leq r \), then \( a_{n-1} + ba_n \in M_i^{\alpha_i} \subseteq M_i^{\alpha_i+1} \), a contradiction. Hence, \( \mu_i = \alpha_i \), \( i = 1, 2, \ldots, t \). Equivalently, \( (a_1, \ldots, a_n, a_{n-1} + ba_n) = A \).

As an immediate consequence, we have:

**Theorem 3.2.** A Dedekind domain is an SB-ring.

**Lemma 3.4.** Let \( R \) be a B-ring, let \( e = e^2 \in R \), and let \( e \in (a_1, \ldots, a_n) \) with \( (a_1, \ldots, a_n) \not\subseteq J(R), n \geq 3 \). Then \( e \in (a_1, \ldots, a_{n-2}, a_{n-1} + ba_n) \) for some \( b \in R \).

**Proof.** Since the case \( e = 1 \) is covered by the hypothesis, we may assume \( e \neq 1 \). Let \( e = \sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} (a_i e)(x_i e) \). Hence, \( 1 = (a_i e + 1 - e)(x_i e + 1 - e) + \sum_{i=2}^{n} (a_i e)(x_i e) \). Thus,

\[
1 \in (a_i e + 1 - e, a_i e, \ldots, a_n e).
\]

If \( a_i e + 1 - e \in J(R) \), then \( 1 - (a_i e + 1 - e) = e(1 - a_i) \) is a unit of \( R \), a contradiction since \( e = e^2, e \neq 1 \). Thus, since \( R \) is a B-ring, we have \( 1 \in (a_i e + 1 - e, a_i e, \ldots, a_n e, a_n e + ba_n) \) for some \( b \in R \). Therefore, \( e \in (a_i e, a_i e, \ldots, a_n e, a_n e + ba_n) \subseteq (a_i, a_i, \ldots, a_{n-2}, a_{n-1} + ba_n) \).

**Corollary.** If \( R \) is a regular ring then \( R \) is an SB-ring.

**Proof.** The result is immediate from Theorem 2.5 and Lemma 3.4; since, for every \( r \in R \), \( r \) is an associate of some idempotent \( e \in R \) ([1, Lemma 10]).

**Theorem 3.3.** If a B-ring \( R \) is also an Hermite ring, then \( R \) is an SB-ring.
Proof. Let $d \in (a_1, a_2, a_3) = (a), a_i \in J(R)$. By Corollary 5 of [1], there exist $b_1, b_2, b_3$ such that $a_i = b_i a, a_2 = b_2 a, a_3 = b_3 a$, and $(b_1, b_2, b_3) = (1)$. Since $R$ is a $B$-ring and since $b_i \in J(R)$, there exists a $q \in R$ such that $(b_1, b_2 + qb_3) = (1)$. Therefore, $(a) = (b_1 a, b_2 a + qb_3 a) = (a_1, a_2 + q a_3)$. Hence, $d \in (a_1, a_2 + q a_3)$.

COROLLARY. Every adequate domain is an SB-ring.

Proof. An adequate domain is both an $F$-domain and a $B$-ring. Since every $F$-domain is an Hermite ring, the result follows from Theorem 3.3.

COROLLARY. If $R$ is an $F$-ring with infinitely many maximal ideals and, if for every ideal $A \subseteq J(R), Z(A)$ is finite, then $R$ is an SB-ring.

Proof. $R$ is necessarily a $B$-ring by Theorem 2.2. By the proof of Corollary 2 of [2], $R$ is also an Hermite ring. Theorem 3.3 completes the proof.

THEOREM 3.4. $R[X]$ is an SB-ring if and only if $R$ is a field.

Proof. The sufficiency follows from Theorem 3.2. To prove the necessity, let $r \in R, r \neq 0$. Then $r \in (X^2, X, r)$ and $X^2 \in J(R[X])$. If $R[X]$ is an SB-ring then $r \in (X^2, X, X + rb(X))$ for some $b(X) \in R[X]$. Let $r = X^2 f(X) + (X + rb(X))g(X)$, where $f(X)$ and $g(X) \in R[X]$, and let $f_i, g_i, b_i$ represent the coefficient of $X^i$ in the polynomials $f(X), g(X), b(X)$, respectively. Equating coefficients in the above equation gives $r = rb_0 g_0$ and $0 = g_0 + r(b_0 g_1 + g_0 b_0)$. Hence $r$ divides $g_0$ and therefore $r = r^2 k$ for some $k \in R$. Hence $rk = (rk)^2$; therefore, $rk$ is an idempotent of $R$. Since $R[X]$ is a $B$-ring, $R$ must be a completely primary ring by Theorem 2.7. It follows that the idempotent $rk$ is either 0 or 1. Since $rk = 0$ and $r = r^2 k$ imply $r = 0$, we conclude that $rk = 1$; i.e., $r$ is a unit of $R$. Hence, $R$ is a field and the proof is complete.

REFERENCES


Received October 30, 1969.

University of Texas at Arlington
University of New Mexico
Charles Compton Alexander, *Semi-developable spaces and quotient images of metric spaces* ................................................................. 277
John A. Beekman and Ralph A. Kallman, *Gaussian Markov expectations and related integral equations* .............................................. 303
Frank Michael Cholewinski and Deborah Tepper Haimo, *Inversion of the Hankel potential transform* .................................................. 319
John H. E. Cohn, *The diophantine equation* ........................................ 331
Philip C. Curtis, Jr. and Henrik Stetkaer, *A factorization theorem for analytic functions operating in a Banach algebra* ............................. 337
Doyle Otis Cutler and Paul F. Dubois, *Generalized final rank for arbitrary limit ordinals* ................................................................. 345
Keith A. Ekblaw, *The functions of bounded index as a subspace of a space of entire functions* .......................................................... 353
Dennis Michael Girard, *The asymptotic behavior of norms of powers of absolutely convergent Fourier series* ......................................... 357
Paul C. Kainen, *Universal coefficient theorems for generalized homology and stable cohomotopy* ................................................ 397
Aldo Joram Lazar and James Ronald Retherford, *Nuclear spaces, Schauder bases, and Choquet simplexes* ............................................ 409
David Lowell Lovelady, *Algebraic structure for a set of nonlinear integral operations* ................................................................. 421
John McDonald, *Compact convex sets with the equal support property* ........ 429
Forrest Miller, *Quasivector topologies* .................................................. 445
Marion Edward Moore and Arthur Steger, *Some results on completability in commutative rings* .................................................... 453
A. P. Morse, *Taylor’s theorem* ............................................................. 461
Richard E. Phillips, Derek J. S. Robinson and James Edward Roseblade, *Maximal subgroups and chief factors of certain generalized soluble groups* ............................................................................. 475
Doron Ravdin, *On extensions of homeomorphisms to homeomorphisms* ...... 481
John William Rosenthal, *Relations not determining the structure of L* .......... 497
Prem Lal Sharma, *Proximity bases and subbases* ..................................... 515
Larry Smith, *On ideals in $\Omega^\omega$* .................................................... 527
Warren R. Wogen, *von Neumann algebras generated by operators similar to normal operators* .............................................................. 539
R. Grant Woods, *Co-absolutes of remainders of Stone-Čech compactifications* .................................................................................... 545